Mon, Feb 3

Last time: Defined quasi-isometric map, quasi-isometry, Showed $\mathcal{C}(G,S)$ is quasi-isometric to $\mathcal{C}(G,S')$. Now want to show hyperbolicity is a quasi-isometry invariant. This requires some preparation

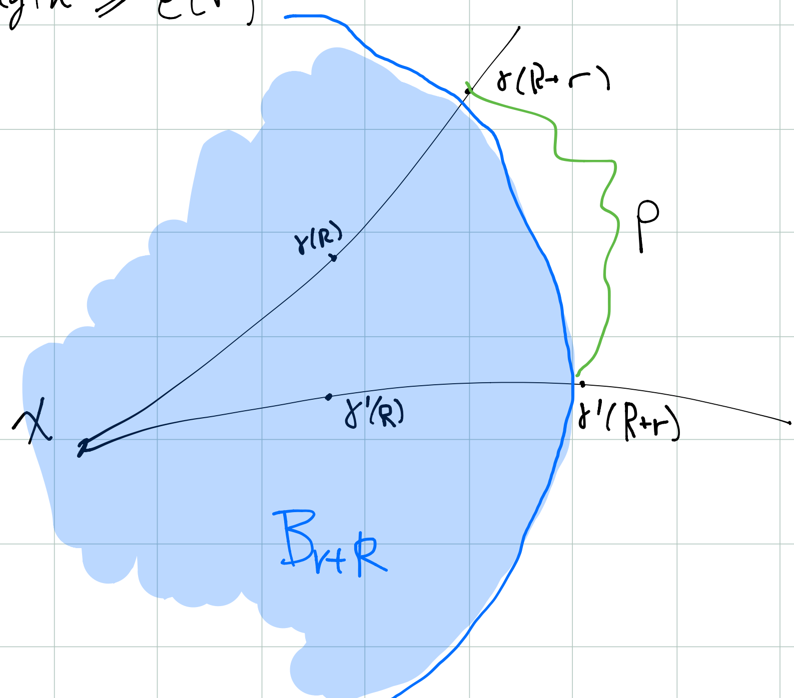
Recall: In $\mathbb{H}^2$, geodesics diverge exponentially fast. We claim this is true in any hyperbolic metric space, if we define "diverge" appropriately. (Basically, the geodesics have to get a certain distance away before you can be confident they will diverge.)

Def $e: \mathbb{N} \to \mathbb{R}$ is a *divergence function* for a metric space X if $\forall x \in X$, γ and γ' geodesic rays with

0) $= x$ and R $r \in \mathbb{N}$ with d (R γ'(R $> e($0

Any path ρ from $\gamma(R+r)$ to $\gamma'(R+r)$ outside the ball $B_{r+R}(x)$ has length $\geq e(r)$

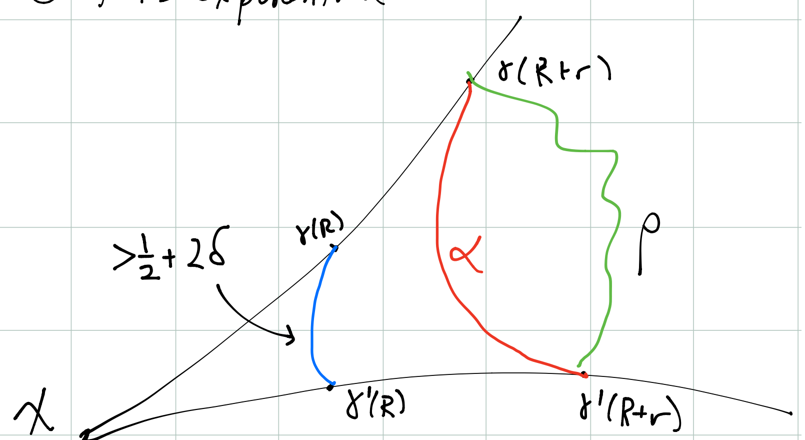
Picture



<u>Thm</u> X δ-hyperbolic, e a divergence function with $e(0) > \frac{1}{2} + 2\delta \Rightarrow e(r)$ is exponential in r.

