Tues, Feb. 11

We showed hyperbolic groups are finitely presented

Similar ideas prove:

Thm: G hyperbolic. Then G has only finitely many conjugacy classes of finite-order elements.

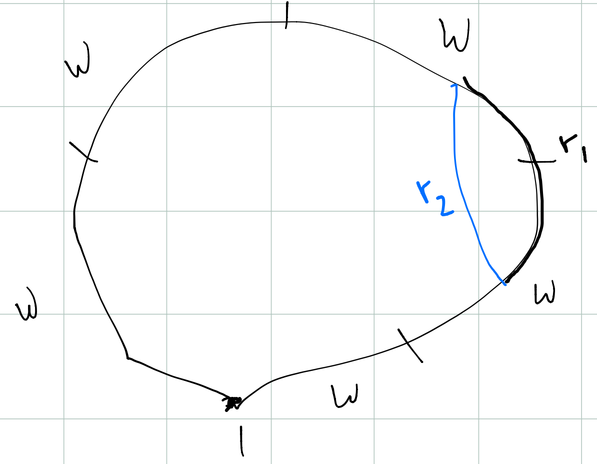
pf Let $g \in G$ with $g^n = 1$, and write $[g]$ for the conjugacy class of g.

We have $G = \langle S \mid R \rangle$, where R = all loops of length $< 16\delta$.

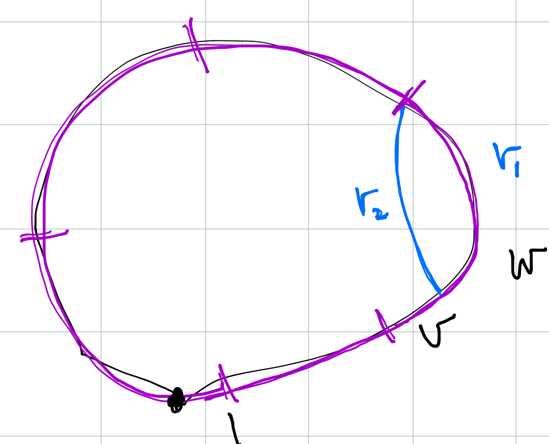
Choose $w \in F(S)$ shortest with $\overline{w} \in [g]$

Since $\overline{w}^n = 1$, w^n gives a loop in the Cayley graph. If this loop has length $> 16\delta$ we know there is a "shortcut", ie a segment of $\leq 8\delta$ which can be replaced by a shorter segment (and still reduce to 1.)

In particular, if $|w| > 8\delta$ we can find a shortcut:



Replacing w by a conjugate of itself, we may assume r_1 is a subword of w.



Now $\overline{w} = \overline{r_1 v} = \overline{r_1 r_2^{-1} r_2 v} = \overline{r_2 v}$,

contradicting minimality of $|w|$.

So all finite-order elts of G have conjugates of length $< 16\delta$, so there are only finitely many conjugacy classes.

How can you prove a group is hyperbolic?
Need it to be finitely generated, and the Cayley graph δ-hyperbolic for some δ.

We had example of a <u>surface group</u>, which acts properly and cocompactly on $\mathbb{H}^2$, which we proved is $\underline{\log_2 3}$-hyperbolic, waved our hands about why its Cayley graph is hyperbolic. Now want to do this right, and in more generality:

<u>Def</u>: A metric space is <u>proper</u> if closed balls are compact.

<u>Def</u>: G acts <u>properly</u> on metric space X if $\forall$ compact $K \subset X$, $\{g \mid gK \cap K \neq \emptyset\}$ is finite.

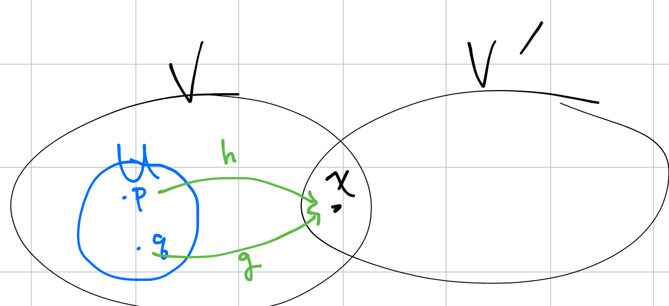
Thm: If G acts properly and cocompactly on a proper geodesic metric space X, then G is finitely generated.

pf. Choose D compact s.t. the orbit of $U = \text{interior}(D)$ covers X.

