Thurs, Feb 27

If $f: X \to Y$ is a continuous quasi-isometry, the proof that ∂f is continuous also works to show $\hat{f}: \hat{X} \to \hat{Y}$ is continuous

The fact that $\hat{X}, \hat{Y}$ are compact then gives that $\hat{f}$ is uniformly continuous

This has a nice application to automorphisms of free groups, called bounded cancellation:

Let $\phi: F_n \to F_n$ $(n \geq 2)$ be an automorphism

Recall elements of F_n are reduced words, but to multiply two you may have to cancel:

$x = x_1 \cdots x_k$, $y = y_1 \cdots y_K$. Then

$xy = x_1 \dots x_k y_1 \cdots y_\ell = x_1 \dots x_{k-r} y_{r+1} \cdots y_\ell$, where

$r = r(x,y) = \max \{ i \mid y_i = \overline{x}_{k-i+1} \}$

<u>Thm:</u> Let x, y be reduced words in F_n with xy also reduced, and let $\phi: F_n \to F_n$ be an automorphism. Let ϕ_x and ϕ_y be the (reduced) images of x and y. There is a constant $R = R(\phi)$ s.t. the amount of cancellation between ϕ_x and ϕ_y is at most R: $r(\phi_x, \phi_y) \leq R$.

So if ϕ_x and ϕ_y are long, so is $\phi(xy)$

Eg if x is cyclically reduced and $\phi_x > R$ then $|\phi(x^2)| \geq 2(\phi(x) - R)$

$$|\phi(x^k)| \geq k(\phi(x) - R) \to \infty \text{ as } k \to \infty$$

<u>Proof of Theorem:</u> Let G be a Cayley graph for F_n with respect to a basis $x_1 \ldots x_n$

ϕ induces a continuous quasi-isometry $f: \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}$, with $f(g) = \phi(g)$ for g a vertex of $\mathcal{G}$ and $f(g - gs) =$ geodesic from $\phi(g)$ to $\phi(gs)$.

Since $\hat{\mathcal{G}}$ is compact, f is uniformly continuous.

Similarly, ϕ^{-1} induces a unif. continuous map $f': \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}$ with $f'(g) = \phi^{-1}(g)$ for $g \in V(\mathcal{G})$.

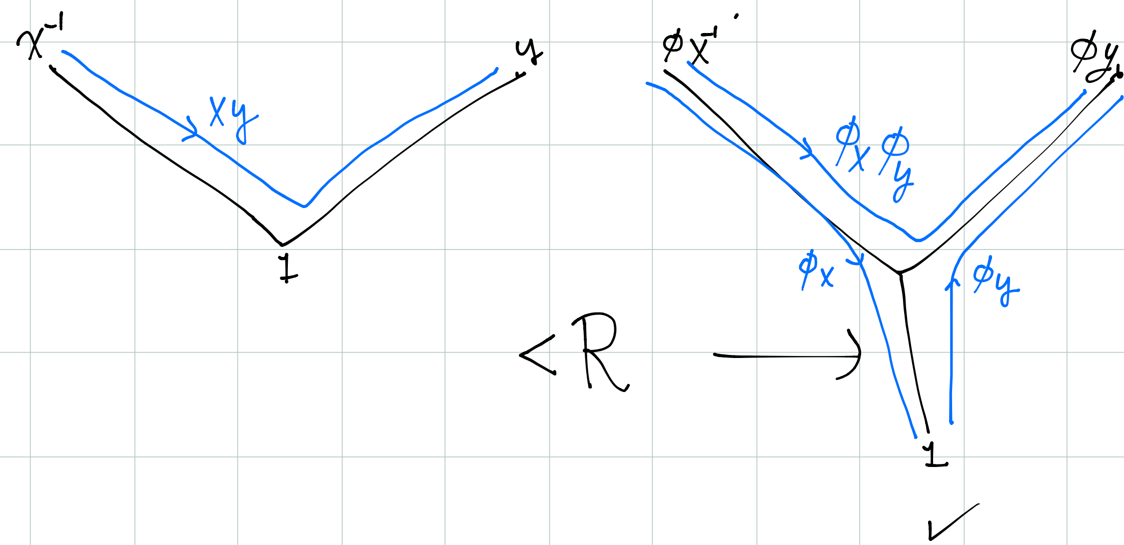
So, for $n = 1$, $\exists R$ (indep of g,h) s.t.

$$(g,h)_1 > R \Rightarrow \underset{\substack{\| \\ (\phi^{-1}g,\, \phi^{-1}h)_1 > 1}}{(f'g, f'h)_1} > 1 \qquad (\text{note } 1 = f'(1))$$

In particular, $(\phi g, \phi h)_1 > R \Rightarrow (g,h)_1 > 1$

So if $(g,h)_1 = 0$, we must have $(\phi g, \phi h)_1 \leq R$

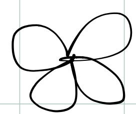
Apply this to $g = x^{-1}$, $h = y$:



Application: Let $\phi : F_n \to F_n$ be an automorphism, and let $\mathrm{Fix}(\phi)$ denote the set of $w \in F_n$ s.t. $\phi w = w$

Note $\mathrm{Fix}(\phi)$ is a subgroup.

