Tues, Feb. 18

Last time: Gave 3 definitions of ∂X for X a hyperbolic metric space

$\partial_r X$ ① Equivalence classes of geodesic rays at x_0

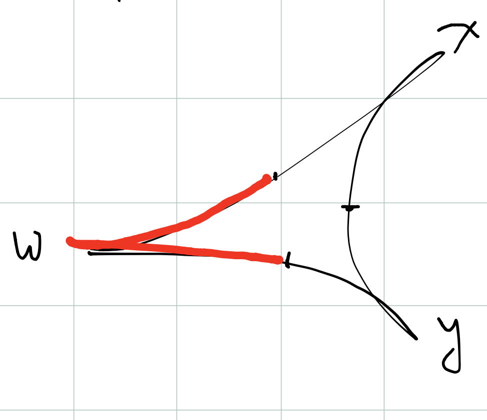
$\partial_q X$ ② Equivalence classes of quasi-geodesic rays

$\partial_s X$ ③ Equivalence classes of sequences $\{a_i\}$ s.t. the Gromov product $(a_i . a_j)_w \to \infty$

Notation: $a_i \to \infty$

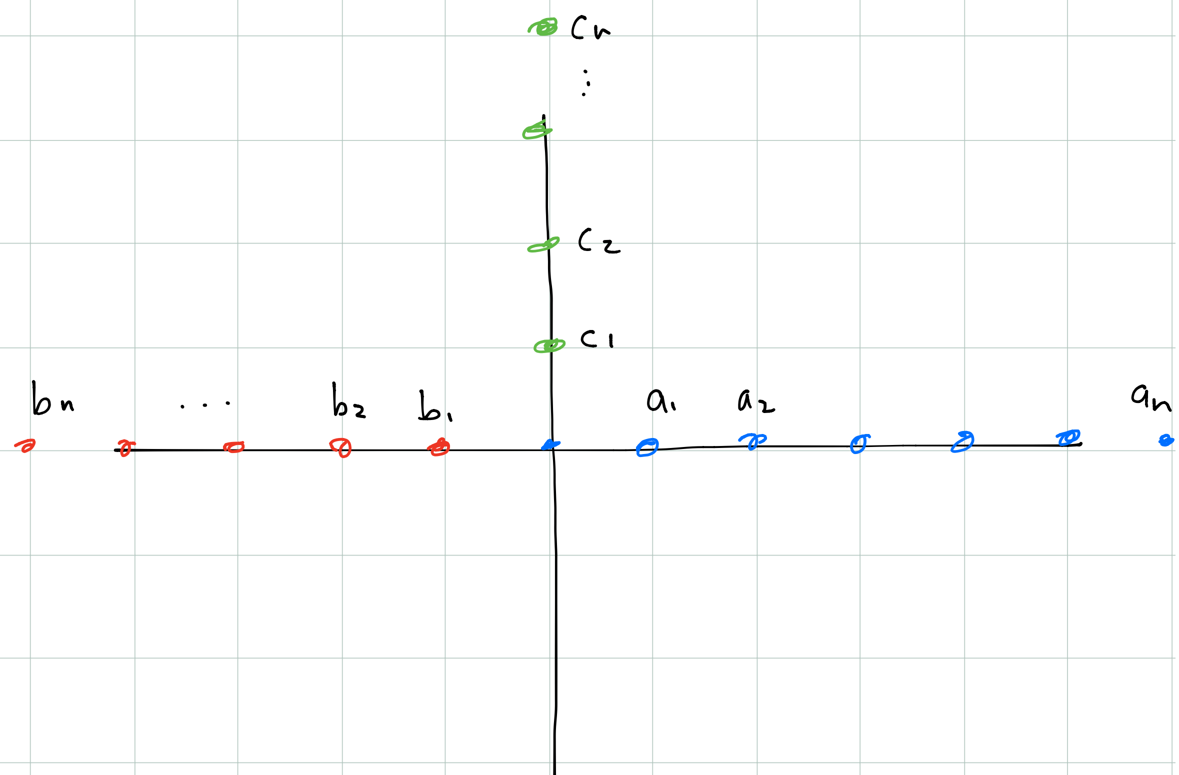
Gromov product:

$$(x,y)_w = \frac{1}{2}\left(d(w,x) + d(w,y) - d(x,y)\right)$$



measures length of red part of $[w,x]$ or $[w,y]$

<u>Remark:</u> If X is not hyperbolic, this may not be an equivalence relation: eg in $\mathbb{R}^2$:



