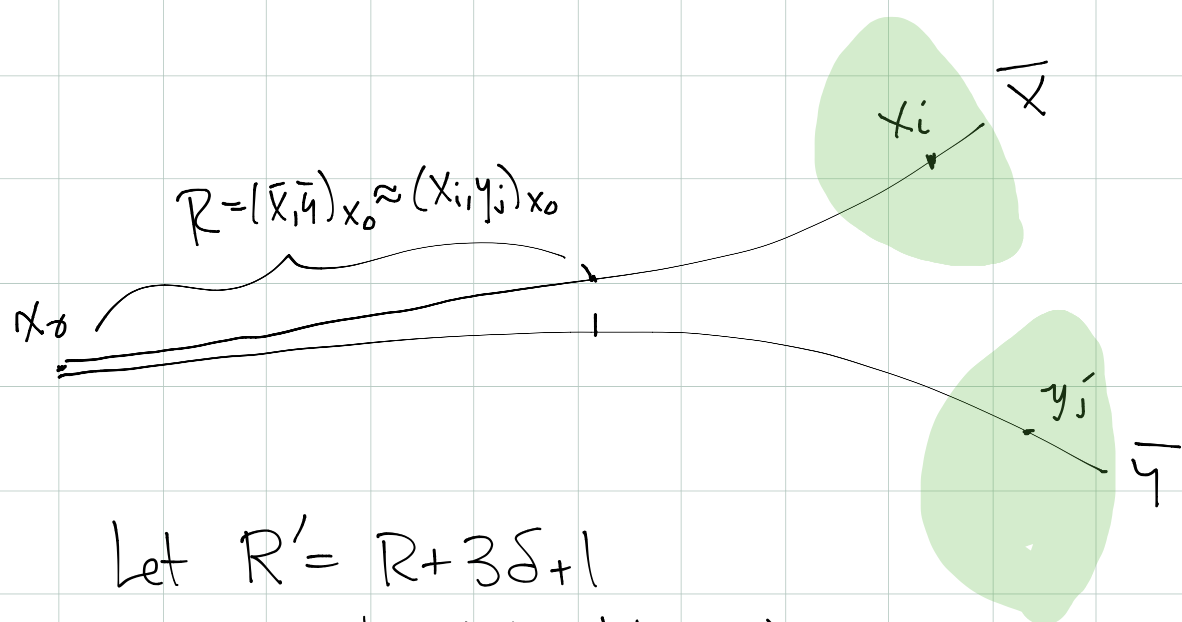
Tues Feb. 25

Last time: Showed $\hat{X} = X \cup \partial X$ is sequentially compact, which $\Rightarrow$ compact if $\hat{X}$ is metrizable.

To show $\hat{X}$ is metrizable, needed to check separation properties: $\hat{X}$ is regular Hausdorff: can separate points from points and points from closed sets.

Eg: To separate $\bar{x}, \bar{y} \in \partial X$



Let $R' = R + 3\delta + 1$

We claim $N_{R'}(\bar{x}) \cap N_{R'}(\bar{y}) = \emptyset$.

We will make use of Lemma proved Feb 18:

For any $x, y, z \in X$, $(x,y)_w \geq \min((x,z)_w, (y,z)_w) - 2\delta$

Suppose $z \in N_{R+3\delta+1}(\bar{x}) \cap N_{R+3\delta+1}(\bar{y})$

ie $(x_i, z)_{x_0}, (y_j, z)_{x_0} > R+3\delta+1$ for i, j large

Lemma $\Rightarrow$ $R = (\bar{x}, \bar{y})_{x_0} \geq \min\{(x,z)_{x_0}, (y,z)_{x_0}\} - 2\delta$

$\geq R+3\delta+1-2\delta = R+\delta+1$

$\times$

Other separation properties were left as exercises.

Prop: If X and X' are quasi-isometric, then ∂X is homeomorphic to $\partial X'$.

Pf Choose a quasi-isometry $f: X \to X'$ with quasi-inverse $g: X' \to X$.

Represent $\bar{x} \in \partial X$ by a quasi-geodesic ray γ. Then $f(\gamma)$ is also a quasi-geodesic ray, so we can define $\partial f(\bar{x}) = [f(\gamma)] \in \partial X'$.

If $d_H(\gamma_1, \gamma_2) < K$, then $d_H(f(\gamma_1), f(\gamma_2)) < \lambda K + C$, ie $f(\gamma_1) \sim f(\gamma_2)$ so ∂f is well-defined

Similarly, $d_H(g\circ f(\gamma), \gamma))$ is bounded, so $g\circ f(\gamma) \sim \gamma$, and $\partial g \circ \partial f = id$. (Similarly, $\partial f \circ \partial g = id$). It remains to verify that ∂f is continuous.

ie $\forall n\ \exists R$ st. $\bar{y} \in N_R(\bar{x}) \Rightarrow \partial f(\bar{y}) \in N_n(\partial f(\bar{x}))$

We use <u>Lemma</u> (Feb18)

$$(x,y)_z \leq d(z,[x,y]) \leq (x,y)_z + \delta$$

Then $\big(f(x_i), f(y_i)\big)_{fx_0} \geqslant d\big(f(x_0), [f(x_i), f(y_i)]\big) - \delta$

