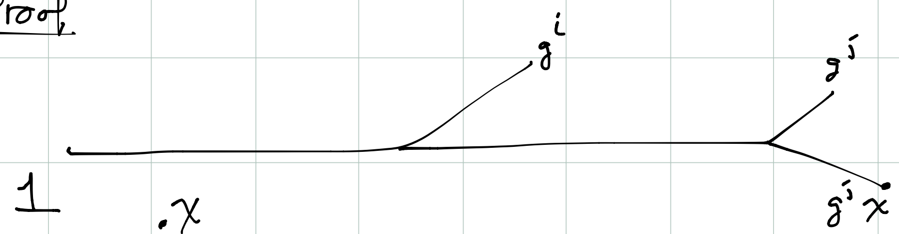
Mon. March 10

Support class today: H3.58

Last time: Given an infinite-order element g of a hyperbolic group G, found a bi-infinite geodesic joining $g_{-\infty} = \{g^{-i}\}$ to $g_\infty = \{g^i\}$

Claim: x any point of G. Then the orbit $\{g^i x\}$ $\longrightarrow g_\infty$. (and $\{g^{-i}x\} \to g_{-\infty}$)

Proof.



We claim $\{g^i\} = \{g^j x\}$, so need to calculate $(g^i, g^j x)_1$

$$(g^i, g^j x)_1 = d(1, g^i) + d(1, g^j x) - d(g^i, g^j x)$$

$$\geq d(1,g^i) + [d(1,g^j) - d(g^j, g^j x)] - [d(g^i, g^j) + d(g^j, g^j x)]$$

Since g acts by isometries, $d(g^j, g^j x) = d(1,x)$,

so this is

$$= d(1,g^i) + d(1,g^j) - d(g^i,g^j) - 2d(1,x)$$

$$= (g^i,g^j)_1 - 2d(1,x) \longrightarrow \infty \text{ as } i,j \to \infty$$

ie $\{g^i\} \sim \{g^i x\}$, so both $\longrightarrow g_\infty$

(similarly, $\{g^{-i}x\} \to g_{-\infty}$)

What about the action of g on ∂G?

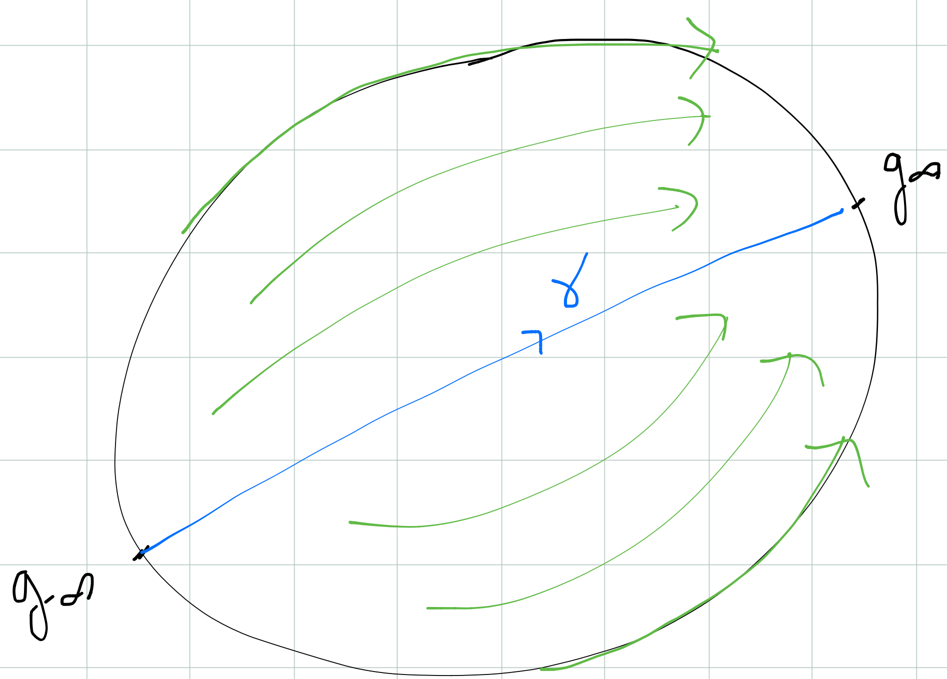
$g \cdot \{g^i\} = \{g^{i+1}\} \longrightarrow g_\infty$ so $g \cdot g_\infty = g_\infty$

$g \cdot \{g^{-i}\} = \{g^{-i+1}\} \longrightarrow g_{-\infty}$ so $g \cdot g_{-\infty} = g_{-\infty}$

<u>Exercise</u>: If $x \in \partial G$, $x \neq g_{-\infty}$ then

$\{g^i x\} \longrightarrow g_\infty$.

