Tues, Feb 4

We're proving Thm: X hyperbolic $\Rightarrow$ quasi-geodesics stay a uniformly bounded distance from geodesics

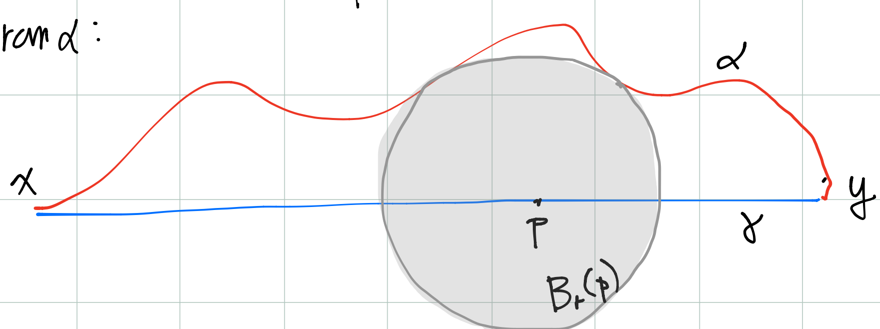
Thm $x, y \in X$, α a (λ, c)-quasi-geodesic joining x and y, γ a geodesic joining x & y

Then $\exists\, D = D(\lambda, c, \delta)$ st

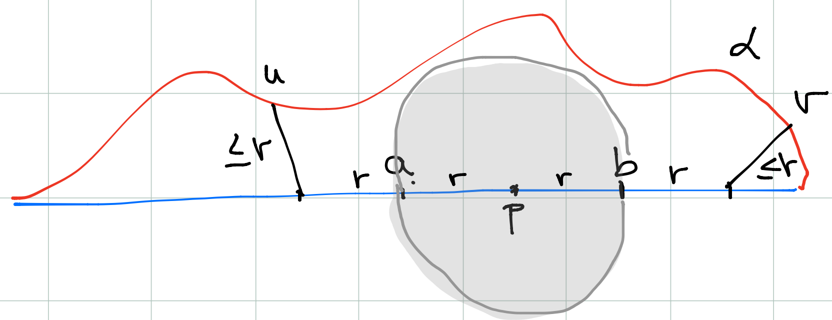
$$\alpha \subseteq N_D(\gamma) \text{ and } \gamma \subseteq N_D(\alpha)$$

First consider the case f is continuous (then show how to get around it)

$\gamma \subseteq N_D(\alpha)$: Find $p \in \gamma$ with maximal ball $B_r(p)$ disjoint from α:



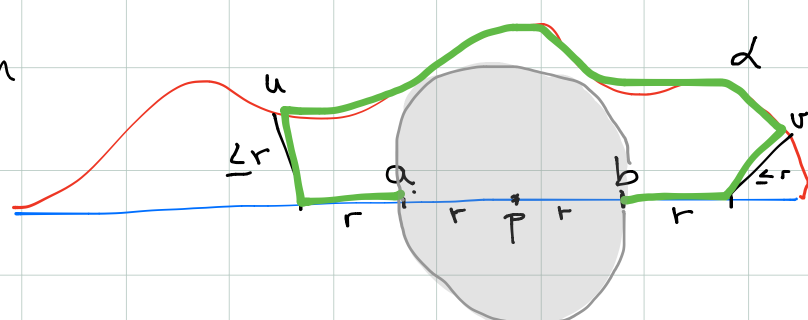
Now go out $2r$ from p in each direction, and find $u, v \in \alpha$ at distance $\leq r$ from those points:



Then $d(u,v) \leq 6r$, so length of segment of α from u to v is $\leq \lambda' \cdot 6r + C'$

(This needs to be justified – λ' and C' depend on λ, C)

The green path:

