Mon, March 3

Next: Application of ∂X to understanding subgroups of hyperbolic groups.

Goal: Play ping-pong w/ group elements.

First need to understand how elements act on $\mathcal{C}(G,S)$

Theorem: Let G be a hyperbolic group with Cayley graph $\mathcal{C}$, and let $g \in G$ have infinite order. Then the map $\mathbb{Z} \longrightarrow \mathcal{C}$ sending $k \longmapsto g^k$ is a quasi-geodesic.

Proof: We need to find λ, C st.

$$\frac{1}{\lambda}|i-j| - C \le d(g^i, g^j) \le \lambda |i-j| + C$$

it suffices to do this for $i=0$, ie

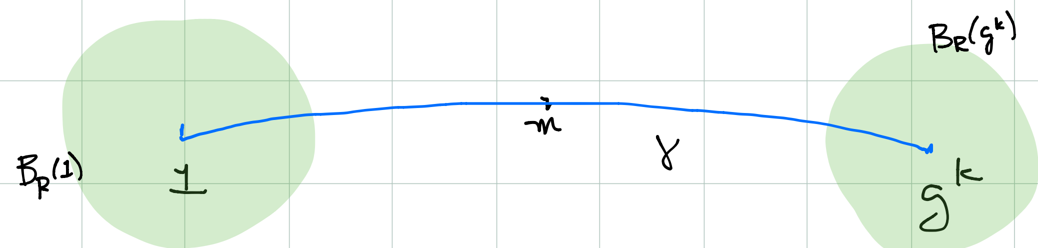
$$\frac{1}{\lambda} s - C \le d(1, g^s) \le \lambda s + C$$

The right-hand inequality is just the triangle inequality, with $\lambda = d(1,g)$, $C=0$

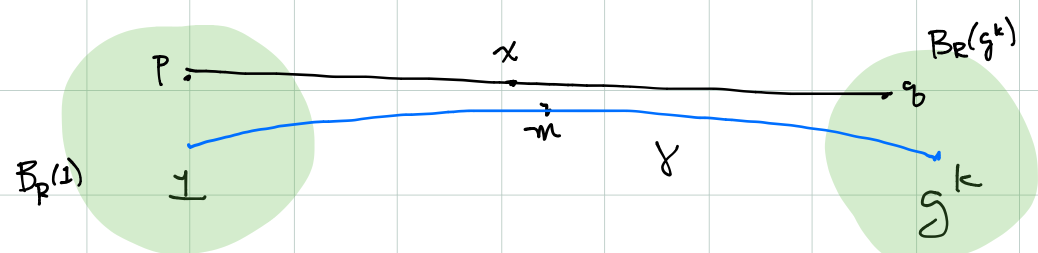
so need to show $d(1, g^s) \geq \frac{1}{\lambda} s - c$, ie it takes $\{g^i\}$ only a linear amount of time to escape the ball of radius s.

Fix a ball of radius R,

Choose k s.t. $d(1, g^k) > 8R + 12\delta$, let γ be the geodesic from 1 to g^k and m its midpoint.

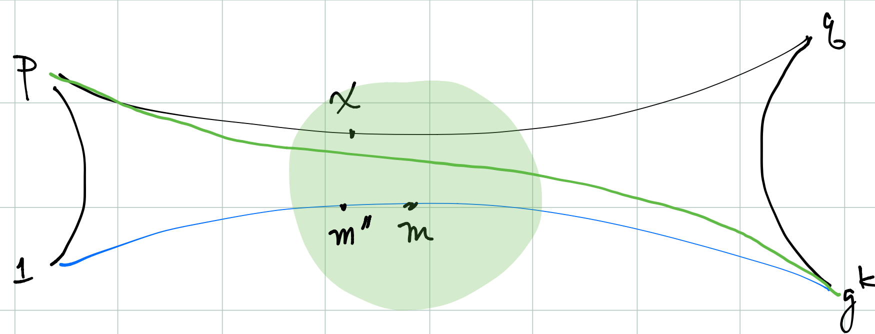


Let $p \in B_R(1)$ and $q \in B_R(g^k)$ be vertices of γ and x the midpoint of a geodesic connecting them.