So we have a good picture of the dynamics of the action of G on $\overline{G} = G \cup \partial G$:



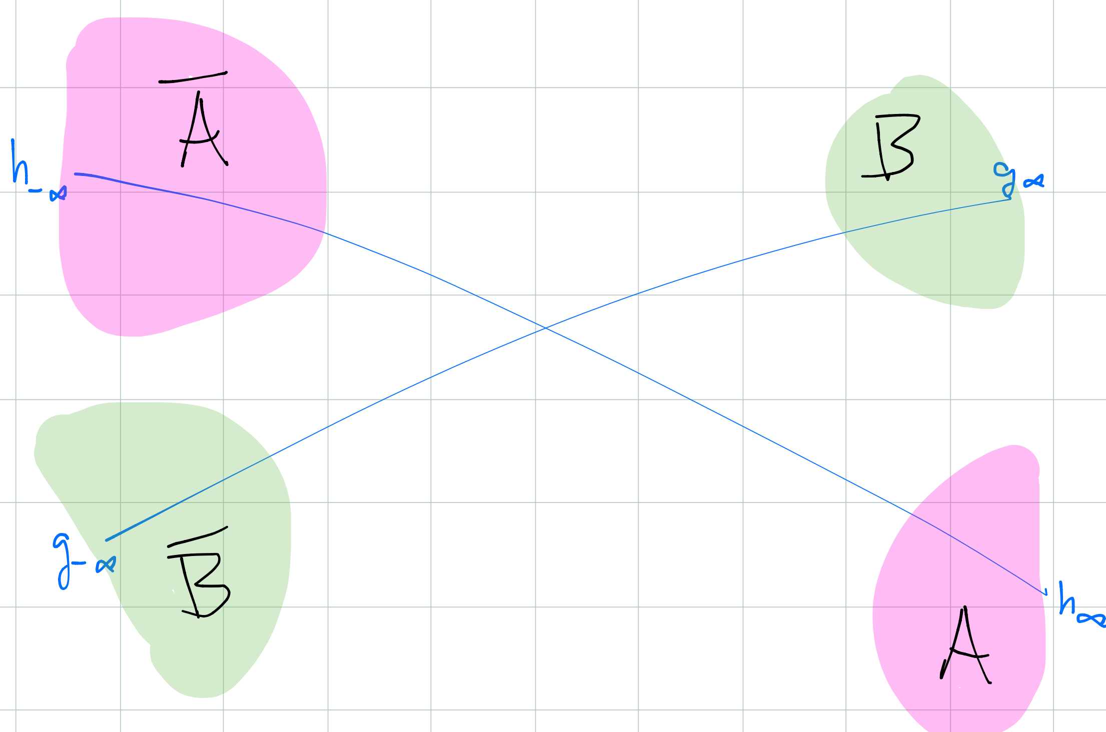
The action of g moves everything in $G \cup \partial X$ <u>except</u> $g_{-\infty}$, towards g_∞, away from $g_{-\infty}$

We say g acts on G with "North-South dynamics"

Now take two infinite order elements g and h with $h_\infty \neq g_\infty, g_{-\infty}$

Choose R with $N_R(h_\infty) \cap N_R(g_\infty) = \emptyset$,
ie for i,j sufficiently large, $(g^i, h^j)_1 < R$

Then $(g^{-i}, h^{-j})_1 < R$ too, so $g_{-\infty}$ and $h_{-\infty}$ are also distinct and we can make $N_R(g_{-\infty}), N_R(h_{-\infty})$ $N_R(g_\infty), N_R(h_\infty)$ all distinct:

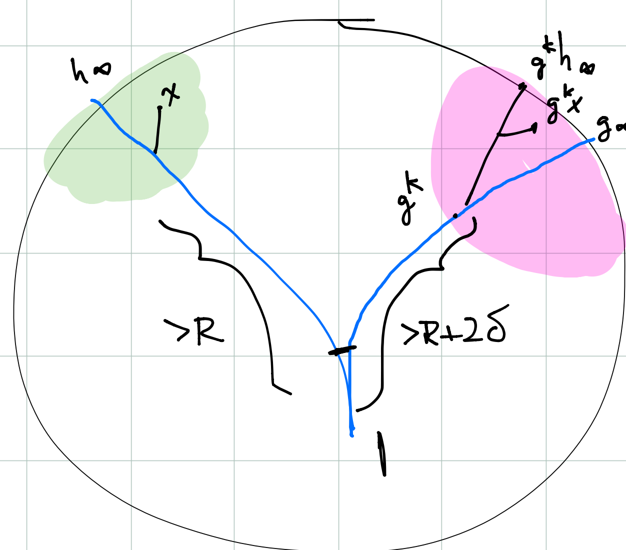


<u>Claim</u> For k sufficiently large,

$$g^k(A \cup \bar{A} \cup B) \subset B$$

Choose k large enough so that
$g^k, g^k h_\infty, g^k h_{-\infty} \in N_{R+2\delta}(g_\infty)$

