Tokyo, July 31, 2014

Automorphisms of RAAGs

Part I – untwistable RAAGs

RAAGs = Right-angled Artin groups

- finitely generated groups
- some generators may commute but no other defining relations

First example F_n (no generators commute, no relations)

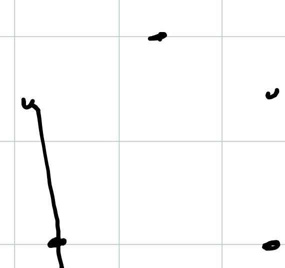
Usually defined by drawing a graph: Γ

Vertices = generators

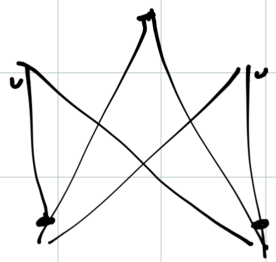
edge $\overset{v}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{w}{\bullet} \iff [v,w]=1$

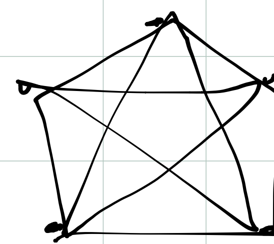
<u>eg</u>

$A_\Gamma = F_5$

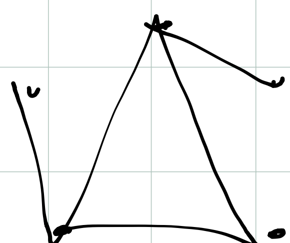


$A_\Gamma = \mathbb{Z}^2 * F_3$





$F_3 \times F_2 = A_\Gamma$ $\qquad$ $A_\Gamma = \mathbb{Z}^5$



A_Γ

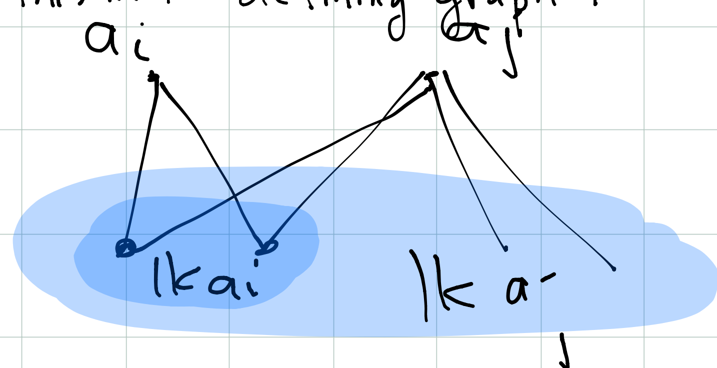
Automorphisms of RAAGs:

$\varepsilon_i : a_i \to a_i^{-1}$

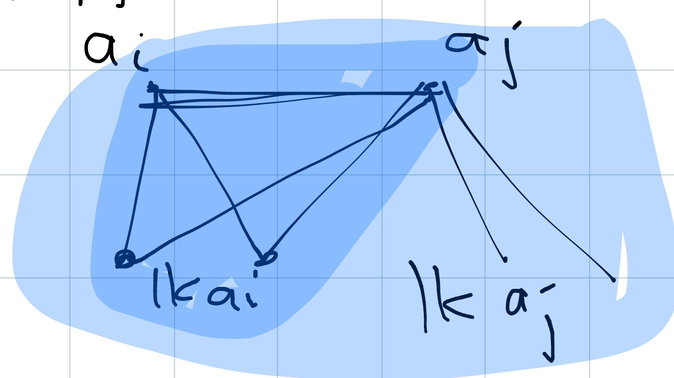
$\rho_{ij} : a_i \mapsto a_i a_j$ is <u>not</u> always an automorphism

need: $C(a_i) \subseteq C(a_j)$

Can see this in the defining graph Γ:



