EXERCISES FOR MA4J7 ALGEBRAIC TOPOLOGY II

1. Week 1

(1) Read over your notes from Algebraic Topology I.
(2) Show that cohomology is a contravariant functor from the category of chain complexes to the category of abelian groups.
(3) Fix an abelian group G and let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. If C is free, show that
  (a) $B \cong A \oplus C$
  (b) $Hom(B, G) \cong Hom(A, G) \oplus Hom(C, G)$
  (c) $0 \to Hom(C, G) \to Hom(B, G) \to Hom(A, G) \to 0$ is exact.

2. Week 2

(1) Let $X = RP^2$. As we saw in class, X is obtained from the circle S^1 by attaching a disk D^2 by a map $\partial D^2 \to S^1$ of degree 2. Show that the quotient map $X \to X/S^1 = S^2$ induces the trivial map on reduced homology in all dimensions, but not on $H^2(\_\_;\mathbb{Z})$. Deduce that the splitting in the universal coefficient theorem for cohomology cannot be natural. (This is adapted from Exercise 11, p. 205, in Hatcher).
(2) Use the cell structure on the 3-dimensional lens space $L_m(\ell_1, \ell_2)$ described in Example 2.43 of Hatcher to compute the cochain complex and cohomology of $L_m(\ell_1, \ell_2)$ with coefficients in $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Z}/m\mathbb{Z}$. (This is adapted from Exercise 10, p. 205, in Hatcher).
(3) Derive the Mayer-Vietoris sequence in cohomology from the Excision property. (This is done in Hatcher).

3. Week 3

(1) Compute the cohomology and cup product structure of the Kline bottle.
(2) Use the relative cup product $H^k(X, A; R) \times H^\ell(X, B : R) \to H^{k+\ell}(X, A \cup B; R)$ to show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. (First part of Exercise 2, p. 228)

4. Week 4

(1) Check the formula $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi$, where k is the degree of ϕ.
(2) Compute the cup product structure of $S^1 \times S_g$, where S_g is an orientable surface of genus g.

5. Week 5

(1) Let $0 \to A \to A \oplus A \to B \to 0$ be short exact sequence of abelian groups. If the first map is $a \mapsto (a, a)$, show that the restriction of the second map to either copy of A is an isomorphism.
(2) Fill in the details of the cup product structure of complex projective space, with cohomology coefficients in $\mathbb{Z}$. Point out each place where the argument needs to be modified from the argument for real projective space.

6. Week 6

(1) Compute $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$ for arbitrary integers n and m.
(2) If N is a finitely generated free R-module, show that

$$(\prod_{\alpha} M) \otimes_R N \cong \prod_{\alpha} (M \otimes_R N).$$

Give an example to show that this is not true for arbitrary N.
(3) Using the cup product $H^k(X, A; R) \times H^\ell(X, B; R) \to H^{k+\ell}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero.
(4) Using the cup product structure, show there is no map $RP^n \to RP^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ if $n > m$.
(5) The Borsuk-Ulam theorem says every continuous function $f\colon S^n \to R^n$ maps some pair of antipodal points to the same point. Prove this by the following argument. Suppose on the contrary that $f\colon S^n \to R^n$ satisfies $f(x) \neq (-x)$ for all x . Then define $g\colon S^n \to S^{n-1}$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

so $g(-x) = -g(-x)$ and g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Then apply the previous exercise.

7. Week 7

(1) Show that every covering space of an orientable manifold is orientable.
(2) Show that $M \times N$ is orientable if and only if M and N are both orientable.

8. Week 8

(1) For a map $f\colon M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the *degree* of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental class. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

(2) For a map $f\colon M \to N$ between connected closed orientable n-manifolds, suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B . Show the degree of f is ϵ_i, where ϵ_i is +1 or -1 according to whether $f\colon B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

9. Week 9

(1) Show that $H^0_c(X) = 0$ for a non-compact path-connected space X.
(2) Let $\{G_i\}_{i\in\mathbb{N}}$ be the directed system of groups in which each G_i is a copy of $\mathbb{Z}$ and for $i < j$, the homomorphism f_{ij} is multiplication by 2^{j-i}.

$$G_1 \xrightarrow{\times 2} G_2 \xrightarrow{\times 2} G_3 \xrightarrow{\times 2} \dots$$

   (a) Let $G_1 \to \lim_{\to} G_i$ be the homomorphism which takes k to $[k]$. Is this map injective? Is it surjective? Explain.
   (b) Is there a non-trivial homomorphism $\lim_{\to} G_i \to \mathbb{Z}$? If so, define it. If not, why not?

(3) Show that a direct limit of torsion-free abelian groups G_α is torsion-free.
(4) Show that a direct limit of short exact sequences is short exact.
(5) Show that the second square in the Mayer-Vietoris diagram of Lemma 3.36 is commutative.

10. Week 10

(1) (Exercise 25 on p. 260 of Hatcher) Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsion free, then $H_k(M;\mathbb{Z})$ is also torsion free.
(2) (Exercise 27 on p. 260 of Hatcher). Show that after a suitable change of basis, a skew-symmetric nonsingular bilinear form over $\mathbb{Z}$ can be represented by a matrix consisting of 2×2 blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ along the diagonal and zeros elsewhere. [For the matrix of a bilinear form, the following operation can be realized by a change of basis: Add an integer multiple of the i-th row to the j-th row and add the same integer multiple of the i-th column to the j-th column. Use this to fix up each column in turn. Note that a skew-symmetric matrix must have zeros on the diagonal.]