Last time:

Briefly discussed fundamental groups and covering spaces, "Galois correspondence". Given X,

$$\exists \quad \begin{matrix} \tilde{X} & \longrightarrow & X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \tilde{b} & \longmapsto & b \end{matrix} \quad \text{covering space, } \pi_1(\tilde{X},\tilde{b}) = 1$$

and $\pi_1(X,b)$ acts freely on $\tilde{X}$.

Then subgroups of $\pi_1(X,b) \overset{1\text{-}1}{\longleftrightarrow}$ covering spaces of X

$$G \longmapsto [\tilde{X}/G \longrightarrow X]$$

$$\pi_1(Y) \longleftarrow\!\!\mapsfrom [Y \longrightarrow X]$$

TA will go over this in more detail for those who want a review (or have never seen it) on Thursday

Application to free groups: $F = F\langle S \rangle$, S finite

We've seen F acts freely on a tree T

Quotient by action is a graph X, $\pi_1(X,b) \cong F$

$T \longrightarrow X$ covering space, $\pi_1 T = 1$, $\pi_1 X$ acts freely

So subgps of F $\leftrightarrow$ covering spaces of X

Now claim:

(1) A covering space of a graph is a graph

(2) The fundamental group of any graph is free

This gives <u>Theorem</u> subgroups of free groups are free.

Now want to introduce more geometry.

Will first go over some features of $\mathbb{H}^2$, the hyperbolic plane

$\longrightarrow$ Classical example of a metric space of constant negative curvature, prototype for negative curvature ideas in groups.

<u>Reference</u>. Casson and Bleiler, Autos of surfaces after Thurston

Define $\mathbb{H}^2 = \mathbb{H} =$ interior of unit disk D in $\mathbb{R}^2$ <u>as a set</u> $S^1 = \partial D$ is the <u>circle at ∞</u> S_∞

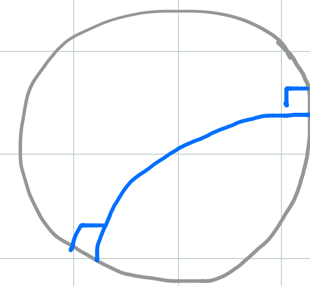
We will put a different metric on D to get $\mathbb{H}$

But we <u>first</u> define geodesics and isometries.

A geodesic in $\mathbb{H}$ is $C \cap D$,

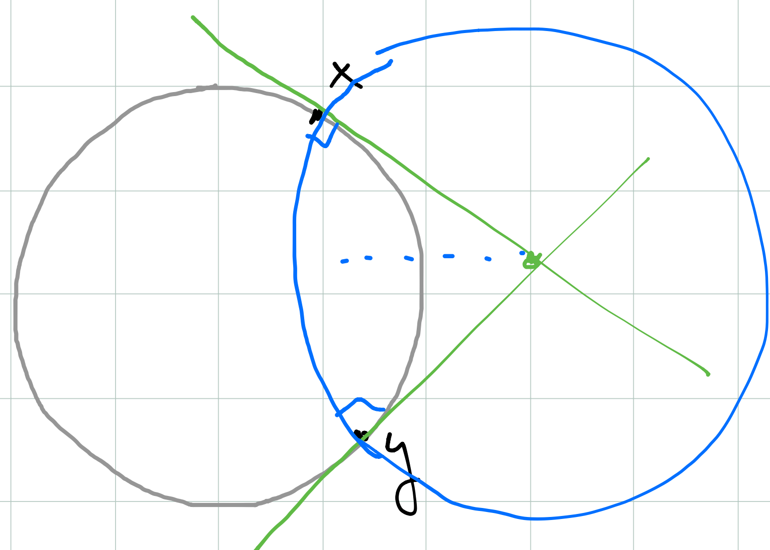
where C = circle $\perp$ to S_∞

(possibly line thru center)

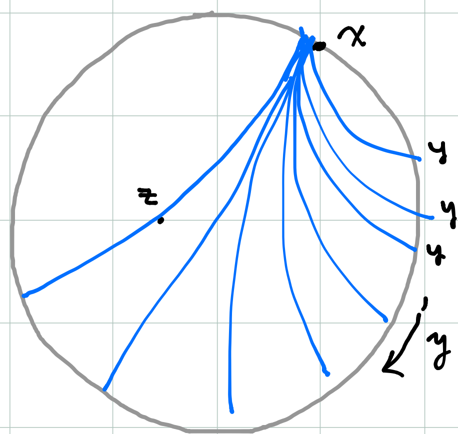


Note: There is a ! geodesic between any two points of $\mathbb{H}$

pf: First note $\exists!$ geodesic connecting two pts of S_∞:

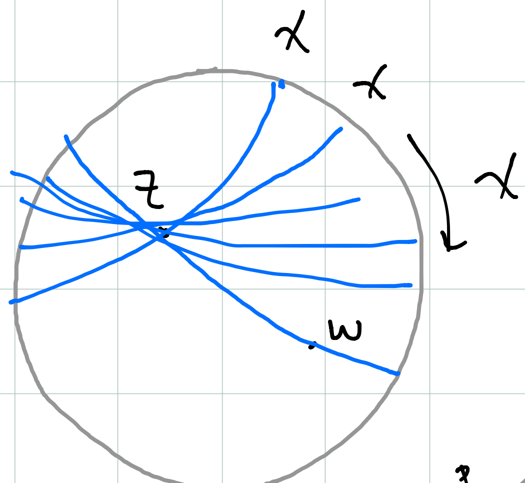


Then connect $x \in S_\infty$ with $z \in \mathbb{H}$:

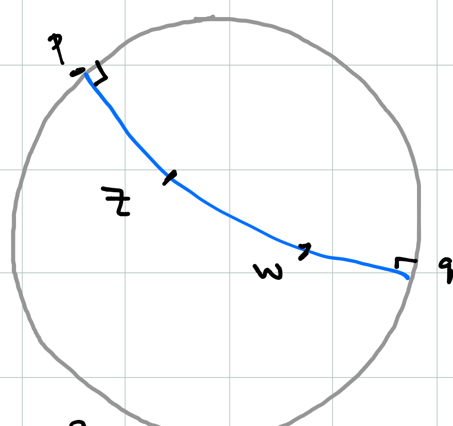


vary y: the circles fill D, one hits z

Then connect $z \in \mathbb{H}$ with $w \in \mathbb{H}$:



vary x, the blue circles fill $\mathbb{H}$, one meets w

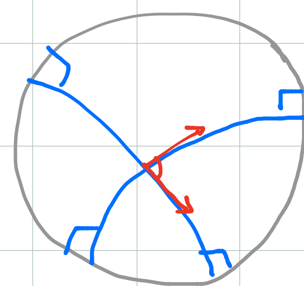


Unique?

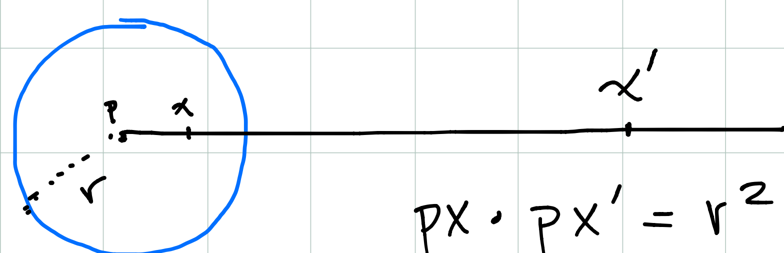
$\exists!$ circle thru p, z, w -

changing p alters $\angle$ of $\cap$ w/ S_∞.

Angles = Euclidean angle between tangent vectors



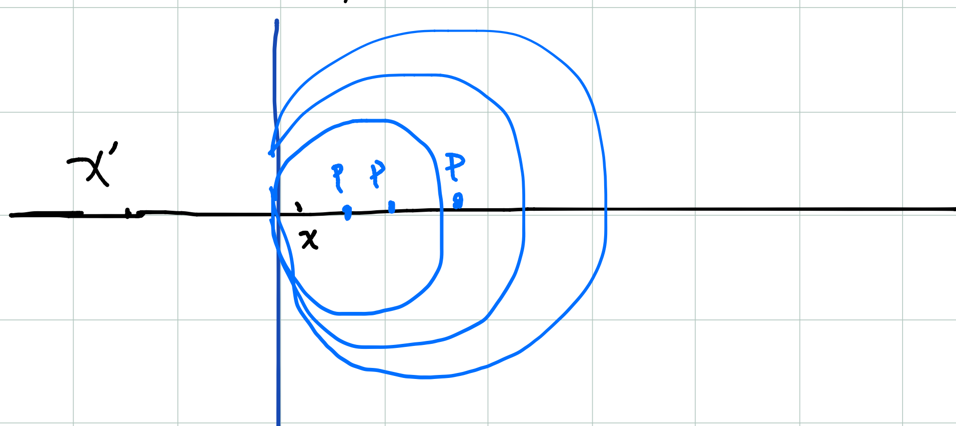
Reflection in a geodesic = inversion in C



$$px \cdot px' = r^2$$

(note this fixes C)