If $[a_i, a_j] \neq 1$ need $\mathrm{lk}\, a_i \subseteq \mathrm{lk}\, a_j$

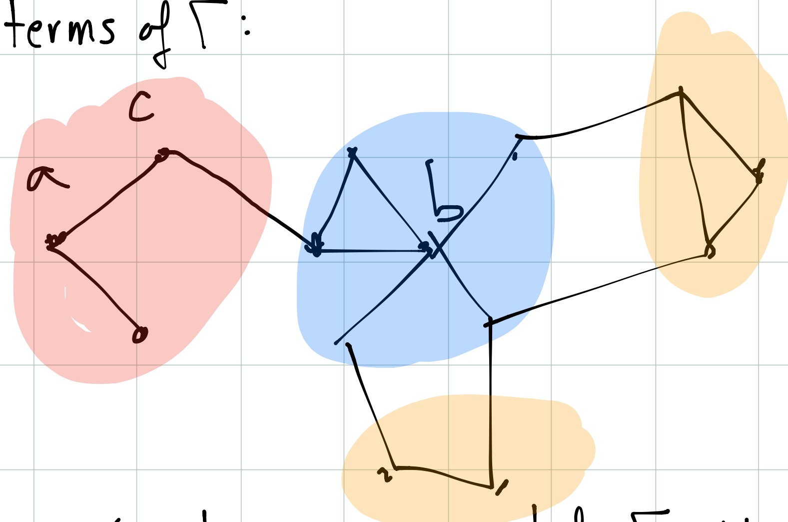


If $[a_i, a_j] = 1$ need $\mathrm{st}\, a_i \subseteq \mathrm{st}\, a_j$

Even if you can't multiply a by b you may be able to conjugate a by b: $a \longmapsto bab^{-1}$

- as long as you also conjugate $C(a)$:

In terms of Γ:



can conjugate any component of $\Gamma - \mathrm{st}\, b$

Since you don't have all p_{ij}, you don't have all permutations of the generators - just autos of Γ

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Thm (Laurence, Servatius) These generate $\mathrm{Aut}(A_\Gamma)$:

- inversions

- graph automorphisms
- partial conjugations
- transvections $a \mapsto ab$
  - if $ab \neq ba$ (folds)
  - if $ab = ba$ (twists)

Charney-Farber

Remark: A random graph has no automorphisms, stars don't disconnect them, $\mathrm{lk}\, a \subseteq \mathrm{st}\, b$ doesn't happen, So $\mathrm{Aut}(A_\Gamma)$ is just inversions!

We could just stop here — but there are still plenty of Γ with interesting $\mathrm{Out}(A_\Gamma)$, starting from

$$F_n : \mathrm{Out}(F_n)$$
$$\mathbb{Z}^n : \mathrm{Aut}(\mathbb{Z}^n) = GL_n\mathbb{Z}$$

Many A_Γ exhibit characteristics of both.

<u>Today</u>: Look at RAAGs with no twists

(alternatively, look at the subgroup $U(A_\Gamma) \subset Out(A_\Gamma)$ generated by all generators except twists)

We will construct an "Outer space" for those modeled on Outer space for F_n.

For F_n

- free minimal actions of F_n on metric simplicial trees
- equiv classes of marked metric graphs

$$R_n \xrightarrow{\simeq} G \qquad (\pi_1(R_n) \cong F_n)$$

For A_Γ?

- free minimal actions of A_Γ on metric simplicial trees ?
- equiv classes of marked metric graphs ?

