Thurs, Feb. 20

We're showing $\partial_r X \overset{\checkmark}{=} \partial_q X = \partial_s X$.

Next: Map $\partial_r X \longrightarrow \partial_s X$ by $\gamma \longmapsto \{\gamma(i)\}_{i \in \mathbb{N}}$

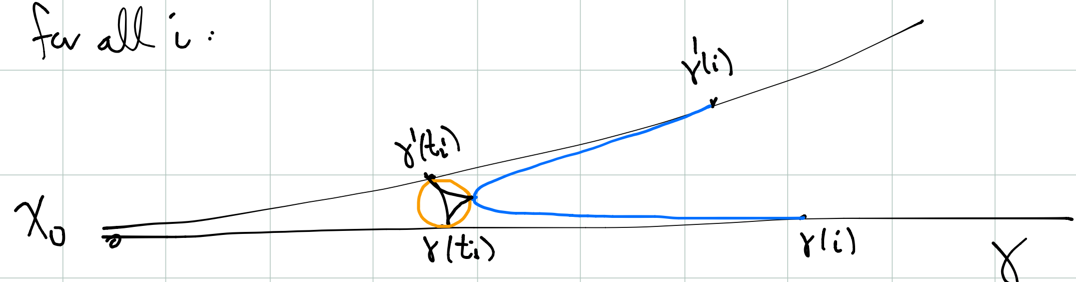
Exercise: $\gamma(n) \longrightarrow \infty$ and $\gamma \sim \gamma' \Rightarrow \{\gamma(i)\} \sim \{\gamma'(i)\}$

So the map is well-defined.

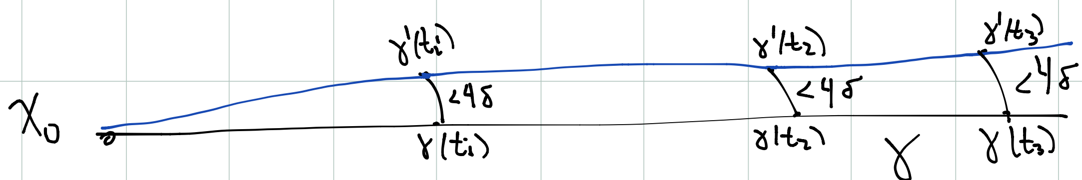
Next: Show it's injective:

If $(\gamma(i), \gamma'(i))_{x_0} \longrightarrow \infty$, then $d(\gamma(t), \gamma'(t))$ is bounded

Pf: Let $t_i = (\gamma(i), \gamma'(i))_{x_0}$. Then $d(\gamma(t_i), \gamma'(t_i)) < 4\delta$ for all i:

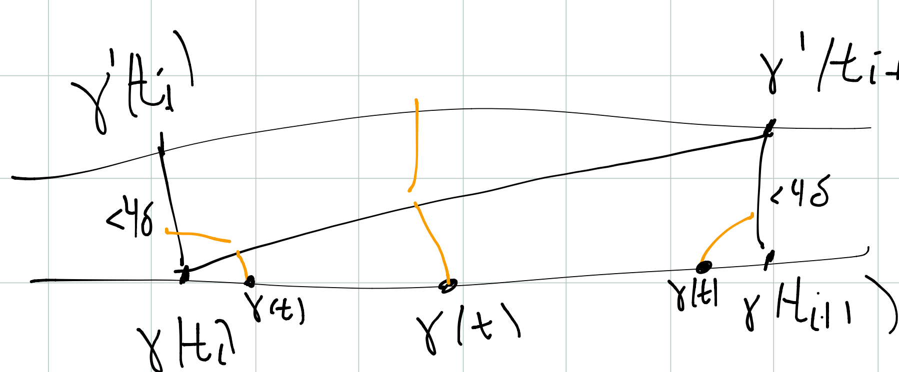


So picture is:



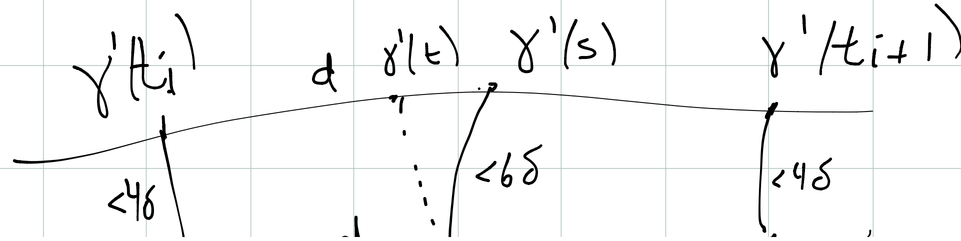
Still have to show $d(\gamma(t), \gamma'(t))$ bounded for $t_i < t < t_{i+1}$.

