Tokyo - August 1, 2014

Automorphisms of RAAGs

Part II - Twisted RAAGs

Recall $\mathrm{Out}(A_\Gamma)$ generated by

U
- inversions $a_i \longmapsto a_i^{-1}$
- graph automorphisms
- partial conjugations $X \mapsto bXb^{-1}$
- folds $a \longmapsto ab \neq ba$

$T(A_\Gamma)$
- twists $a \longmapsto ab = ba$

$U(A_\Gamma) \subseteq \mathrm{Out}(A_\Gamma)$ generated by all but twists

$\boxed{\underline{\text{Define}}: \ T(A_\Gamma) = \text{subgroup generated by twists}}$

$\underline{\text{eg}}$ $A_\Gamma = \mathbb{Z}^n$

there are no folds or partial conjugations, so

$$U(\mathbb{Z}^n) = S_n \times (\mathbb{Z}/2)^n$$

$$T(\mathbb{Z}^n) = SL_n(\mathbb{Z})$$

$$P_{ij} = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix} = E_{ij}$$

Outer space for $T(\mathbb{Z}^n)$?

Contractible space with proper $SL_n\mathbb{Z}$-action?

$$SO_n \backslash SL(n,\mathbb{R})$$

Action of $A \in SL_n\mathbb{Z}$:

$$(SO_n \cdot M) \cdot A = SO_n (MA)$$

(right action)

Another way to describe $SO_n \backslash SL_n\mathbb{R}$:

As a space of marked metric Salvetti complexes!

For $A_\Gamma = \mathbb{Z}^n$, $S_\Gamma = T^n$ = n-torus.

We will use flat metrics, volume 1

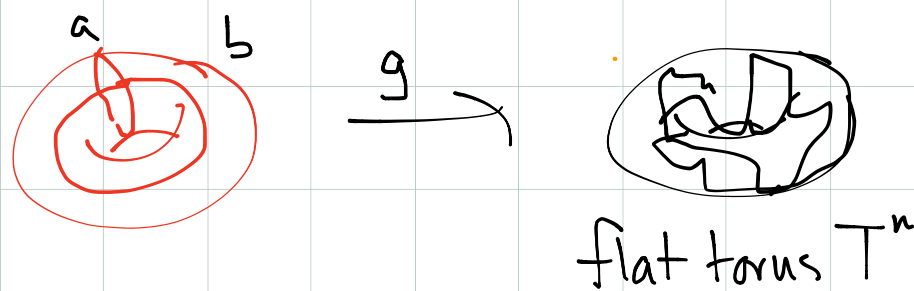
A marking is a homotopy equiv

$$S_\Gamma \xrightarrow{g} T^n$$

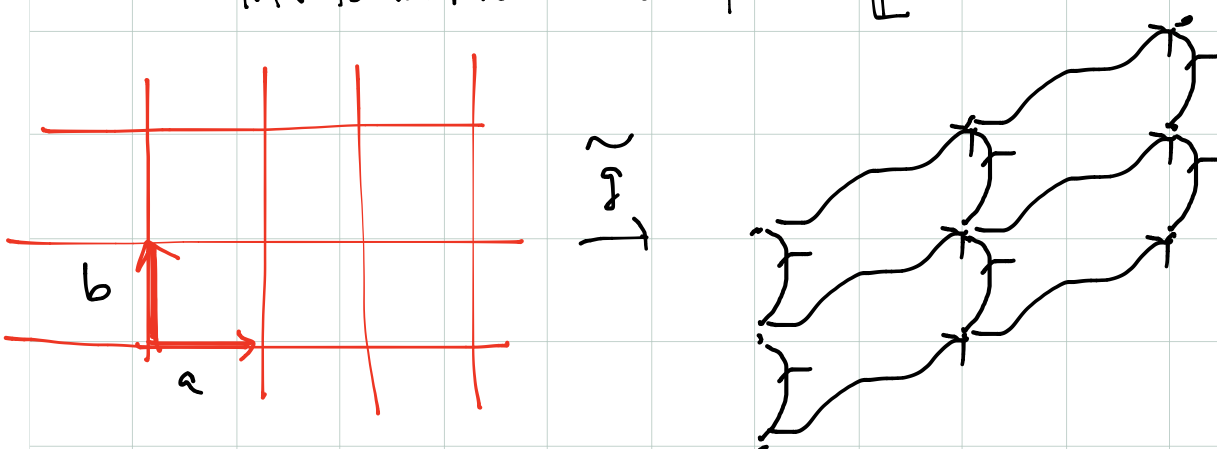
any such g is homotopic to a homeomorphism,