$$\text{?} \xrightarrow{\simeq} X \qquad (\pi_1(\text{?}) \cong A_\Gamma)$$

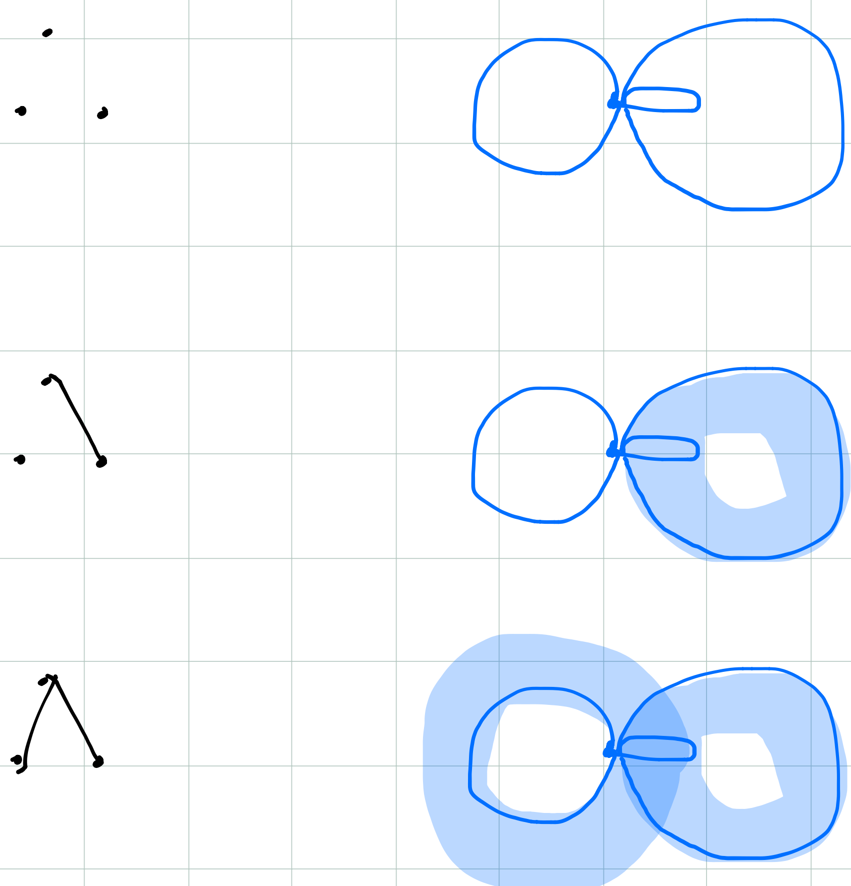
A nice space with $\pi_1 \cong A_\Gamma$: Salvetti complex

S_Γ = NPC cube complex with

- one vertex
- one k-cube for each k-clique in Γ

Examples:

Γ	S_Γ



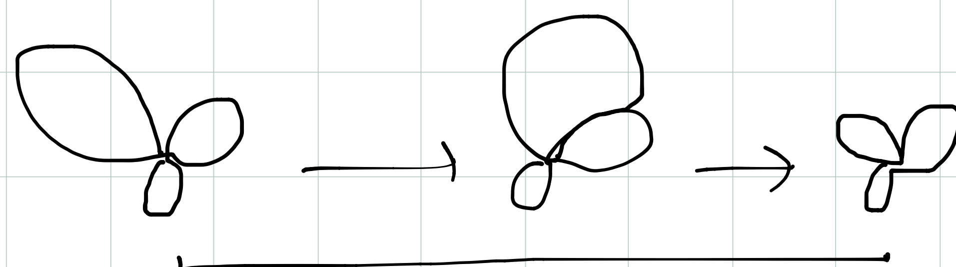
S_Γ is a NPC (locally CAT(0)) special cube complex

This gives us <u>roses</u> for $A_\Gamma = F_n$

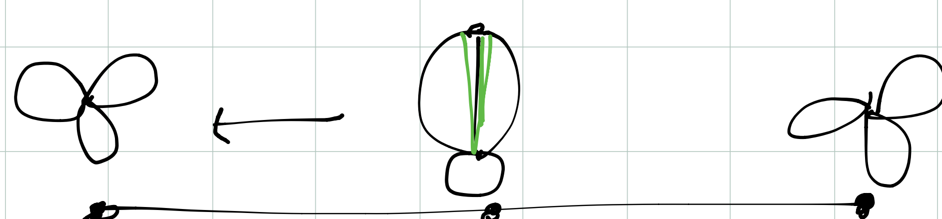
Roses are a discrete subset of Outer space.