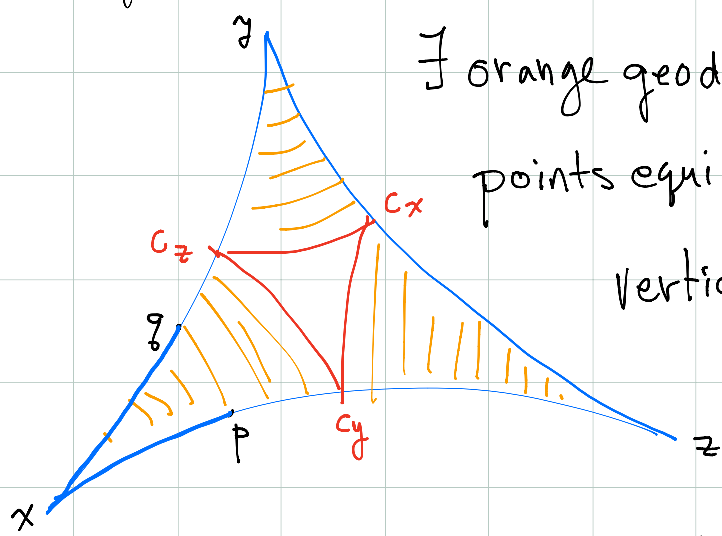
Claim x lies close to $m'' \in B_R(m) \cap \gamma$

Draw geodesic p to g^k:



We'll study the two triangles.

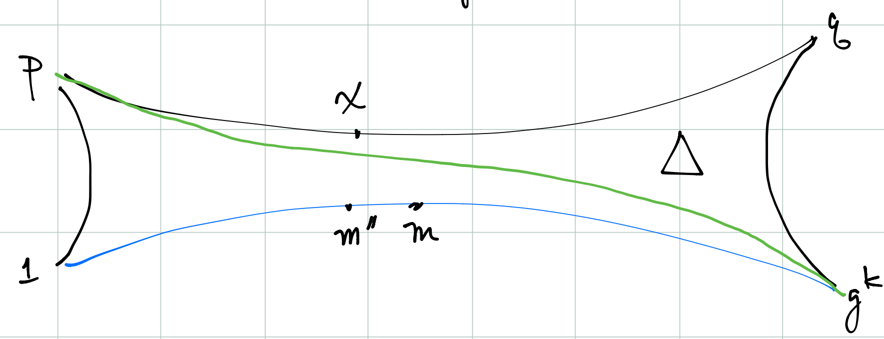
Recall our picture of a triangle in a hyperbolic space (from Feb 10)



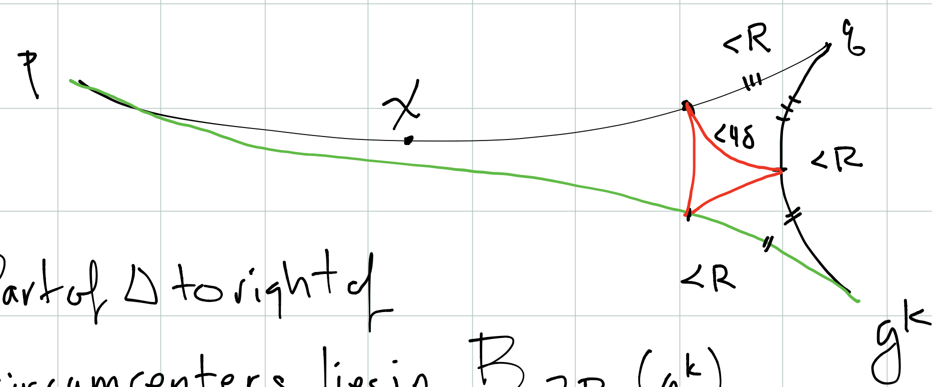
$\exists$ orange geodesics joining points equidistant from vertices, all of length $\leq 6\delta$

and $d(c_x, c_y) \leq 4\delta$

First look at top triangle $\Delta(p, q, g^k)$



We have:

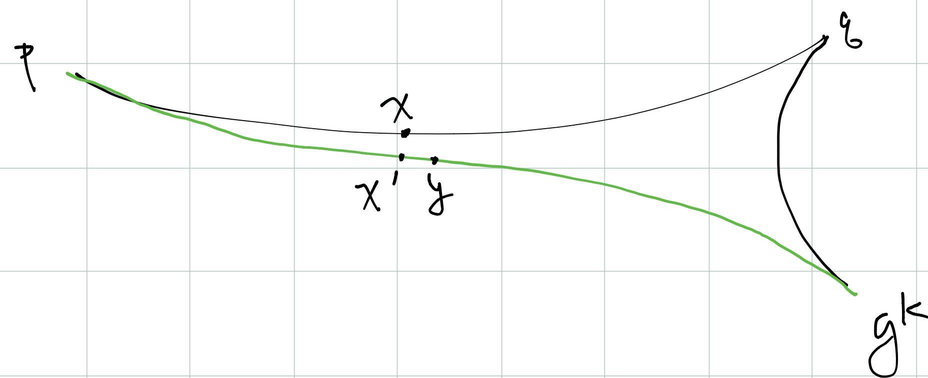


Part of Δ to right of circumcenters lies in $B_{2R}(g^k)$

But
$$d(x, g^k) \geq d(x, q) - d(q, g^k)$$
$$\geq \frac{1}{2} d(p, q) - R$$
$$\geq \frac{1}{2}\left(d(1, g^k) - 2R\right) - R = 2R + 6\delta$$

so x is between p and the circumcenter c

So x' (at the same distance from p) has distance $< 6\delta$ from x:



Now let y = midpoint of $[p, g^k]$

$$d(p,q) \le d(p,g^k) + d(g^k,q)$$

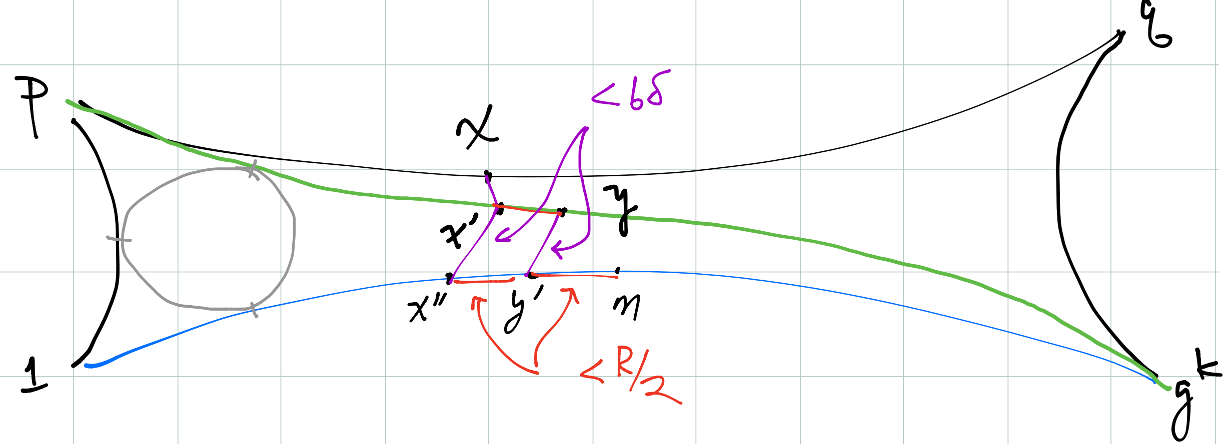
$$\begin{array}{c} \| \\ 2d(p,x) \le 2d(p,y) + R \\ \| \end{array}$$

