Tues March 11

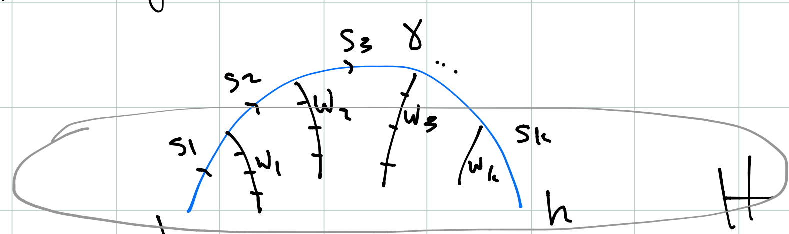
We defined quasi-convex subgroup of a finitely generated group.

Showed q.c. subgroups of hyperbolic groups are finitely generated.

In fact they are hyperbolic:

Prop G hyperbolic, $H < G$ quasi-convex. Then H is hyperbolic.

Proof Let S = finite generating set for G and T the gen. set for H described last time:



elts of T have length $\leq 2C+1$, and
$$d_T(h,h') \leq d_S(h,h').$$

We want to show triangles in $\mathcal{C}(H,T)$ are thin. We do this by showing geodesics in $\mathcal{C}(H,T)$ are quasi-geodesics in $\mathcal{C}(G,S\cup T)$ (note $S\cup T$ still generates G!)

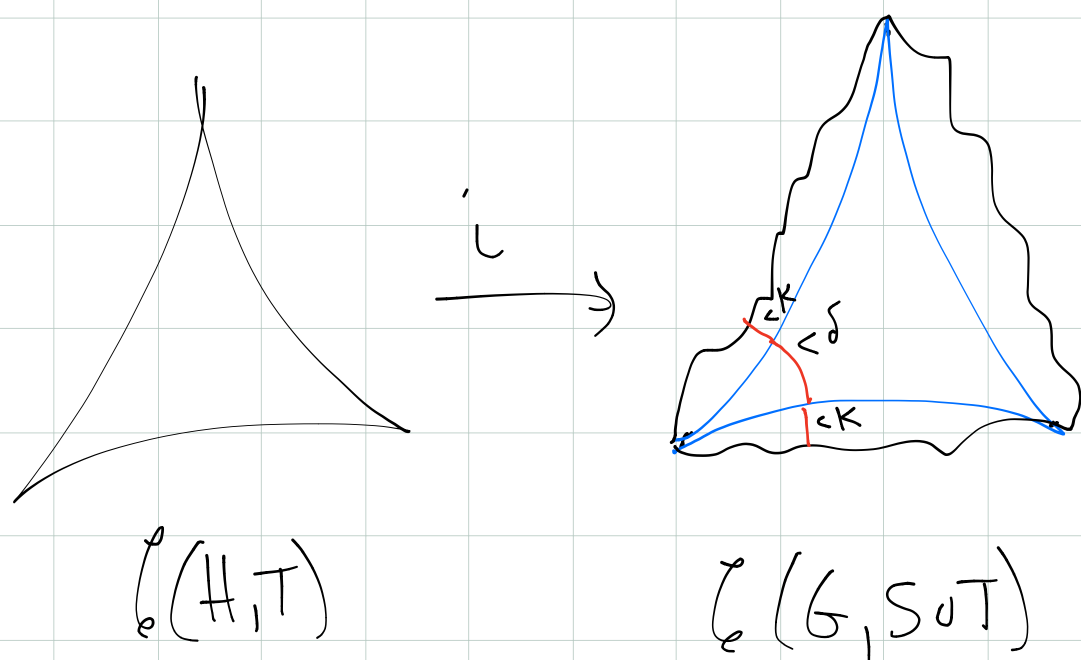
$d_{S\cup T}(h,h') \leq d_S(h,h')$ since adding more generators can only shorten distances.