Let $S = \{ g \in G \mid gD \cap D \neq \emptyset \}$ – S is finite

Let $H < G$ be the subgroup generated by S, and $V = H \cdot U$. Set $V' = (G \setminus H) \cdot U$. Both V and V' are open, and $V \cup V' = X$. Since X is connected, and $V \neq \emptyset$, either $V' = \emptyset$ ($\Rightarrow G = H$) or $V \cap V' \neq \emptyset$

Suppose $V \cap V' \neq \emptyset$, choose $x \in V \cap V'$



$x = hp = gq$ for $h \in H$, $g \in G \setminus H$, $p, q \in U$

So $g^{-1}hp = q$, so $g^{-1}hU \cap U \neq \emptyset$ so $g^{-1}h \in S$

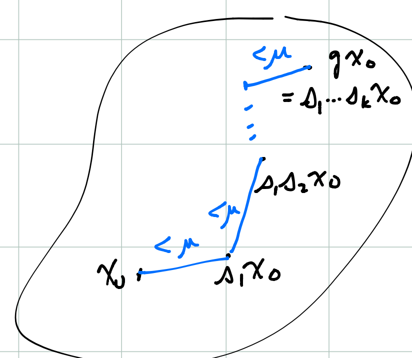
so $g^{-1}h \in H$ so $g \in H$ $\ast$

We're now ready for the Švarc-Milnor Lemma.

<u>Thm:</u> G acts properly and cocompactly on a proper geodesic metric space $X \Rightarrow G$ is quasi-isometric to X.

<u>Pf</u> Choose $x_0 \in X$ and map $G \longrightarrow X$ by $g \mapsto gx_0$

We claim this is a quasi-isometry. One direction is immediate:



(1) Let $\mu = \max_{s \in S} d(x_0, sx_0)$

write $g = s_1 \cdots s_k$

Then Δ-inequality gives $d_X(x_0, gx_0) \leq k \cdot \mu = \mu \cdot d_S(1, g)$

so $d_X(gx_0, hx_0) \leq \mu \cdot d_S(g, h)$

Other direction:

(2) Let $D \subset X$ be compact, $GD = X$, $x_0 \in X$.

Choose r s.t. $B_r = B_r(x_0) \supset D$, and (as before) let

$S = \{ g \in G \mid gB_{3r} \cap B_{3r} \neq \emptyset \}$ - (finite)

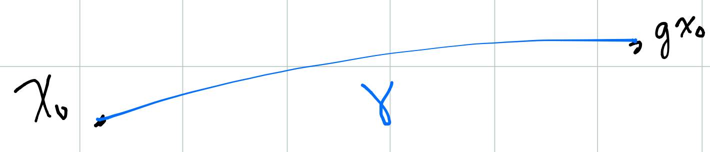
We're trying to find an upper bound on $d_S(g', g)$

in terms of $d(g'x_0, gx_0)$

Since G is acting by isometries, we may as well

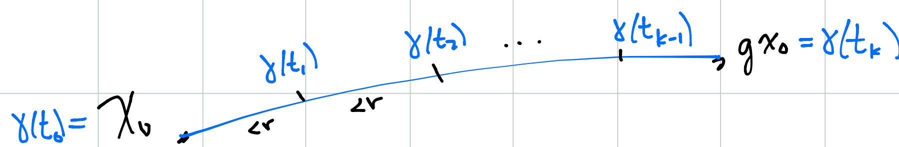
assume $g' = 1$.

Let γ be a geodesic from x_0 to gx_0.

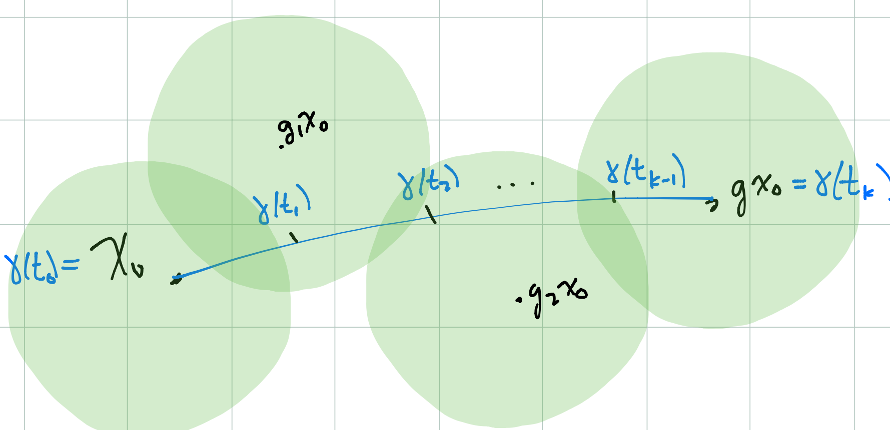


Divide γ into $k = \left[\frac{d(x_0, gx_0)}{r}\right]+1$ equal pieces; so

Each piech has length $\le r$



The balls $B_r(gx_0)$ cover X, so for each t_i, we can find g_i w/ $d(\gamma(t_i), g_i x_0) < r$



Now $d(g_ix_0, g_{i+1}x_0) \le 3r$, so $g_i^{-1}g_{i+1} \in S$

So $g = g_1(g_1^{-1}g_2)(g_2^{-1}g_3)\cdots(g_{k-1}^{-1}g)$

$= s_1 \cdots s_k$,

ie $d_S(1,g) \le k = \frac{d(x_0,gx_0)}{r} + 1$

ie $r \cdot d_S(1,g) - r \le d(x_0, gx_0)$ ✓