If C is a line, inversion = Euclidean reflection:



$$(p-x)(p-x') = p^2$$

$$\cancel{p^2} - px' - px + xx' = \cancel{p^2}$$

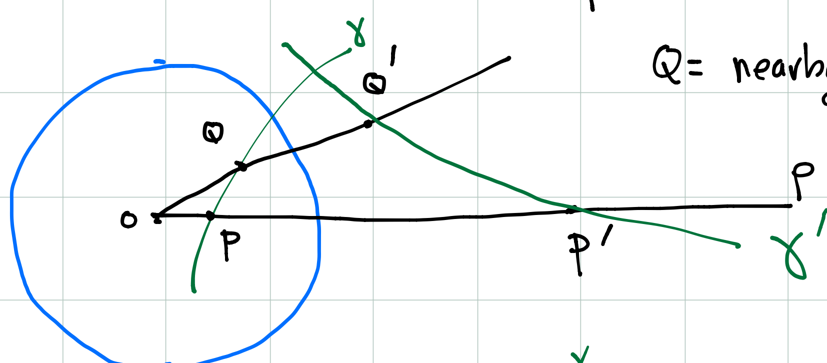
$$x'(x-p) = px$$

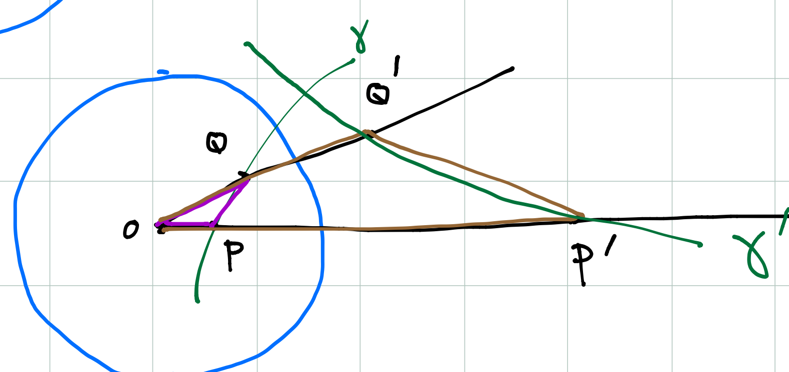
$$x' = \frac{px}{x-p} = \frac{x}{\frac{x}{p}-1} \to -x \text{ as } p \to \infty$$

<u>Lemma</u> Inversion preserves angles

<u>pf</u>: First look at angle γ makes with ray thru center of inversion circle: $P = \gamma \cap \rho$ $P' =$ image of P