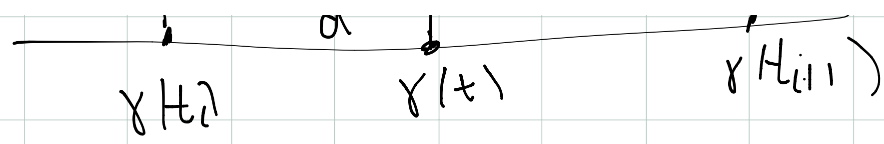
Triangles are δ-thin, so one of the three orange paths below has length $< 2\delta$, depending on where $\gamma(t)$ is:



In any case, $d(\gamma(t), \gamma|_{[t_i, t_{i+1}]}) < 6\delta$

Say $d(\gamma(t), \gamma'(s)) < 6\delta$:





Then $d(\gamma(t),\gamma'(t)) < 6\delta + |s-t|$

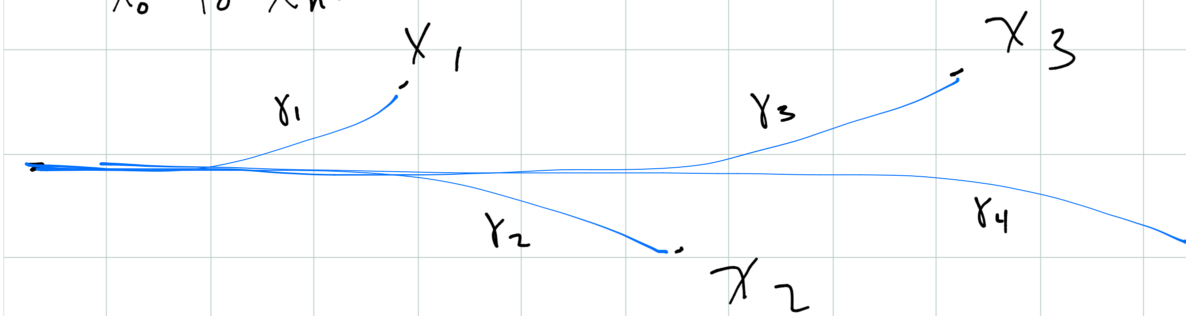
Let $d = t - t_i = d(\gamma(t_i),\gamma(t))$

Then $d + |s-t| \le d + 10\delta \Rightarrow s-t < 10\delta$

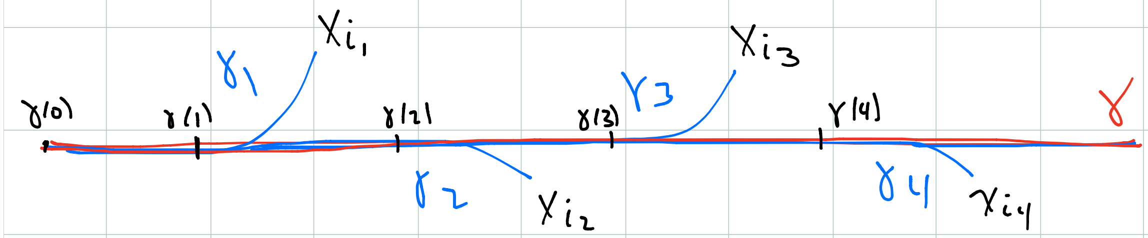
so $d(\gamma(t),\gamma'(t)) \le 16\delta$ ✓

Now show
$$\begin{aligned} \partial_r X &\longrightarrow \partial_s X \\ \gamma &\longmapsto \{\gamma(i)\} \end{aligned}$$
is surjective.

Suppose $x_i \to \infty$. Choose geodesics γ_n from x_0 to x_n:



As before, after passing to a subsequence $\{x_{i_k}\}$ we can find a geodesic ray γ st $\gamma|_{[0,n]} = \gamma_{i_k}|_{[0,n]}$ for $k \ge n$



So for $k \geq n$, $(x_{i_k}, \gamma(n))_{x_0} = d(x_0, \gamma(n)) = n \longrightarrow \infty$,

and $\{\gamma(n)\} \sim \{x_{i_k}\}$

Now observe that a subsequence of $\{x_i\}$ is equivalent to $\{x_i\}$: $(x_{i_k}, x_k)_{x_0} \longrightarrow \infty$ since x_k large

$\Rightarrow$ x_{i_k} large and we know $(x_i, x_j)_{x_0} \longrightarrow \infty$

So $\{x_i\} \sim \{x_{i_k}\} \sim \{\gamma(k)\}$

So the map $\partial_r X \longrightarrow \partial_s X$ is surjective

$$\gamma \longmapsto \{\gamma(k)\}$$

✓

Now want to glue ∂X to X to get a compact topological space $\hat{X} = X \cup \partial X$