What's the analog of an arbitrary graph?

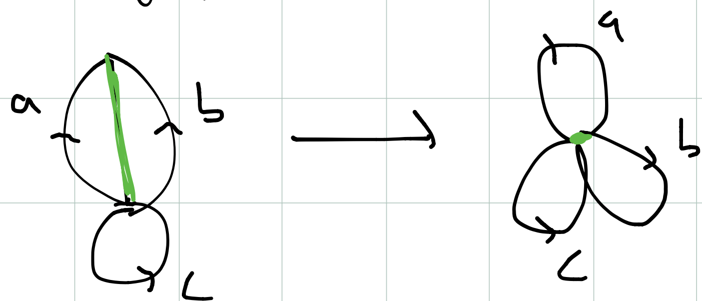
Recall to connect roses we folded edges together



Recall also that to construct the spine, we could ignore metrics:



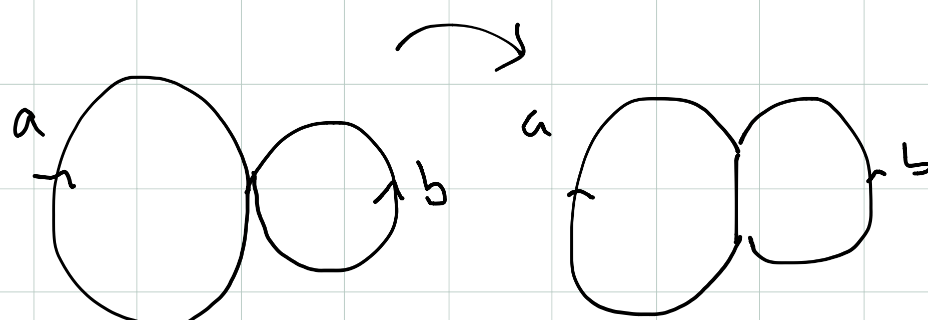
To obtain (either) rose, we collapsed an edge of the middle graph



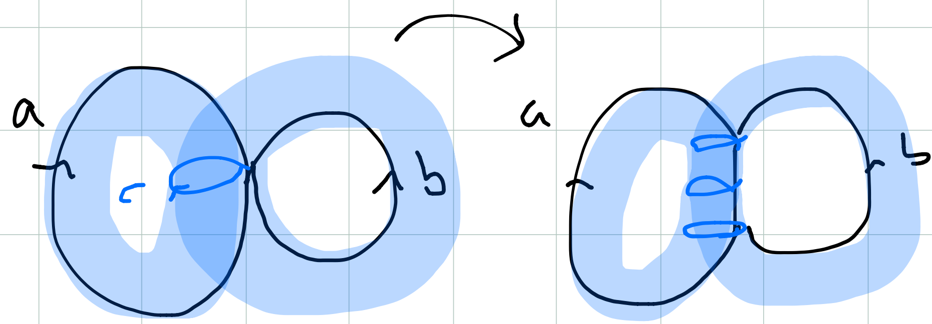
$G \longrightarrow R$

The collapsed edge $\longleftrightarrow$ partition of the half-edges of R

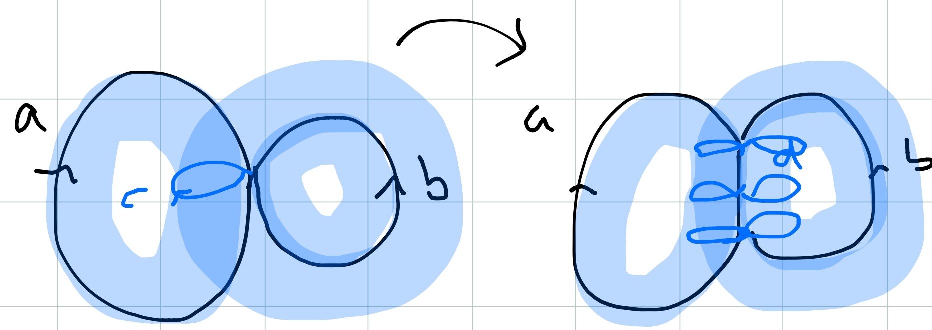
Now try to fold in S_Γ:



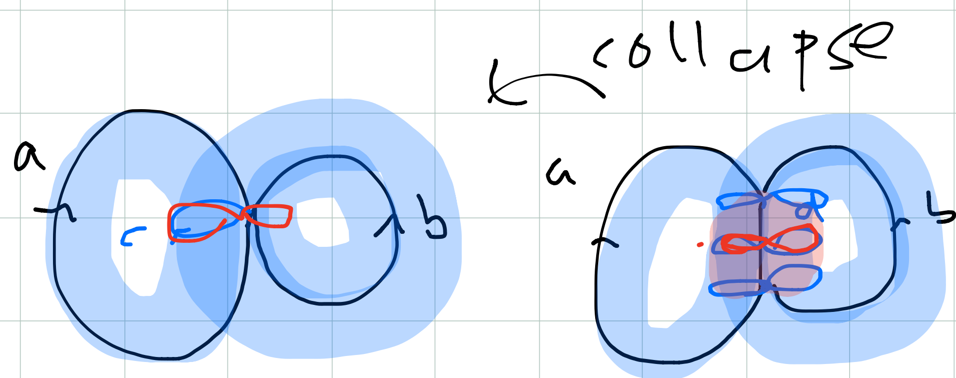
but a & b carry "baggage"



$[c,a]=1 \Rightarrow [c,b]=1$



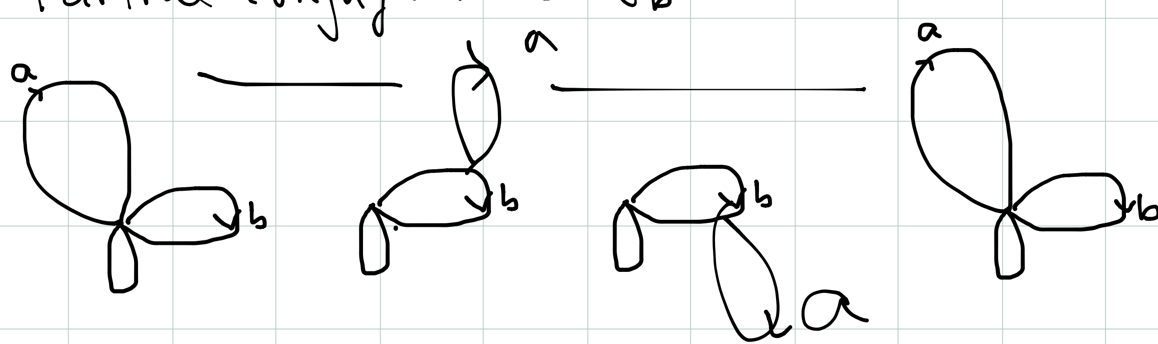
so need to fold the whole ac-torus



hyperplane collapse

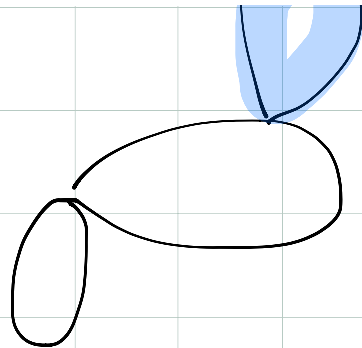
Get back to the original S_Γ by collapsing (to) a hyperplane