Also $\quad d_T(h,h') \leq d_S(h,h') \leq (2C+1)\, d_{S\cup T}(h,h')$

since any word in $S\cup T$ can be written as a word in S alone at a cost of multiplying length by at most $2C+1$. So

$$d_{S\cup T}(h,h') \leq d_T(h,h') \leq (2C+1)\, d_{S\cup T}(h,h')$$

ie a geodesic in $\mathcal{C}(H,T)$ is a $(2C+1)$-quasi-geodesic in $\mathcal{C}(G,S\cup T)$



$\Rightarrow$ Triangles in $\mathcal{C}(H,T)$ are $(2K+\delta)$-thin in $\mathcal{C}(G, S\cup T) \Rightarrow (2K+\delta)$-thin in $\mathcal{C}(H,T)$

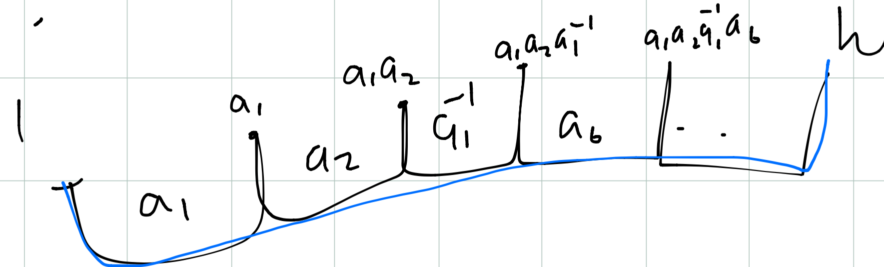
$(d_S(h,h') \geq d_T(h,h'))$ //

I set <u>exercice</u>: Fin. gen. subgroups of F_n are quasi-convex

Since it's the last week of class I'll do it

$A \subset F_n$ w/ basis $\{a_1, \dots, a_k\}$

$w =$ geodesic in F_n from 1 to $h \in A$



$\mathcal{C}(F_n)$ is a tree, geodesic from 1 to h is the blue path; stays within $\max\{\ell(a_i)\}$ of A. ✓

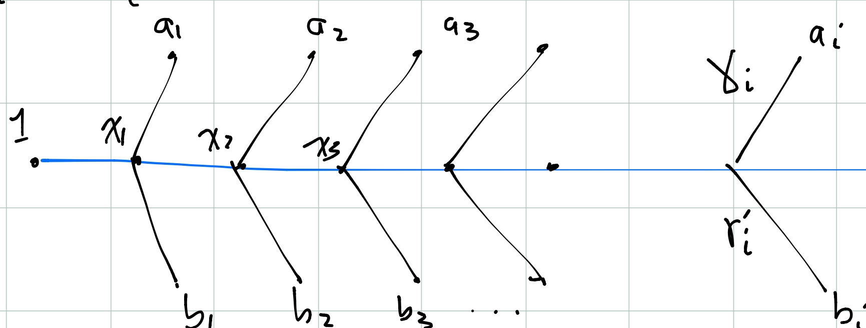
Thm (Howson) The intersection of two finitely-generated subgroups of F_n is finitely-generated

Follows from

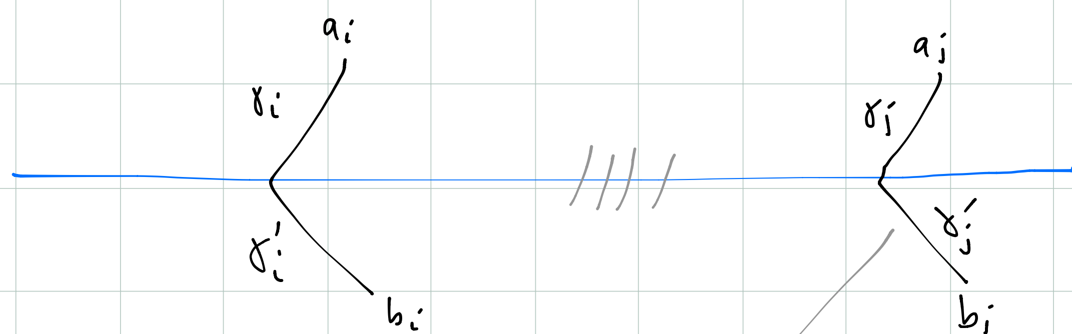
Prop The intersection of two quasi-convex subgroups is quasi-convex.