$(a_n, c_n)_0 = n + n - \sqrt{2}n = (2-\sqrt{2})n \longrightarrow \infty \quad \Rightarrow \{a_n\} \sim \{c_n\}$

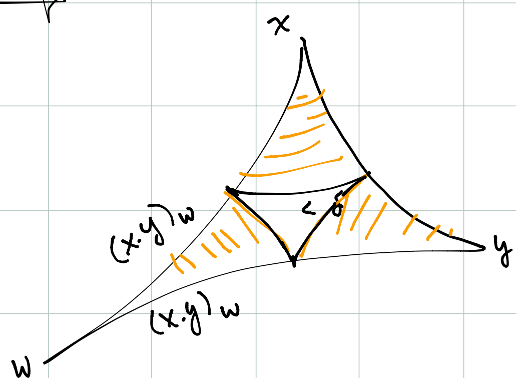
$(c_n, b_n) = (2-\sqrt{2})n \longrightarrow \infty \quad \Rightarrow \{c_n\} \sim \{b_n\}$

But $(a_n, b_n)_0 = n + n - 2n = 0 \not\longrightarrow \infty$ so $\{a_n\} \not\sim \{b_n\}$

2nd remark: To prove this is an equivalence relation in a hyperbolic space, used that in a hyperbolic space, for any x, y, z and w

$$(x,y)_w \geq \min\left((x,z)_w, (y,z)_w\right) - 2\delta$$

Proof: Recall we showed thin triangles $\Rightarrow$

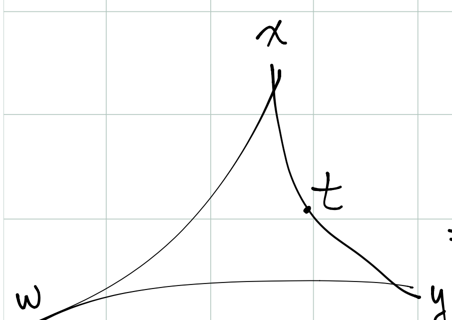


orange segments are $<\delta$

(maybe have to start with $\frac{\delta}{6}$-thin)

so $d(w, [xy]) < (x,y)_w + \delta$

On the other hand $(x,y)_w \leq d(w, [x,y])$:



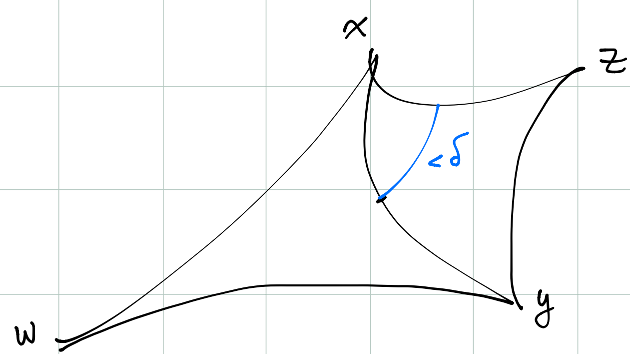
$$d(w,t) + d(t,x) \geq d(w,x)$$

$$d(w,t) + d(t,y) \geq d(w,y)$$

$$\Rightarrow 2d(w,t) + d(x,y) \geq d(w,x) + d(w,y)$$

$$d(w,t) \geq (x,y)_w$$

So $(x,y)_w \le d(w,[x,y]) \le (x,y)_w + \delta$



$$(x,y)_w + 2\delta \geq d(w,[x,y]) + \delta \geq \min\big(d(w,[xz]),\, d(w,[yz])\big)$$

$$\geq \min\big((x,z)_w,\, (y,z)_w\big) \quad \checkmark$$

Today, Show the three definitions are equivalent.