Partial conjugations γ_b

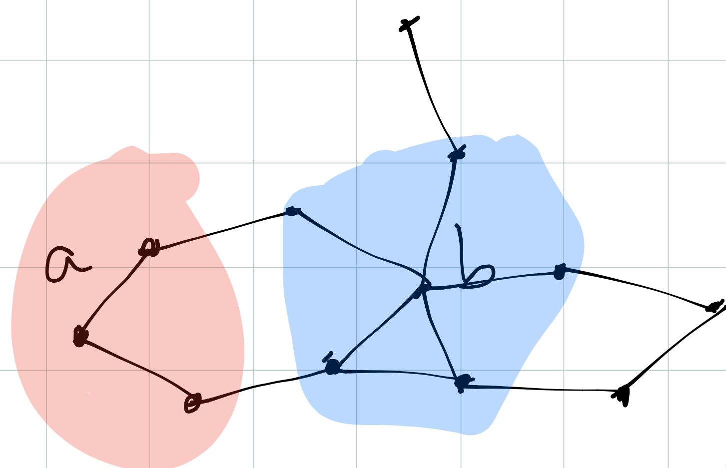


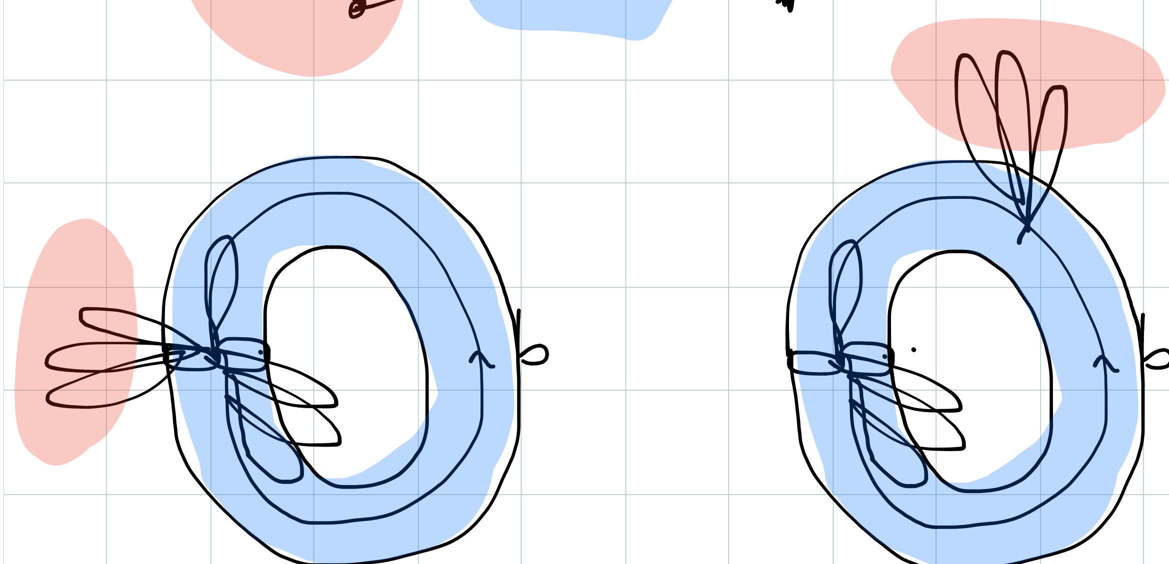
$a \mapsto bab^{-1}$

Again comes with baggage

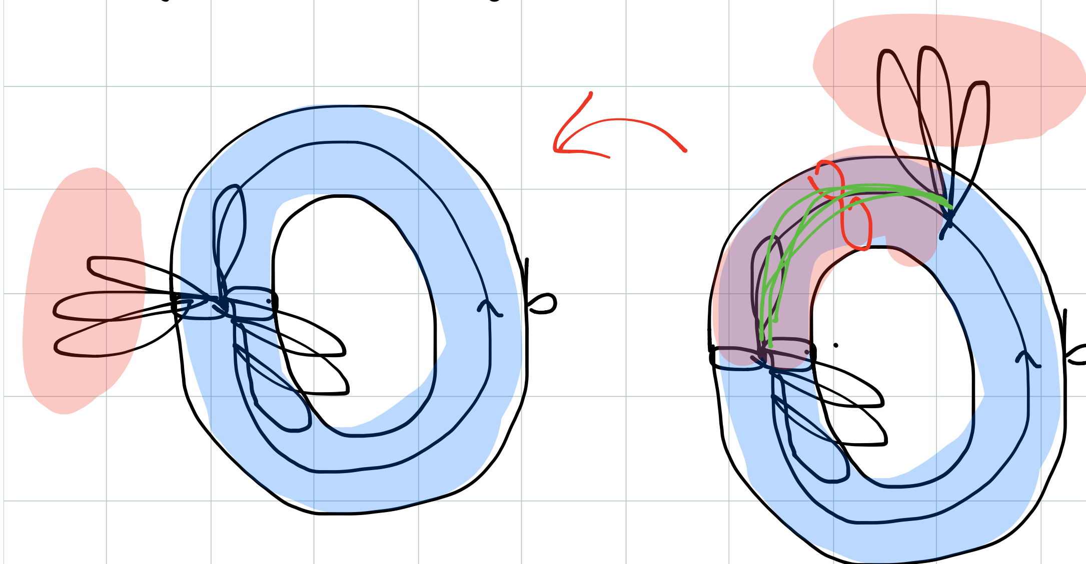


$[a,c]=1 \Rightarrow$ have to drag the whole ac-torus around b.





Again, collapsing a hyperplane recovers S_Γ



For general graphs, collapsing a maximal tree recovers the rose R_n

Analog for A_Γ: X_n = NPC special cube complex s.t. collapsing a hyperplane forest recovers S_Γ.

$\boxed{:= \Gamma\text{-complex}}$

Hyperplane forest: The edges dual to the hyperplanes forms a forest in the 1-skeleton of X

There is also a constructive definition of Γ-complex in terms of partitions of the half-edges of the 1-skeleton of S_Γ

Need this to work with Γ-complexes

Can now define a "spine" K_Γ for

Outer space = realization poset of marked Γ-complexes (X, g)

$g: S_\Gamma \longrightarrow X$ a homotopy equiv

poset relation = hyperplane forest collapse

If $U(A_\Gamma) \neq Out(A_\Gamma)$ need to be more careful:

$$\begin{tikzcd} S_\Gamma \arrow[r, "g"] & X \arrow[d, "c"] \\ S_\Gamma \arrow[u, "\varphi"] \arrow[ur, "g \circ \varphi"] \arrow[r] & S_\Gamma \end{tikzcd}$$