Proof Suppose A and B are quasi-convex subgroups of G, with $K = \max$ of quasi-convexity constants. Let $h \in A \cap B$, and draw the geodesic in G from 1 to h

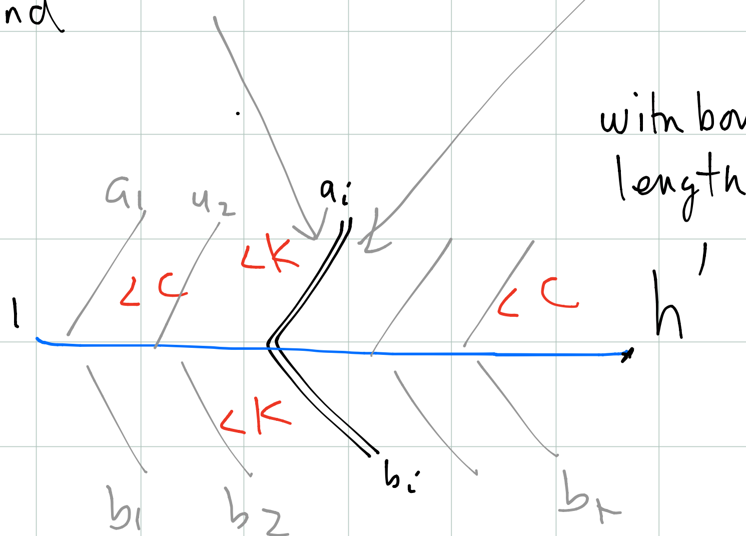
Find $a_i \in A$, $b_i \in B$ at distance $\leq K$ from x_i and geodesics γ_i from x_i to a_i, γ_i' from x_i to b_i.



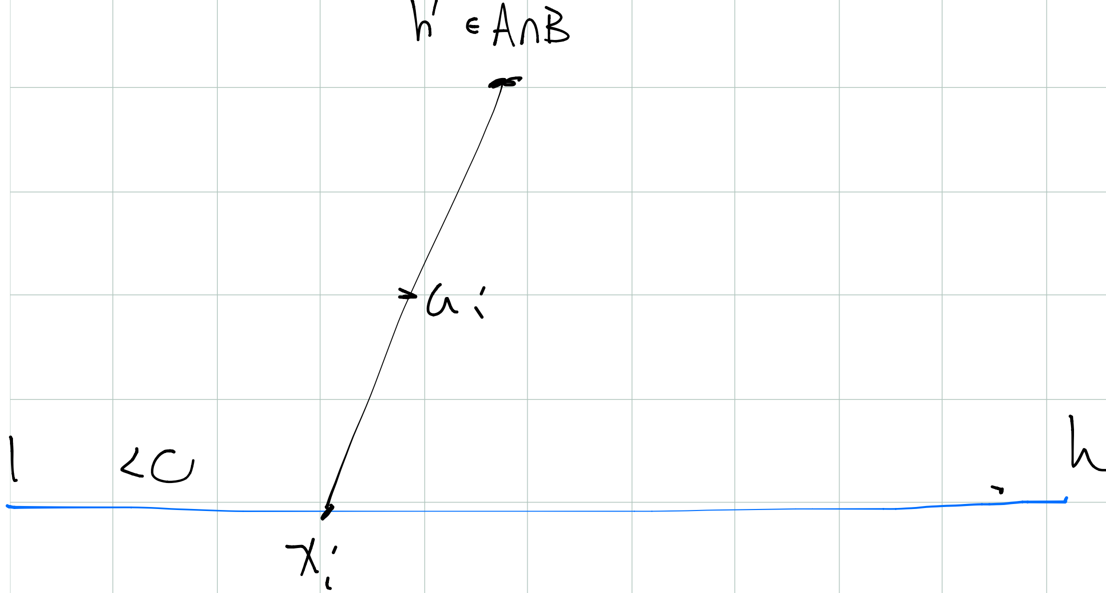
If h is long, $\exists\ i, j$ with $\gamma_i = \gamma_j$, $\gamma_i' = \gamma_j'$



Then can find $h' \in A \cap B$ with bounded length



length, so is a bounded distance from original geodesic.



Start again from x_i to find h'', etc.

Remark: In fact for a free group

$$\operatorname{rank}(H \cap K) - 1 \le (\operatorname{rk} H - 1)(\operatorname{rk} K - 1)$$

Conjectured by Hanna Neumann in 1957

Proved in 2011 (J. Friedman, I. Mineev)