so wma markings are homeomorphisms

Given a marking

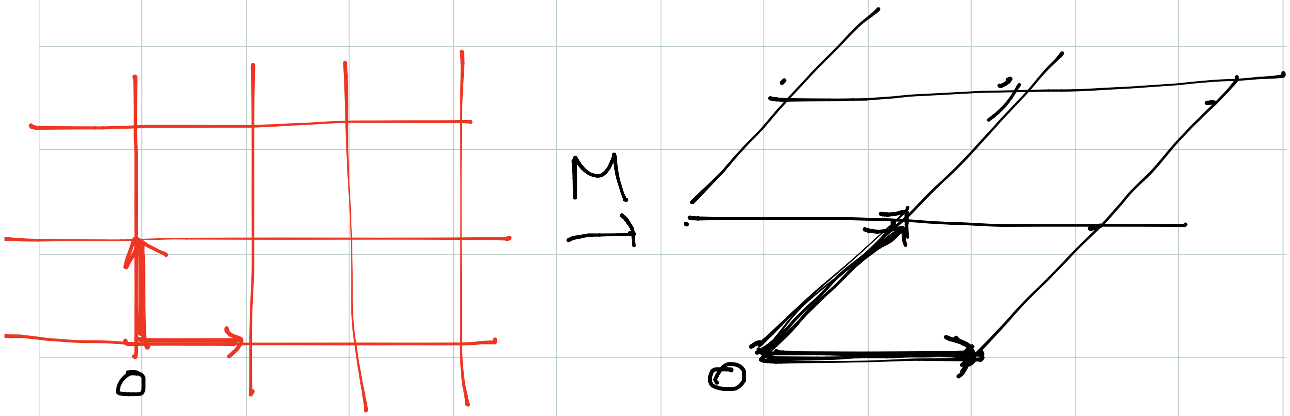


lift to universal cover $\widetilde{T}^n = \mathbb{E}^n$



Straighten out $\tilde{g}$, choose an origin, rotate so one vector is horizontal, other goes upward:

Get a lattice with a marked basis



$$\mathbb{Z}^n \xrightarrow{\quad M \quad} \mathbb{R}^n$$

linear map, given by matrix $M \in SL(n, \mathbb{R})$

Rotating the original picture doesn't change the result

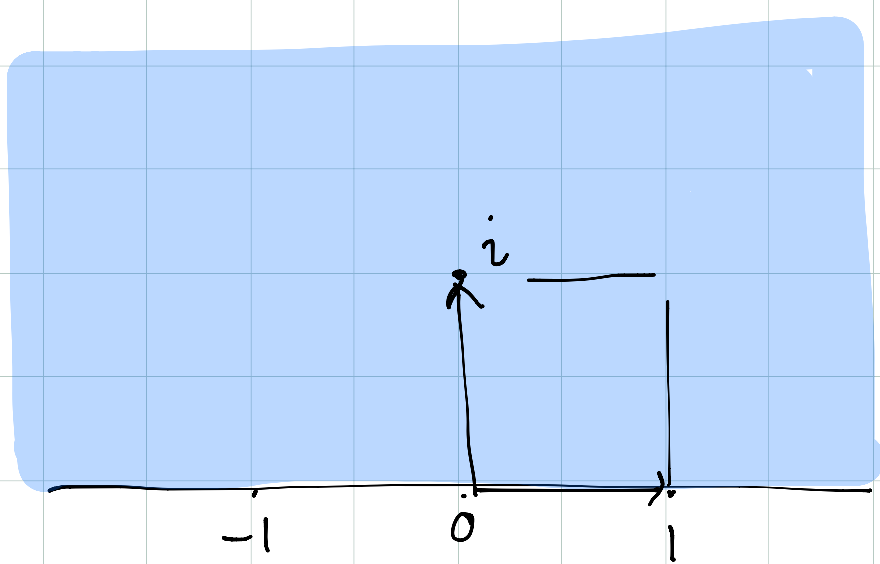
$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, "M"] \arrow[dr, "OM"'] & \mathbb{R}^n \arrow[d, "O"] \\
 & \mathbb{R}^n
\end{tikzcd}$$