<u>Pf</u>: Setup:

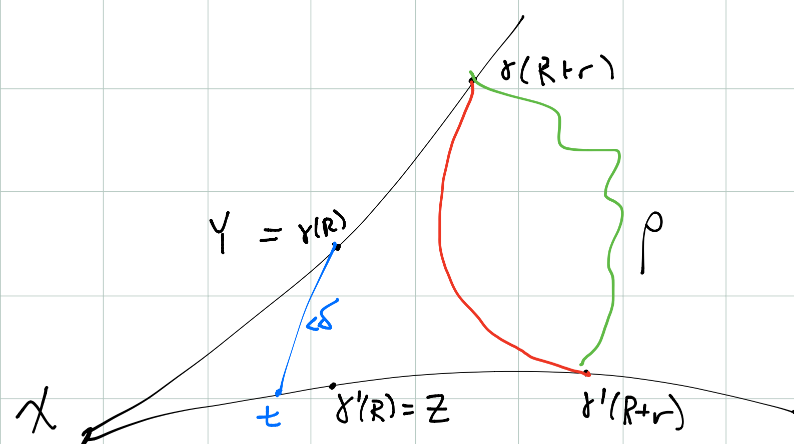
α = geodesic from $\gamma(R+r)$ to $\gamma'(R+r)$



Claim: $\ell(\rho)$ is exponential function of r

First claim is that $d(\gamma(R), \alpha) < \delta$

We know that $\gamma(R)$ is within δ of some side of $\Delta(x, \gamma(R+r), \gamma'(R+r))$. Suppose it is not α:



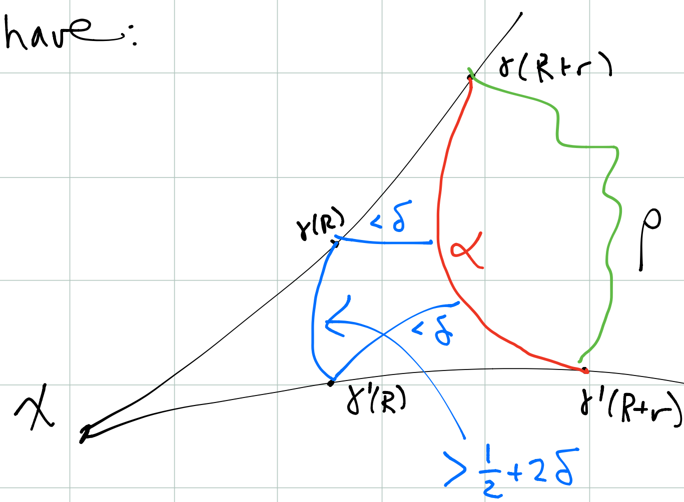
Then $R = d(x, Y) \leq d(x,t) + \delta$, so

$$\cancel{d(x,t)} + d(t,z) = d(x,z) = R \leq \cancel{d(x,t)} + \delta$$

$$\Rightarrow d(t,z) < \delta \Rightarrow d(Y,Z) < 2\delta \text{ ※}$$

Same argument works for $\gamma'(R)$

So we have:



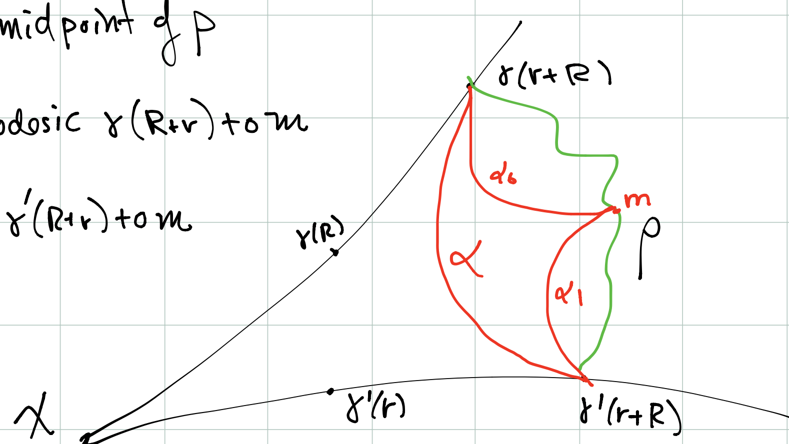
$$d(\gamma(R),\gamma'(R)) > 2\delta+\tfrac{1}{2} \Rightarrow d(\gamma(R+r),\gamma'(R+r)) > \tfrac{1}{2} \Rightarrow \boxed{\ell(\rho) > \tfrac{1}{2}}$$

Now cut ρ into pieces of length $\frac{1}{2} \le \ell \le 1$:

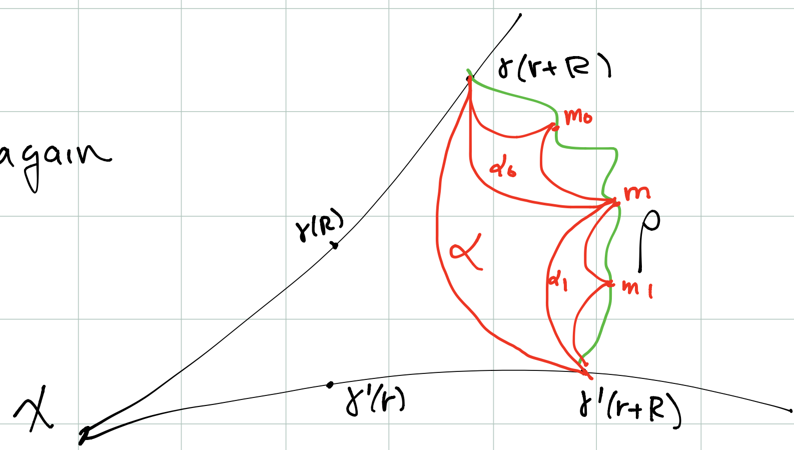
Let m = midpoint of ρ

α_0 = geodesic $\gamma(R+r)$ to m

α_1 = geod. $\gamma'(R+r)$ to m



Subdivide again



Do this n times, until length of each segment is $\frac{1}{2} \leq l \leq 1$

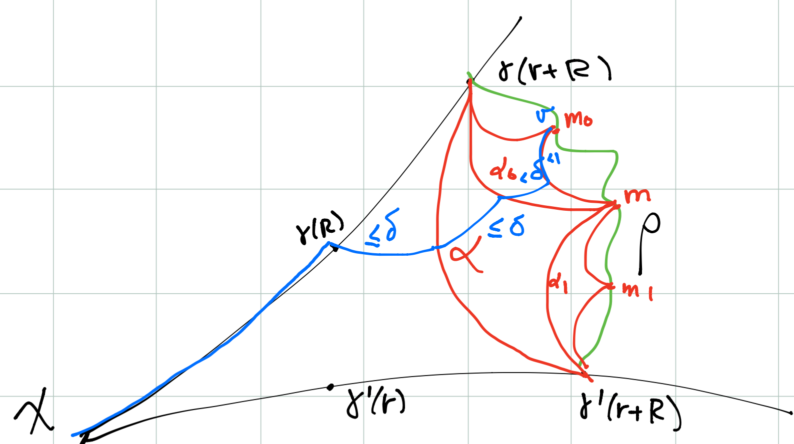
ie $$\frac{1}{2} \leq l = \frac{l(\rho)}{2^n} \leq 1$$

$$2^{n-1} \leq l(\rho) \leq 2^n$$

$$l(\rho) \leq 2^n \leq 2\,l(\rho)$$

$$\log_2 l(\rho) \leq n \leq \log_2 l(\rho) + 1$$

Now notice $\exists$ path x to $v \in \rho$ of length $\leq R + (n+1)\delta + 1$



So $d(x,v) \leq R + (n+1)\delta + 1$

But we know $d(x,v) \geq R + r$

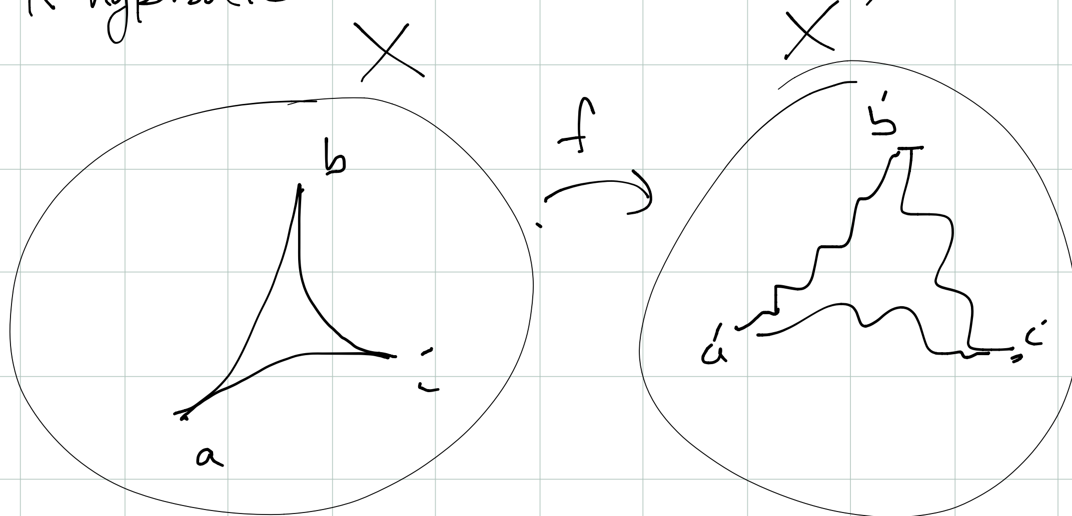
So
$$\begin{aligned} R + r &\leq R + (n+1)\delta + 1 \\ &\leq R + (\log_2 \ell(\rho) + 2)\cdot\delta + 1 \end{aligned}$$