There is an obvious map $\partial_r X \to \partial_g X$.

<u>Exercise:</u> This map is well-defined & injective.

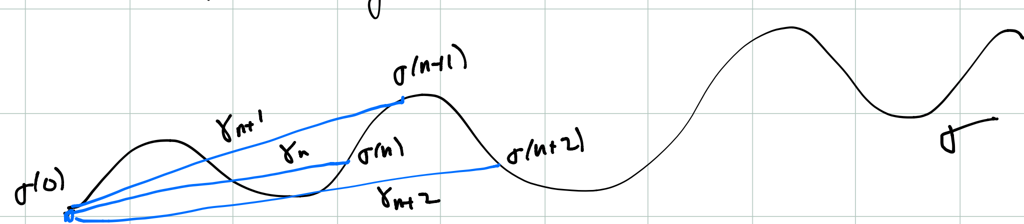
<u>Prop</u>: Let X be a δ-hyperbolic locally finite <u>graph</u> (eg a Cayley graph). Then $\partial_r X \hookrightarrow \partial_g X$ is a bijection.

<u>Lemma</u>: Any quasi-geodesic ray stays within bounded Hausdorff distance of a geodesic

<u>Proof</u> Let $\sigma : [0,\infty)$ be a (λ, C)-quasi-geodesic ray, ie $\sigma|_{[0,t]}$ is a (λ,C)-quasigeodesic for all t.

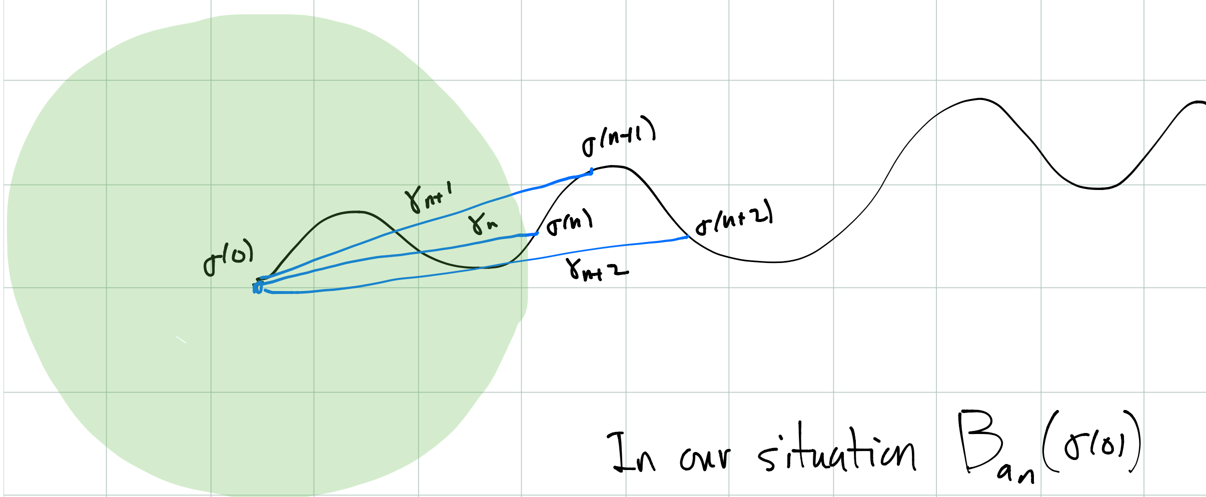
Let $a_n = d(\sigma(0), \sigma(n))$ for each n and

Let γ_n be a geodesic from $\sigma(0)$ to $\sigma(n)$:



We know each γ_n stays within $L = L(\lambda, c, \delta)$ of $\sigma|_{[0,n]}$ where L does <u>not</u> depend on n.