First, extend Gromov product $(\ ,\ )_w$ to ∂X:

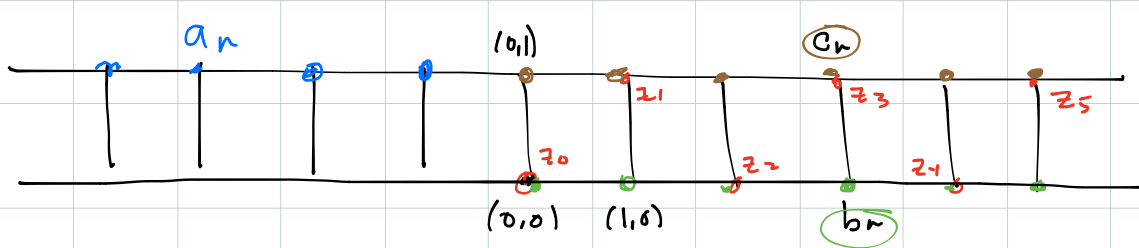
Notation: For x = equivalence class of $\{x_i\}$, write

$$x_i \longrightarrow x$$

Would like to say: $x_i \longrightarrow x$, $y_i \longrightarrow y$

then $(x,y)_w = \lim_{i,j \to \infty} (x_i, y_j)_w$

<u>But:</u> <u>Example</u> $G = \mathbb{Z} \times \mathbb{Z}/2$



This is hyperbolic two $\sim$ classes ∂^+ and ∂^- of unbounded sequences. What is $(\partial^+, \partial^-)_{(1,0)}$?

$a_n = (-n, 1) \longrightarrow \partial^-$

$b_n = (n, 0)$, $c_n = (n, 1)$, $z_n = (n, n \bmod 2) \longrightarrow \partial^+$

$(a_i, b_j) = \frac{1}{2}\Big((i+2) + (j-1) - (i+j+1)\Big) = 0$

$(a_i, c_j) = \frac{1}{2}\big((i+2) + j - (i+j)\big) = 1$

$(a_i, z_j) = \begin{cases} 0 & j \text{ even} \\ 1 & j \text{ odd} \end{cases}$

So $\lim\limits_{i,j \to \infty} (a_i, z_j)$ doesn't exist. Fix this

by taking $\liminf\limits_{i,j \to \infty} (x_i, y_j)$

Not good enough — $\lim (a_i, b_j) = 0 \neq \lim (a_i, c_j)$

Def:

$$(x, y)_w = \sup_{\substack{x_i \to x \\ y_i \to y}} \{ \liminf (x_i, y_i)_w \}$$

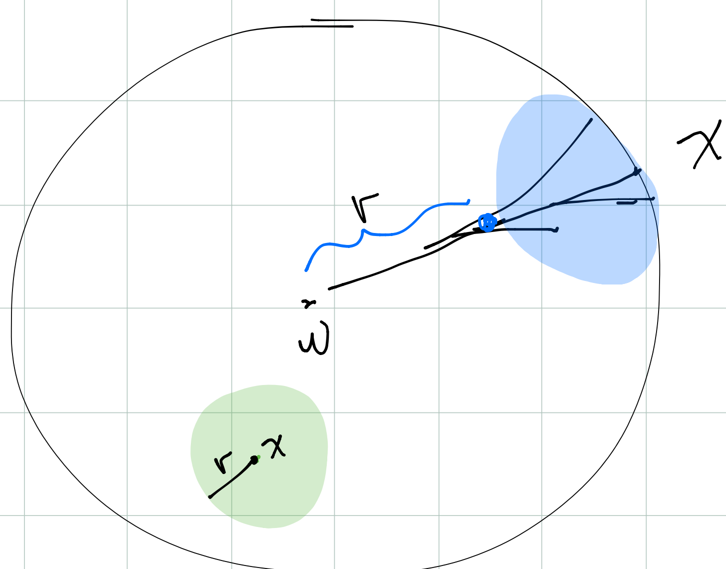
Topology on $X \cup \partial X = \hat{X}$

Want a basis for the topology. Fix w = basept

For $x \in X$, take balls of radius $r \in \mathbb{Q}_+$

For $x \in \partial X$, define $N_r(x) = \{y \mid (x.y) > r\}$

where if $y \in X$, $(x,y)_w = \sup_{x_i \to x} \left(\liminf (x_i.y) \right)$



Have to show: This is a neighborhood base, countable ($r \in \mathbb{Q}_+$), satisfies the separation axiom T_3 (regular Hausdorff) so $\hat{X}$ is metrizable.

Prop. $\hat{X} = X \cup \partial X$ is compact

Pf $\hat{X}$ metrizable $\Rightarrow$

compact is equivalent to sequentially compact.

So viewing points of ∂X as sequences will be convenient.