GGT course - Warwick Sp 2014

Intro

GGT - primarily studies infinite discrete groups - algebraic objects - by studying their actions on geometric objects

Action: each $g \in G$ determines a symmetry of X

multiplying elts $\leftrightarrow$ composing symmetries

Uses tools from geometry, topology, algebra, analysis, ... to study these actions and draw conclusions about the groups.

Historically, group theory has always been linked with geometry -

Klein's Erlangen program: the geometry of a metric space is encoded in its group of isometries. Study the group algebraically in order to deduce geometric properties of the space.

GGT is a "backwards Erlangen program": study the geometry of the space to deduce algebraic properties of the group.

Outline

Need background in

- Metric spaces (assumed)
- Free groups and presentations
- Fundamental groups and covering spaces
- Hyperbolic geometry

Then can do

- Gromov hyperbolic metric spaces
- quasi-isometries
- Cayley graphs
- Hyperbolic groups
  - Examples
  - Properties
- Genericity of hyperbolic groups

Hyperbolic spaces related to <u>non</u>-hyperbolic groups

Curve complex and mapping class group

Sphere complex and $Out(F_n)$

Applications of curve complex hyperbolicity

Asymptotic dimension (Bestvina-Bromberg-Fujiwara)

Geometry of Outer space and sphere complex

CAT(0) spaces

CAT(0) groups

Boundary of a hyperbolic space, hyperbolic group

RAAG example: no CAT(0)-boundary

Applications of ∂

Free groups

It is often possible to write the identity as a product of non-trivial elements of a group G

eg $G = \mathbb{Z}$, $7 - 11 + 4 = 0$

Any such equality $a_1 \dots a_k = 1$, $\underset{i}{a_i \neq 1}$ is called a relation in G

Here's an obvious relation that holds in any group: $aa^{-1} = 1$

Informally, a free group is a group which has no other relations.

Here's the formal definition:

F is a free group if there is a subset $S \subset F$ such that every set map from S to a group G extends to a unique homomorphism $F \longrightarrow G$

Picture:

$$\begin{tikzcd} S \arrow[r] \arrow[d, phantom, "\cap"] & G \\ F \arrow[ur, dotted, "\exists! \text{ homomorphism}"'] & \end{tikzcd}$$

We say F is <u>free on S</u>, or <u>free w/ basis S</u>,

and write $F = F\langle S \rangle$.

This is an example of a <u>universal</u> <u>property</u>,
which is a common notion in algebra

eg you may have seen tensor product defined
by a universal property: for vector spaces
V and W, the tensor product is a vector space $V \otimes W$
and bilinear map $V \times W \longrightarrow V \otimes W$
s.t. any bilinear map

$$\begin{tikzcd} V \times W \arrow[r] \arrow[d] & U \\ V \otimes W \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

extends uniquely to a linear map

Why does this say what we want?

(1) <u>Every</u> elt of F is a product of elts of $S \cup S^{-1}$:
if $x \in F$ were <u>not</u> such a product, then you could find a homomorphism $f' : F \to G$ sending x to $g' \neq f(x)$, contradicting uniqueness of f.

(2) Suppose there was a non-trivial relation $a_1 \cdots a_k = 1$

Take any group G and elements $g_1, \ldots, g_k$ which do not satisfy $g_1 \cdots g_k = 1$. Consider the set map $x_i \mapsto g_i$
This does not extend to a homomorphism of F! $\divideontimes$

<u>Exercise</u>: verify statements (1) and (2)

Next question: Do free groups exist?
To answer this, we construct a free group starting with a set S

S = set. Take two copies S, S' ← (for inverses)

$i: S \to S'$ a bijection

Define an involution $x \mapsto \bar{x}$ on $S \cup S'$

by $\bar{s} = i(s)$, $\overline{i(s)} = s$

A <u>word</u> is a finite string $s_1 \ldots s_k$ of elements of $S \cup S'$

A word is <u>reduced</u> if $s_{i+1} \neq \bar{s}_i$ for all i

$F\langle S \rangle$: (Group axioms: set, operation, rules)

<u>set</u> = all reduced words, including $\emptyset$

<u>operation</u>: $x_1 \ldots x_k \, y_1 \ldots y_\ell = x_1 \ldots x_{k-r} y_{r+1} \ldots y$

where $r = \max\{i \mid y_i = \bar{x}_{k-i+1}\}$ $(0 \le r \le \min(k, \ell))$

note ↑

<u>identity</u>: $\emptyset$

<u>inverse</u>: $(x_1 \ldots x_k)^{-1} = \bar{x}_k \ldots \bar{x}_1$

<u>associative law</u>: (this is the hard one)

Done next time