only allow g if $c_x \circ g \in U(A_\Gamma)$

Then $\underset{\substack{\cup \\ \psi}}{U(A_\Gamma)}$ acts on K_Γ

<u>Thm</u> (Charney-V.-Stambaugh)
K_Γ is contractible, $U(A_\Gamma)$ acts properly and cocompactly.

Sketch of proof

1. $K_\Gamma = \bigcup st(S_\Gamma)$ ✓

2. Order the Salvettis $(S_{\Gamma, g})$
(lexicographically order by length of conjugacy classes)

3. Prove this is a well-ordering

4. Build K_Γ by attaching stars of Salvettis in order

5. Prove that at each stage you are attaching along a contractible set.

This involves understanding how elementary moves (folds, partial conjugations) affect lengths of conjugacy classes

Recap:

$U(A_\Gamma) \subseteq Out(A_\Gamma)$ subgroup generated by all Laurence-Servatius generators <u>except</u> twists.

K_Γ = simplicial complex
= realization of poset of $U(A_\Gamma)$-marked Γ-complexes
$g: S_\Gamma \xrightarrow{\simeq} X$

Thm K_Γ is contractible, $U(A_\Gamma)$ acts properly, cocompactly.

Cor: For any Γ, $\dim(K_\Gamma)$ is an upper bound on the virtual cohomological dimension of $U(A_\Gamma)$.