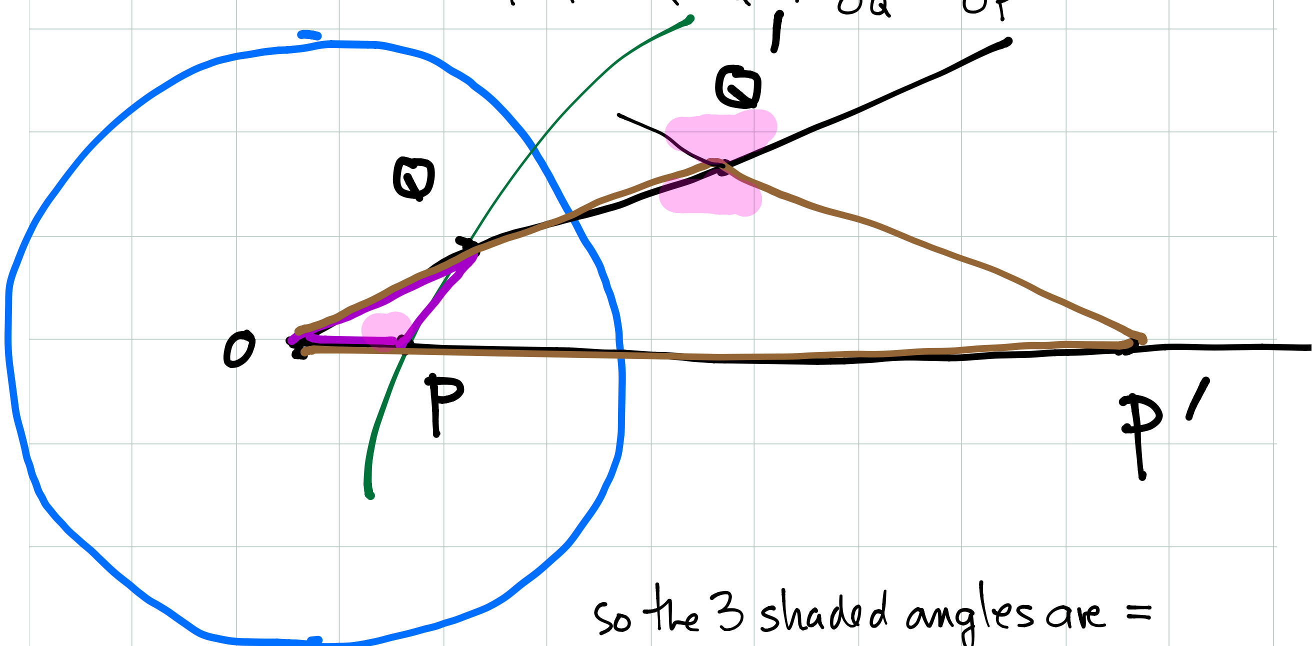
$Q =$ nearby point on γ





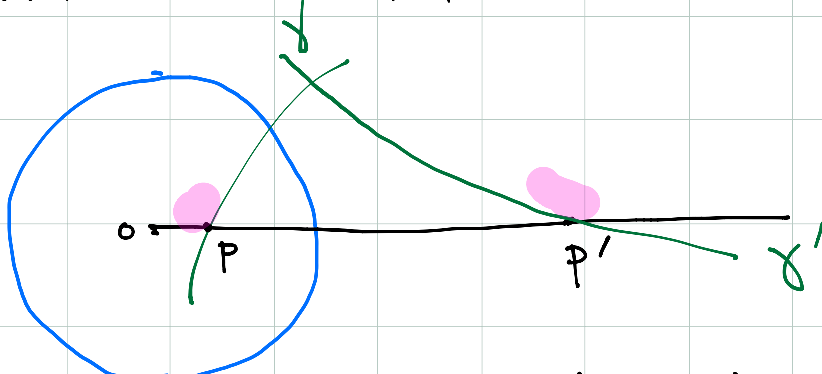
Triangles OPQ and OQ'P' are similar:

$$OP\cdot OP' = OQ\cdot OQ' \Rightarrow \frac{OP}{OQ'} = \frac{OQ}{OP'}$$



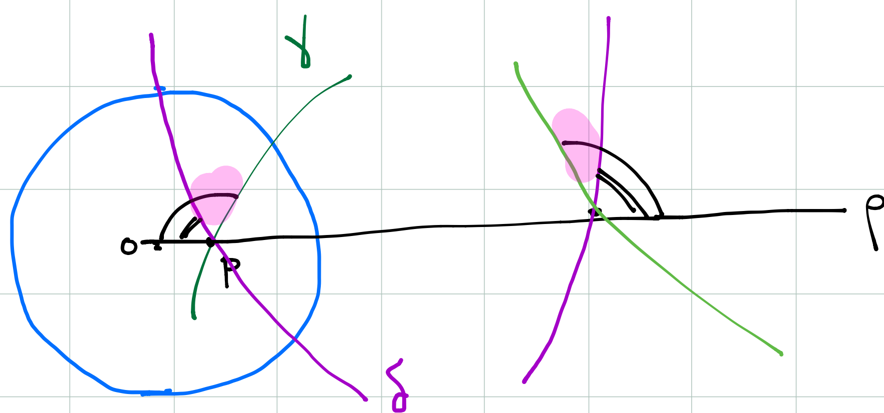
so the 3 shaded angles are =

Now let $Q \to P$. In the limit we have



ie the angle of γ with p is the same

So far the angle between two geodesics γ, δ

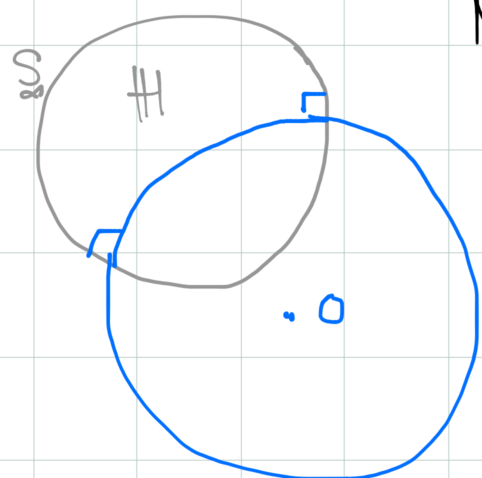


angle of each w/ p is preserved $\Rightarrow$ angle between is too.

Lemma 2 (Exercises) Inversion in C preserves circles (Hint: do this for inversion in unit circle $x^2+y^2=1$, then translate and scale)

Now can say what an isometry of $\mathbb{H}^2$ is:

Define reflection in a geodesic $C \cap \mathbb{H}$ to be inversion in C.



Note inversion in C preserves $\angle$'s, takes circles to circles

$\Rightarrow$ takes S_∞ to a circle orthogonal to C, thru same two pts $\Rightarrow S_\infty \to S_\infty$

$\Rightarrow \mathbb{H} \to \mathbb{H}$

Def An isometry of $\mathbb{H}$ is a product of reflections