$= d(f(x_0), t') - \delta$ for some $t' \in [f(x_i), f(y_i)]$

$\geqslant d(f(x_0), f(t)) - K - \delta$ for some $t \in [x_i, y_i]$

(since $f[x_i, y_i]$ is a quasi-geodesic, it is within $K = K(\lambda, C, \delta)$ of any geodesic)

$\geqslant \frac{1}{\lambda} d(x_0, t) - C - K - \delta$

$\geqslant \frac{1}{\lambda} d(x_0, [x_i, y_i]) - C - K - \delta$

$\geqslant \frac{1}{\lambda} (x_i, y_i)_{x_0} - (C + K + \delta)$

so for $R > \lambda(n + C + K + \delta)$,

$$(fx_i, fy_i)_{fx_0} > R \Rightarrow (x_i, y_i)_{x_0} > n.$$

ie $(f\bar{x}, f\bar{y}) > R \Longrightarrow (\bar{x}, \bar{y})_{x_0} > n$ ✓

<u>Examples</u> of ∂'s of hyperbolic groups:

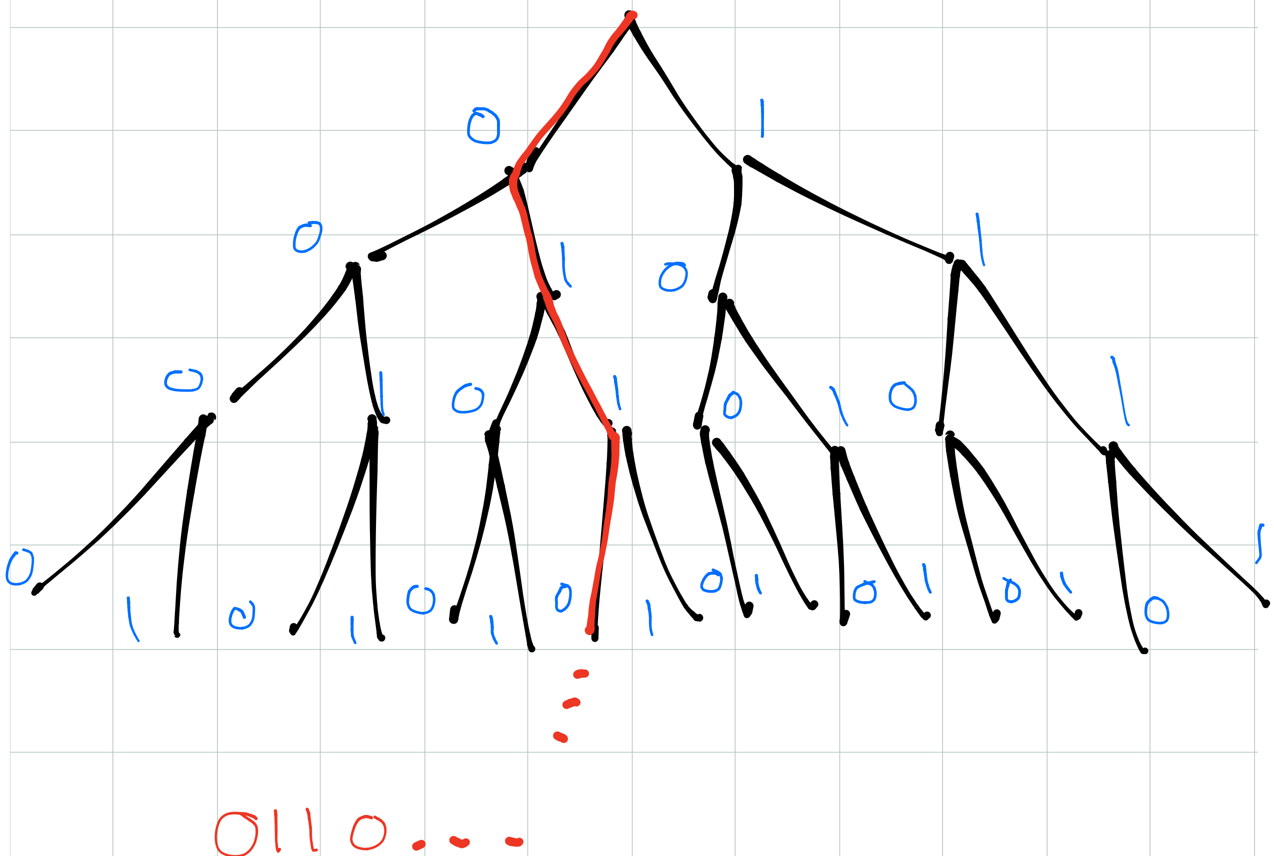
1. $G = \pi_1(S_g)$ $g \geq 2$ $\Rightarrow G \sim \mathbb{H}^2$

   $\Rightarrow \partial G \approx \partial \mathbb{H}^2 = S^1$

2. $G = \pi_1 M^3$ hyperbolic $\Rightarrow \partial G \approx \partial \mathbb{H}^3 = S^2$

3. $G = F_n$. Then $G \sim T_3$

   rays in T_3 at a basepoint x_0 in the middle of an edge correspond to infinite sequences of 0's and 1's



This is one definition of the Cantor set. Another is by removing middle thirds from [0,1]. Sequences which are eventually constant correspond to endpoints of removed intervals, other sequences correspond to other points.