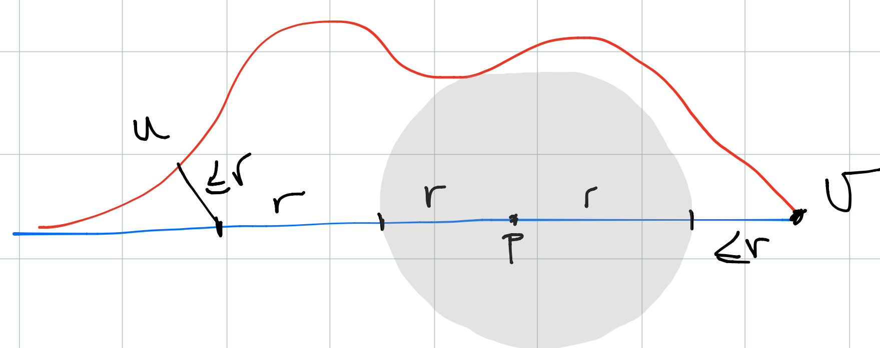


stays outside $B_r(p)$.

Remark:

why did I go out $2r$ from p? I needed the geodesic to u to stay outside $B_r(p)$, which it does by the Δ inequality.

If there is not $2r$ available, this is not a problem: I don't need the extra geodesic segments:



Now back to the proof.

We have a path of length $\le 4r + 6\lambda' r + C'$ from a to b

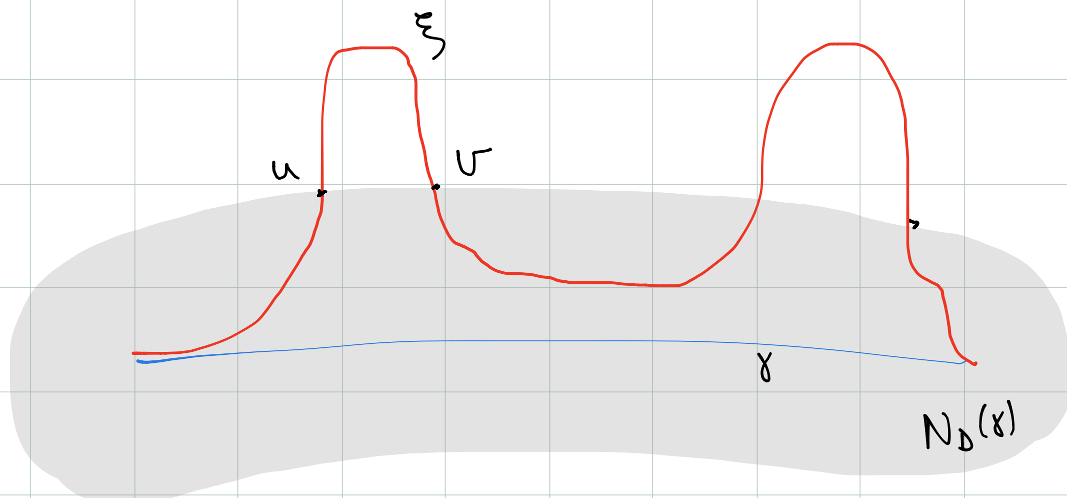
which stays outside $B_r(p)$

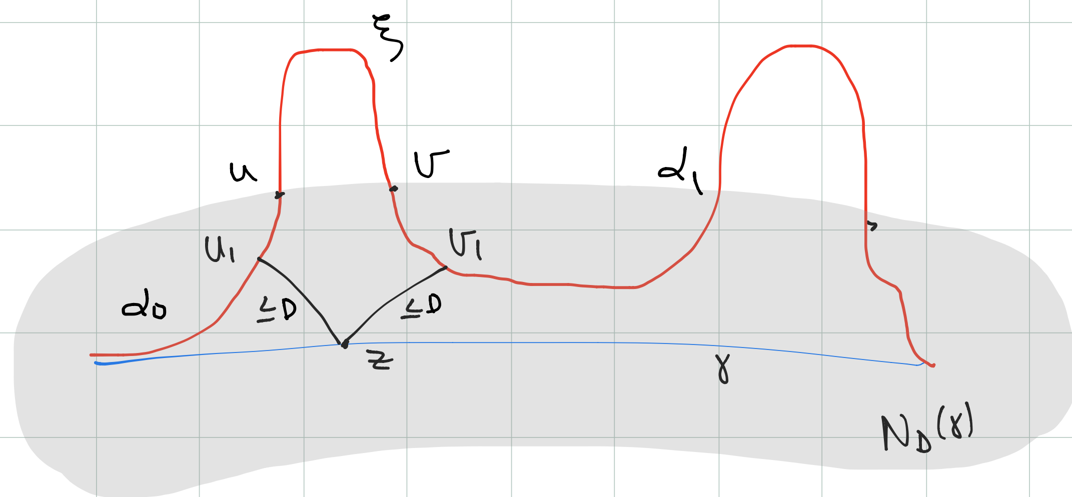
But for r large, any such path must have length