Thm Fix ϕ is finitely-generated.

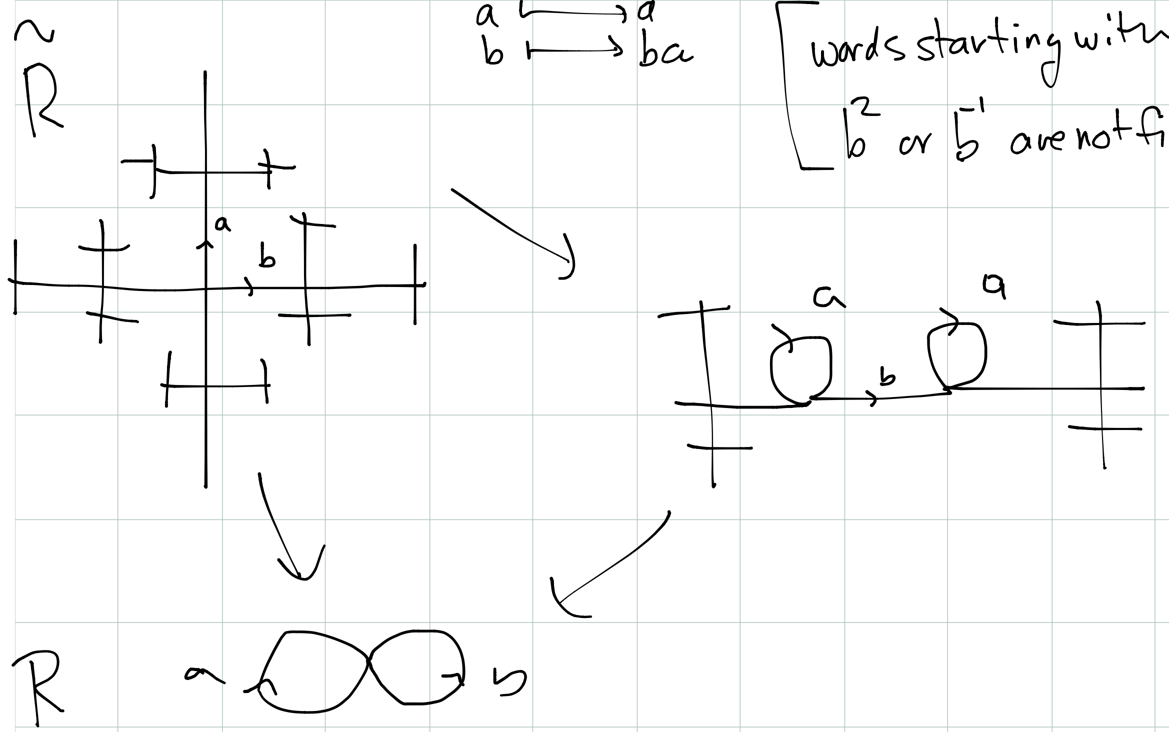
Proof: $F_n = \pi_1$  $= \pi_1 R_n$

Let R_ϕ be the cover corresponding to $\mathrm{Fix}(\phi)$, ie $\pi_1(R_\phi) = \mathrm{Fix}(\phi)$. We want to determine $Nc(R_\phi)$.

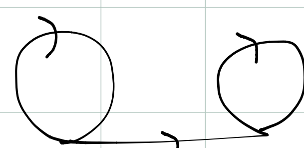
Example: $\phi : F\langle a,b\rangle \longrightarrow F\langle a,b\rangle$. $\quad \mathrm{Fix}\,\phi = \langle a, bab^{-1}\rangle$

$a \longmapsto a$
$b \longmapsto ba$

[words starting with b^2 or b^{-1} are not fixed.]



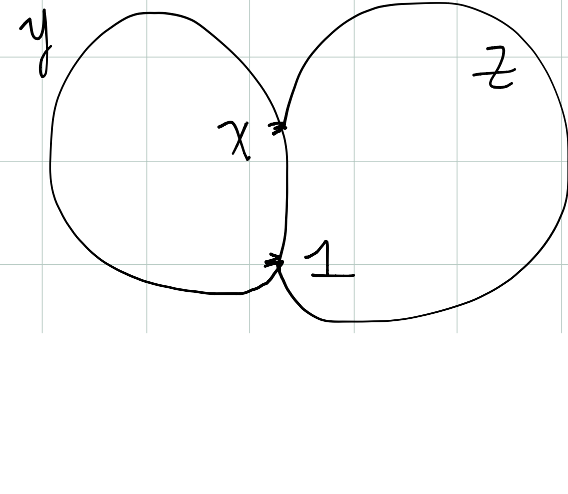
Inside R_ϕ is Γ_ϕ = subgraph spanned by all lifts of $w \in \text{Fix}\,\phi$.