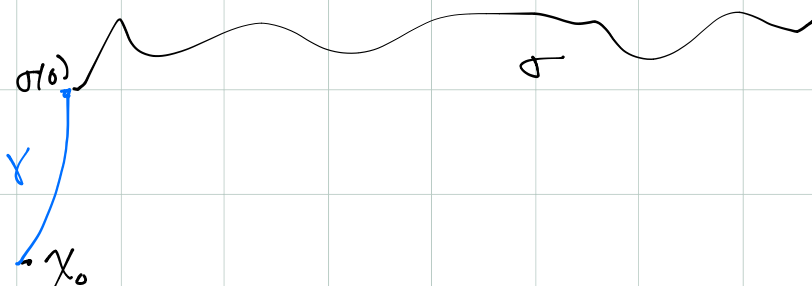
Consider the ball $B_{a_n}(\sigma(0))$:



In our situation $B_{a_n}(\sigma(0))$ is finite, so there are only finitely many paths in it starting at $\sigma(0)$. So an infinite number of γ_n share the same initial segment. Define γ inductively by $\gamma|_{[0,n]}$ = this segment. Keep passing to subsequences as n gets larger. This defines γ for all intervals ✓

<u>Lemma:</u> Any quasi-geodesic ray is within bounded Hausdorff distance of a geodesic <u>from x_0</u>

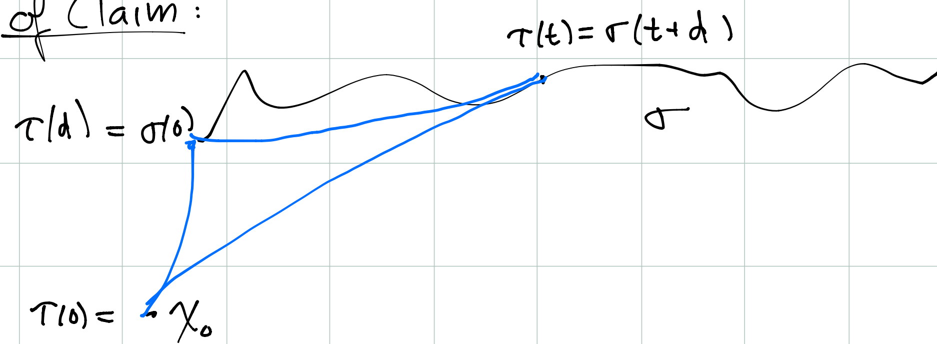
<u>Proof</u> Let σ be a (λ,c)-quasigeodesic, and choose a geodesic γ from x_0 to $\sigma(0)$:



<u>Claim.</u> If $d(x_0,\sigma(0))=d$, then the concatenation of γ and σ is a $\left(\lambda, c+d\left(1+\frac{1}{\lambda}\right)\right)$-quasi-geodesic

Then σ is a bounded Hausdorff distance-rom a geodesic ray from x_0, by the previous lemma.

Pf of Claim:



We are only worried about s,t with $s < d$ and $t > d$

Have to show

$$\frac{1}{\lambda}(t-s) - C' \leq d(\tau(s),\tau(t)) \leq \lambda(t-s) + C'$$

This is just the triangle inequality:

$$d(\tau(s),\tau(t)) \leq d(\tau(s),\tau(d)) + d(\tau(d),\tau(t))$$

$$\leq (d-s) + \lambda(t-d) + C \leq \lambda(t-s) + C \checkmark$$

$$d(\tau(s),\tau(t)) \geq d(\tau(d),\tau(t)) - d(\tau(s),\tau(d))$$

$$\geq \frac{1}{\lambda}(t-d) - C - (d-s)$$

$$= \frac{1}{\lambda}(t-d) + \frac{1}{\lambda}(d-s) - \frac{1}{\lambda}(d-s) - C - (d-s)$$

$$\geq \frac{1}{\lambda}(t-s) - C - \left(1+\frac{1}{\lambda}\right)d \checkmark$$

The lemma shows $\partial_r X \hookrightarrow \partial_g X$ is surjective, so is a bijection.

<u>Remark</u>: This map is a bijection for any proper hyperbolic space. In general you need to appeal to the Arzela-Ascoli theorem (which we haven't proved) to conclude that a subsequence of the γ_n's converges to a geodesic ray. For details, see Bridson-Haefliger's book, Thm 1.3.10 and Corollary 1.3.11.