$> K e^{\mu r}$ for some K, μ depending only on δ

So we get $4r + 6\lambda' r + C' \geq K e^{\mu r}$. Since exponential functions grow faster than linear ones, this puts an upper bound D on r, depending only on λ, C and δ ✓

$\alpha \subseteq N_D(\gamma)$ Suppose not. Let u and v be endpoints of an interval ξ that leaves $N_D(\gamma)$





We have $\gamma \subseteq N_D(\alpha)$, so starting from the left and moving right we can find z as in the picture: at distance $\le D$ from both α_0 and α_1

$$d(u_1, v_1) \le 2D \Rightarrow d_\alpha(u_1, v_1) \le \lambda' 2D + C'$$

$\Rightarrow$ length of ξ is $\le \lambda' 2D + C'$

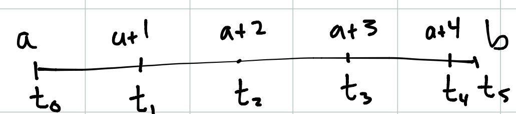
$\Rightarrow$ every point of ξ is within $D + \lambda' D + C'/2$ from a point of γ. So replace D by $D' = D + \lambda' D + C'/2$.

Lemma: Suppose α is a (λ, C)-quasigeodesic $\alpha : [a,b] \longrightarrow X$. Then there is a continuous $(\lambda, 2(\lambda+C))$-quasigeodesic $\beta : [a,b] \longrightarrow X$ s.t.

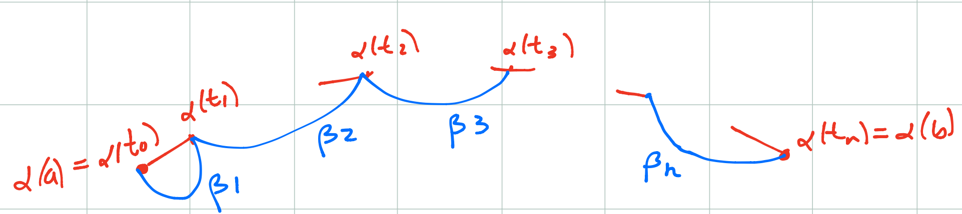
1. $\alpha \subseteq N_{\lambda+C}(\beta)$ and $\beta \subseteq N_{\lambda+C}(\alpha)$
2. $d(\alpha(t), \beta(t)) \leq \lambda + C$
3. $l_\beta(\beta(s), \beta(t)) \leq \lambda' d(\beta(s), \beta(t)) + C'$

where λ', C' depend on λ, C

Proof: Divide $[a,b]$ into n pieces each of size 1 (except at end)



$a = t_0, t_1, \ldots, t_n = b$, then connect $\alpha(t_{i-1})$ to $\alpha(t_i)$ by a reparamaterized geodesic segment β_i:



reparamatrize linearly on each $[t_{i-1}, t_i]$ so that
$\beta_i(0) = \alpha(t_{i-1})$ and $\beta_i(1) = \alpha(t_i)$.

The concatenation of the β_i is a continuous path with
$\beta(t_i) = \alpha(t_i)$ for all i.

Since each segment β_i has length $\leq \lambda + C$,
we get $\beta \subseteq N_{\lambda+C}(\alpha)$ and since α is a (λ, C)-quasigeodosic
we get $\alpha \subseteq N_{\lambda+C}(\beta)$.

<u>Claim</u> β is a $(\lambda, 2(\lambda+C))$-quasigeodosic