In our example $\Gamma_\phi =$ 

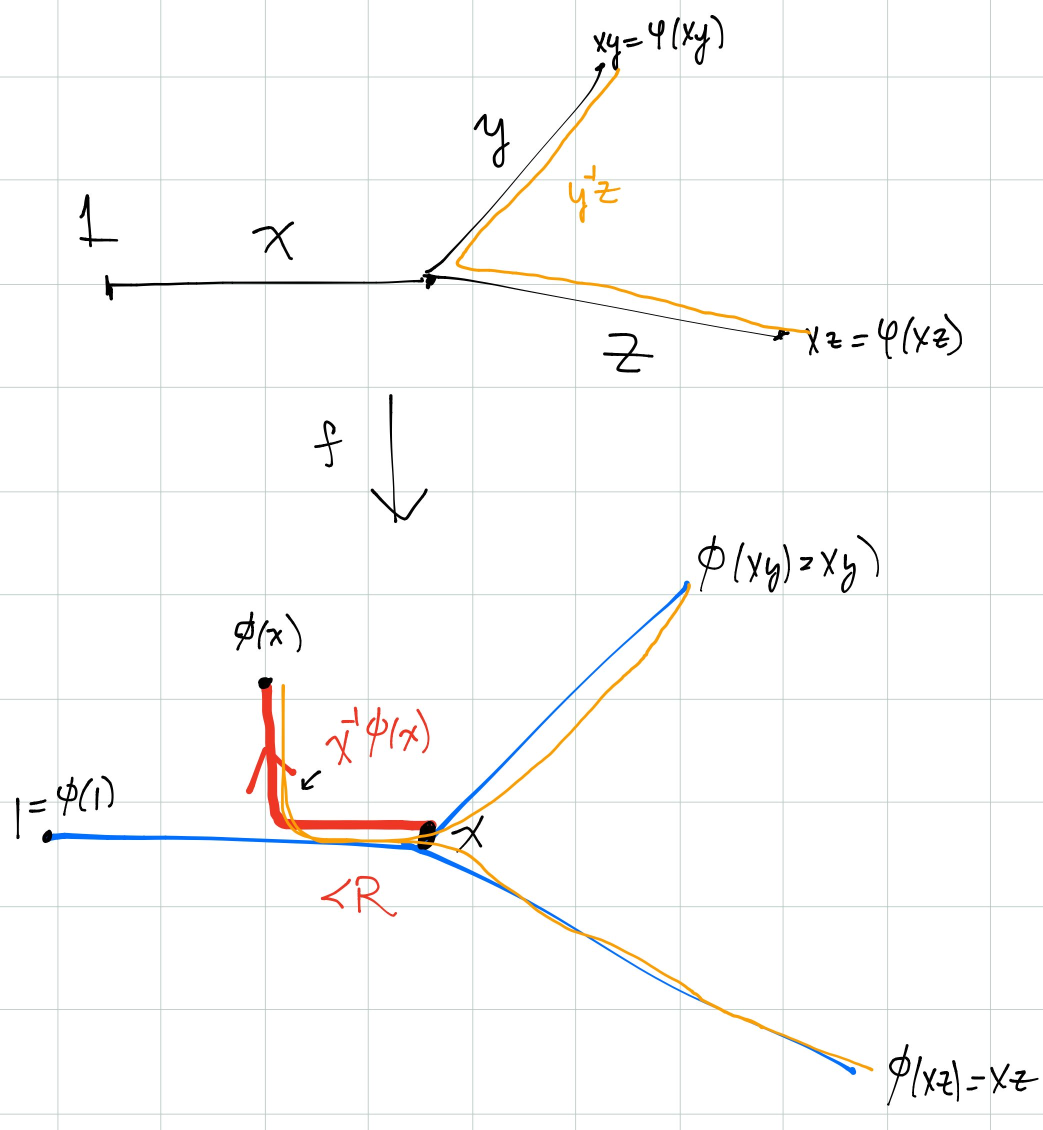
Γ_ϕ contains a loop for every elt of $\text{Fix}\,\phi$, ie $\pi_1(\Gamma_\phi) = \pi_1(R_\phi)$, so suffices to show $\pi_1 \Gamma_\phi$ has finite rank. To do this, it suffices to show Γ_ϕ has only finitely many verties of valence ≥ 3.

Suppose $x \in \Gamma_\phi$ has valence ≥ 3. Then x is contained in at least two different fixed loops

xy and xz



Look upstairs in $T_{2n} = \widetilde{R}_n$



Since $B_R(1)$ is finite there are only finitely many possibilities for $x^{-1}\phi(x)$

But if $x^{-1}\phi(x) = y^{-1}\phi(y) = \varepsilon$

Then $\phi(xy^{-1}) = \phi(x)\phi(y)^{-1} = x\,\varepsilon\,\varepsilon^{-1}y^{-1} = xy^{-1}$

so $x = (xy^{-1}).y$ is the same point in $\Gamma_\phi \subset R_\phi$.

So there are only finitely many vertices of valence ≥ 3 in Γ_ϕ