$\dim(K_\Gamma)$ = maximum number of hyperplanes in any hyperplane forest

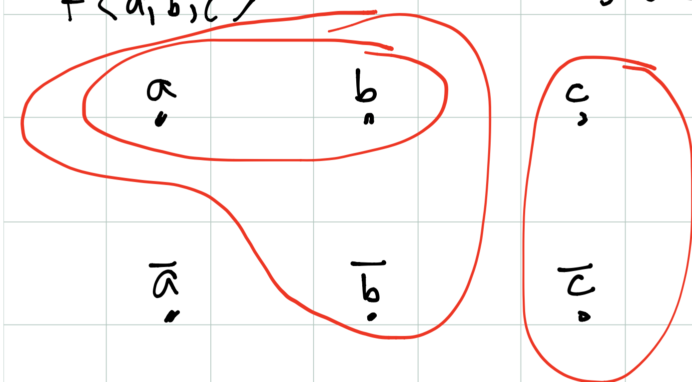
This is easily computable from Γ using the partition description of hyperplanes

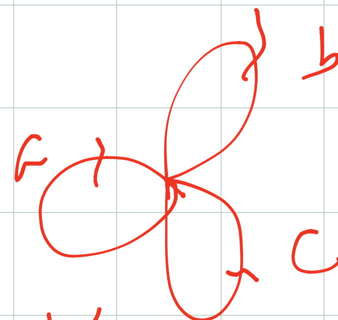
= maximal number of compatible Γ-partitions of half-edges in 1-skeleton of S_Γ

$\to = VCD(Out(F_3))$

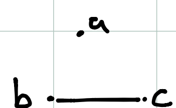
eg $A_\Gamma = F_n$ Γ-partition $\Leftrightarrow$ each side has ≥ 2 elts

$F\langle a, b, c\rangle$

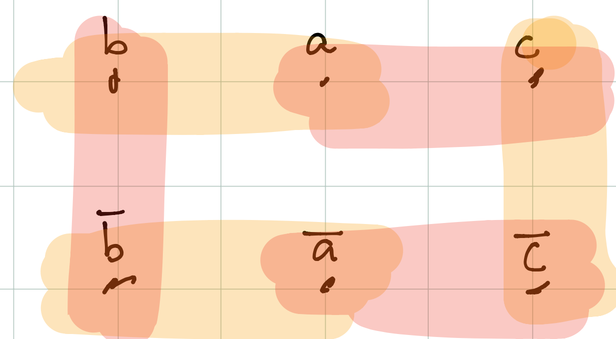




dim $K_\Gamma = 3$

$\Gamma =$ 

Γ-partition has 3 parts:

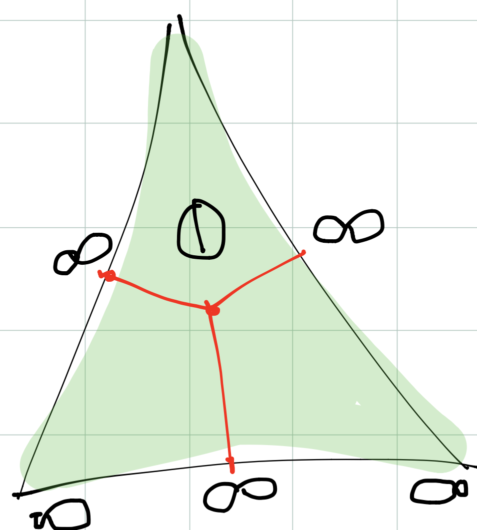


dim $K_\Gamma = 2$.

Lower bounds on the VCD of $U(A_\Gamma)$ (and $Out(A_\Gamma)$) can be found by exhibiting free abelian subgroups of $U(A_\Gamma)$.

For $A_\Gamma = F_n$, K_Γ = spine of Outer space

Can recover all of Outer space by adding information about lengths of edges



Similarly, can enlarge K_Γ by adding metric information:

- allow side lengths of "cubes" to vary (keeping angles orthogonal)
- normalize: take homothety classes