$$\Rightarrow 2\left| d(p,x') - d(p,y) \right| \le R$$

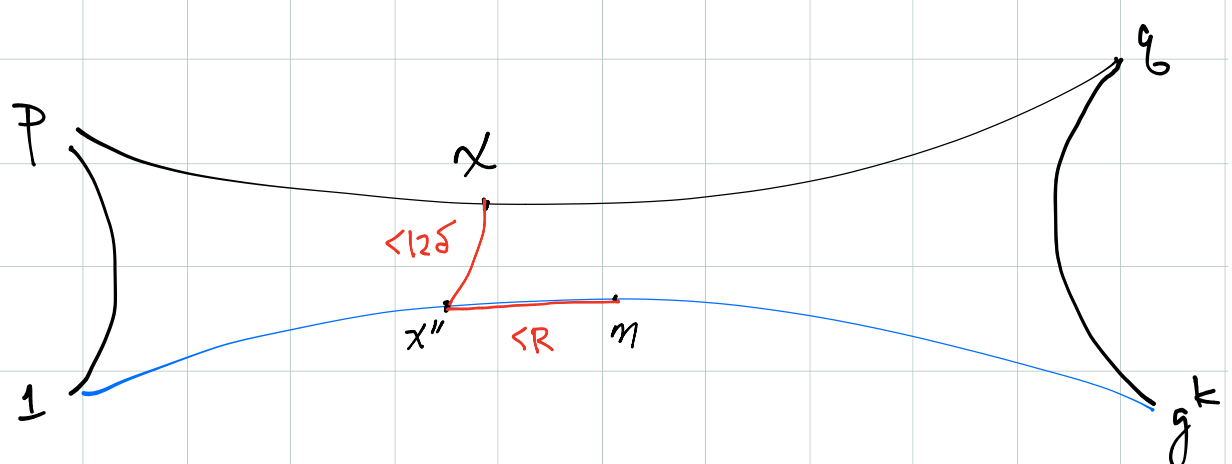
$$\Rightarrow d(x',y) \le R/2$$

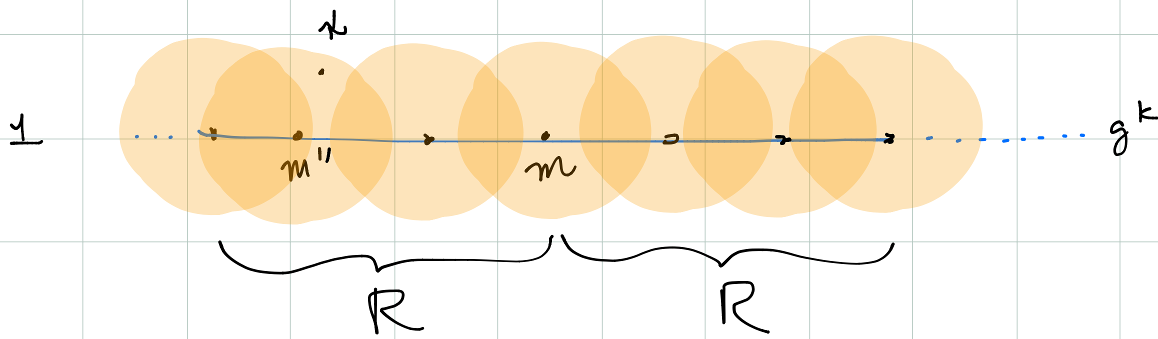
Similarly, find $x'', y' \in [1, g^k]$,

$d(x', x''), d(y, y') < 6\delta$, $d(x'', y'), d(y', m) < R/2$



Now we have:





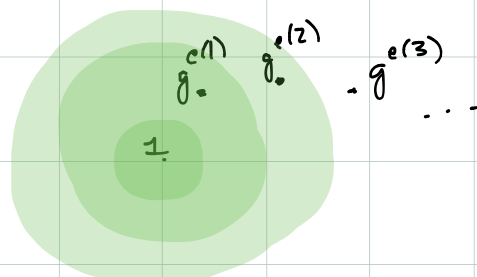
Let $K = \#$ vertices in $B_{12\delta}(1)$

$\exists \leq K\cdot(2R+1)$ vertices in a 2δ-nbd of the interval $[m-R, m+R]$ of γ

Midpoints of $g^i[1, g^k]$ are all different,

So for some $i \leq K(2R+1)$ the endpoints of $g^i[1, g^k]$ are not in $B_R(1)$ and $B_R(g^k)$.

Let $e(R) \leq 2KR+K$ be the "first escape" of g^i from $B_R(1)$.



Note $R \leq d(1, g^{e(R)}) \leq e(R) \cdot d(1,g)$, so

$$R/d(1,g) \leq e(R) \leq 2KR + K$$

If $s = e(R) \leq 2KR + K$ then $R \geq \frac{s}{2k} - \frac{1}{2}$

and $d(1, g^s) \geq \frac{s}{2k} - \frac{1}{2}$ $\Rightarrow \lambda = \frac{1}{2k}$, $C = \frac{1}{2}$ works

If s is not $e(R)$ for any R have to work a little harder