$M \sim OM$, ie $M \in SO(n)\backslash SL_n\mathbb{R}$

Action of $A \in SL_n\mathbb{Z}$:

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "M"] & \mathbb{R}^n \\ \mathbb{Z}^n \arrow[u, "A"] \arrow[ur, "A"'] & \end{tikzcd}$$

For $n=2$, $SO(n)\backslash SL_n\mathbb{R}$ = upper half-plane $\mathrm{Im}(z)>0$



$i \longleftrightarrow$ square torus

So point in $SO_n \backslash SL_n\mathbb{R} \longleftrightarrow$ marked flat torus
volume 1

$\longleftrightarrow$ free action of $\mathbb{Z}^n$ on
the plane

Relation to yesterday's construction?

$\Gamma = \Delta \Rightarrow$ stars don't separate, no folds

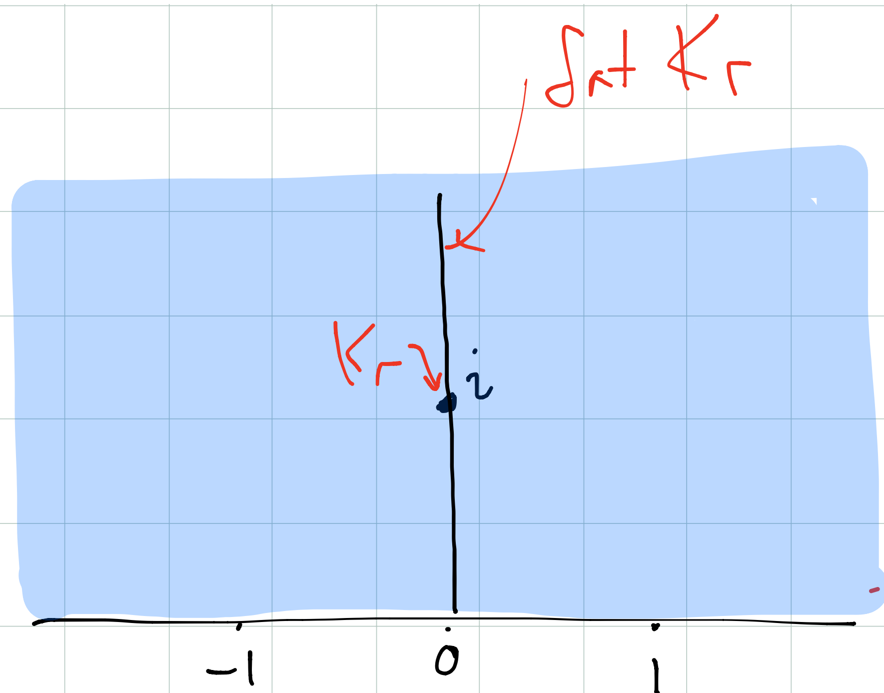
$S_\Gamma = T^n$ is the only Γ-complex.

$K_\Gamma =$ point.