$$\Rightarrow \quad r \leq \delta \log_2 \ell(\rho) + 2\delta + 1$$

$$\frac{r - 2\delta - 1}{\delta} \leq \log_2 \ell(\rho)$$

$$\ell(\rho) \geq 2^{\frac{r-2\delta-1}{\delta}} = C\cdot e^{\lambda r} \qquad \square$$

We want to show: $X \sim X'$, X' hyperbolic $\Rightarrow$ X hyperbolic



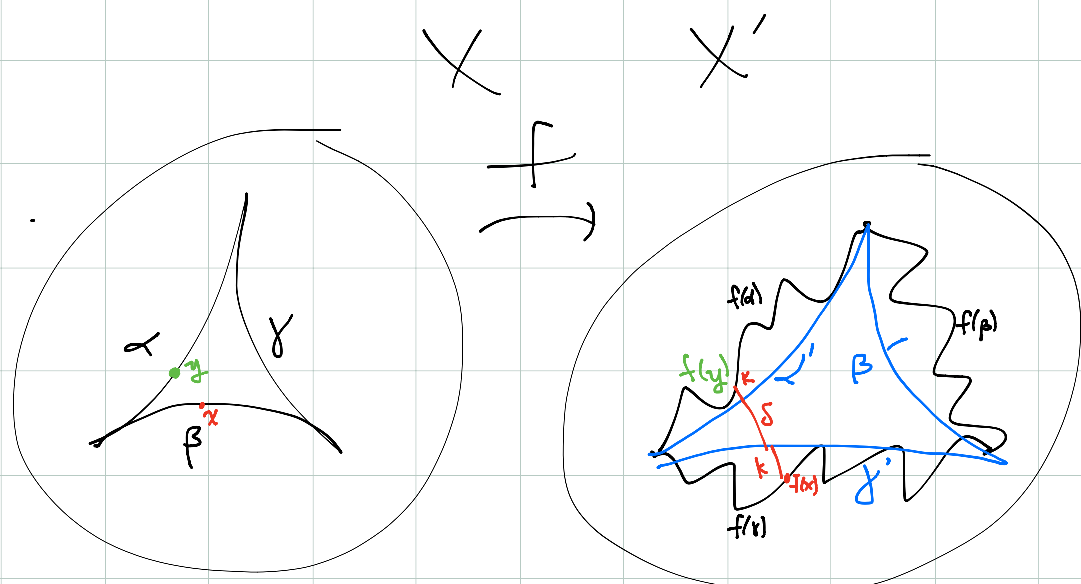
Idea: Take a geodesic triangle in X, map it by f into X', show the image is thin (even though it's not a geodesic triangle, conclude the original Δ had to be thin.

f is a quasi-isometry, so the image of each geodesic is not too distorted

<u>Key claims</u>: Let γ be a geodesic joining the end pts of $f(\alpha)$.

Then $f(\alpha) \subset N_k(\gamma)$ and $\gamma \subset N_k(f(\alpha))$

That will do it:



$x \in \gamma$, $\Rightarrow \exists y$ $\quad d(f(x), f(y)) < 2k + \delta$

$$\Rightarrow \quad d(x,y) < \lambda(2k+\delta) + C$$

Def A (λ, c)-quasi geodesic is a (λ, c)-quasi-isometric map from an interval to X:

$$\gamma : [0, d] \longrightarrow X$$

$$\frac{1}{\lambda} d(x,y) - c \leq d(\gamma(x), \gamma(y)) \leq \lambda\, d(x,y) + c$$

So the sides of our image triangle are quasi-geodesics, and we want to prove

Thm $x, y \in X$, α a (λ, c)-quasi-geodesic joining x and y, γ a geodesic joining x & y

Then $\exists\, D = D(\lambda, c, \delta)$ st

$$\alpha \subseteq N_D(\gamma) \text{ and } \gamma \subseteq N_D(\alpha)$$

First consider the case f is continuous (then show how to get around it)