Say $a_i \to a$ if $\{a_i\}$ is in the equivalence class a

<u>Eg</u> $X = \mathbb{R}$, $w = 0$

$\{a_i\} \in S_\infty \Rightarrow$ $a_i > 0$ for almost all i

or $a_i < 0$ for almost all i

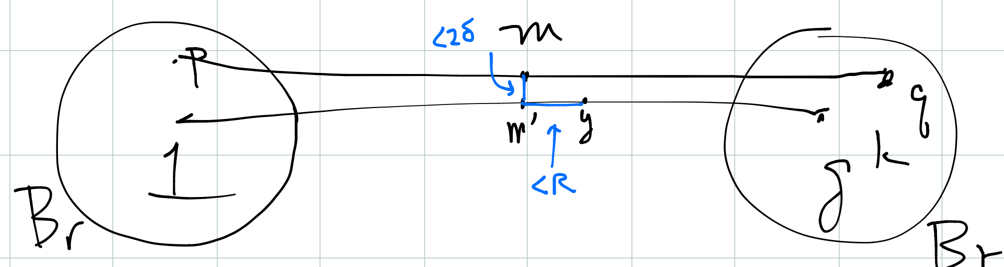
$\Rightarrow$ there are 2 equiv. classes.

$(+\infty, -\infty)$

<u>Prop:</u> $\{g^i\}$ is a quasi-geodesic

First show: For any R, $k > 8\delta + R \Rightarrow$

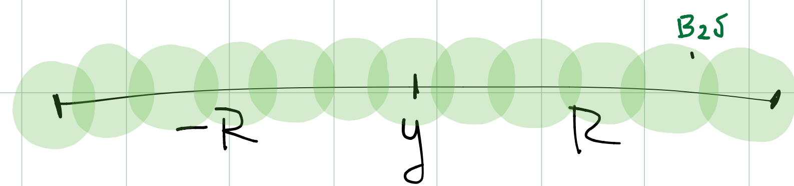
mid



$p \in B_r(1)$, $q \in B_r(g^k)$, $m =$ midpt of geod p to q

$y =$ midpt of geod 1 to g^k $\Rightarrow d(y, m) < 2\delta + R$

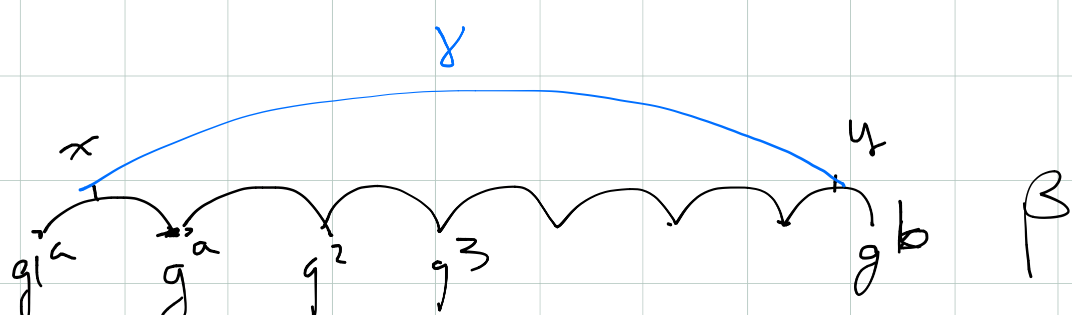
$N = \#$ of pts in $B_{2\delta}(1) \Rightarrow \exists \leq N \cdot 2R$ possible such midpoints m of segments between $B_R(1)$ and $B_R(g^k)$.



In particular, endpts of $g^i[1, g^k]$ are <u>not</u> in $B_r(1) \cup B_r(g^k)$ for some $i = p(R) < 2NR$

<u>Claim</u> g^{2NR} is not in $B_R(1)$ for any R.

This is <u>linear</u> in R, not exponential!



$$d_\beta(x,y) \lesssim |b-a| \cdot \ell(g)$$

$$d_\gamma(x,y)$$

$$\frac{1}{\lambda}|b-a| - c \le d(g^a, g^b) \le \lambda |b-a| + c$$