Let $x \in N_R(h_\infty)$. <u>Claim</u> $(g^k h_\infty, g^k x)_1 > R+2\delta$



Then $(g^k x, g_\infty)_1 \geq \min\{(g^k x, g^k h_\infty)_1, (g^k h_\infty, g_\infty)_1\} - 2\delta$
$$\geq R+2\delta-2\delta = R.$$

So $g^k(N_R(h_\infty)) \subseteq N_R(g_\infty)$, ie $g^k(B) \subseteq A$

Similarly, $g^k(\bar{B}) \subseteq A$. So $g^k(A \cup B \cup \bar{B}) \subset A$.

In the same way we get

$$g^{-k}(A \cup \bar{A} \cup \bar{B}) \subseteq \bar{B}$$

$$h^{k}(A \cup B \cup \bar{B}) \subset A$$

$$h^{-k}(\bar{A} \cup B \cup \bar{B}) \subset \bar{A}$$

So by a ping-pong argument, no word in $a = g^k$, $b = h^k$ is the identity, ie $\langle g^k, h^k \rangle$ is a free subgroup

So we expect to find finitely generated free groups in hyperbolic groups

Those are themselves hyperbolic

Question: Are all subgroups of hyperbolic groups hyperbolic?

Answer: No. eg. F_2 contains infitely-generated free subgroups

Q2: Are all finitely-generated subgps of hyperbolic groups hyperbolic?

A: No: Example due to E. Rips ~1990

Q3: Are all finitely-presented subgps of hyperbolic groups hyperbolic?

A: No (Noel Brady, 1999)

Def: X a geodesic metric space

$A \subset X$ is quasi-convex if $\exists$ constant C

s.t. all geodesics between points of A lie in

$N_C(A)$.

eg quasi-geodesics are quasi-convex

Def A subgroup $H < G$ is quasi-convex if the vertices of H are a quasi-convex subset of the Cayley graph $\mathcal{C}$ of G.

eg G hyperbolic, $H = \langle g \rangle \cong \mathbb{Z}$

Exercise: Quasi-convexity is independent of the set of generators S for G.

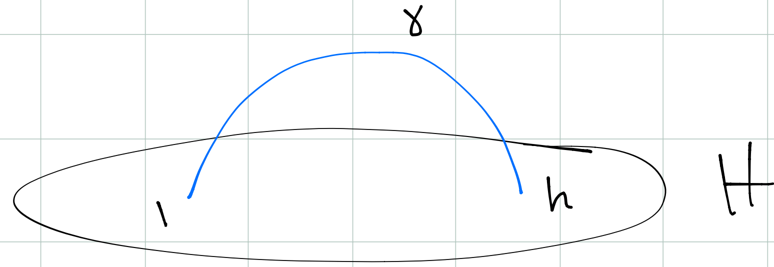
Examples:

1. finite subgroups of any fin. gen. group
2. $n\mathbb{Z} \subseteq \mathbb{Z}$
3. finite index subgroups of any group

Exercise 4. Finitely generated subgroups of free groups

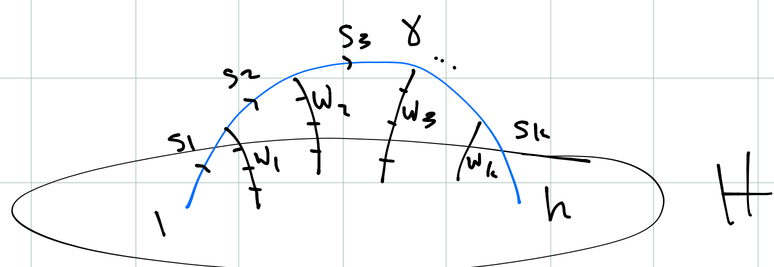
Prop G finitely generated, H a quasi-convex subgroup $\Rightarrow$ H is finitely generated.

Proof:



For $h \in H$, choose a geodesic $\gamma = \gamma_h$ joining 1 to h.

$\exists$ paths of length $\leq C$ joining each vertex of γ to a point in H



Then $h = (s_1 w_1)(w_1^{-1} s_2 w_2) \dots (w_k^{-1} s_k)$

so H is generated by {words of length $\leq 2C+1$ in S}, which is finite.