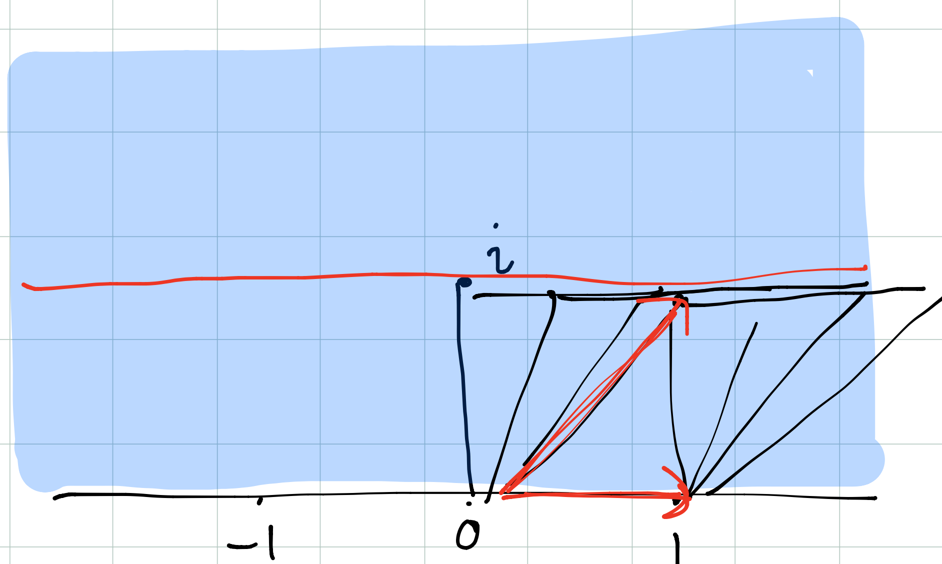
If we want to equip S_Γ with a metric, we
construct S_Γ from a unit cube

So for $n=2$ $K_\Gamma \longleftrightarrow i \in \mathbb{H}$

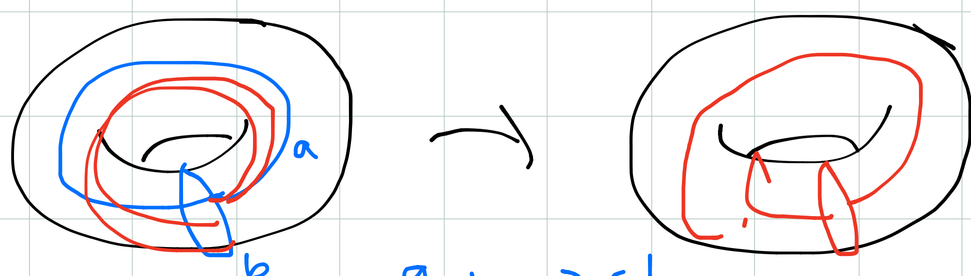
Replacing cubes by rectangles
"fattens" K_Γ to the line $(\mathbb{R}_{>0}) \cdot i$



To move horizontally in $\mathbb{H}$ we need to shear



On the torus this has the effect of doing a Dehn twist



$a \mapsto ab$

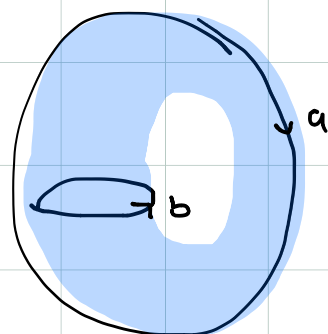
$b \mapsto b$

(hence the name "twist")

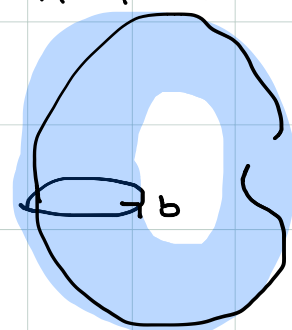
Outer space for $T(A_\Gamma)$, arbitrary Γ?

How should $\tau : a \mapsto ab = ba$ affect the Salvetti S_Γ?

ab=ba so there is an ab-torus in S_Γ

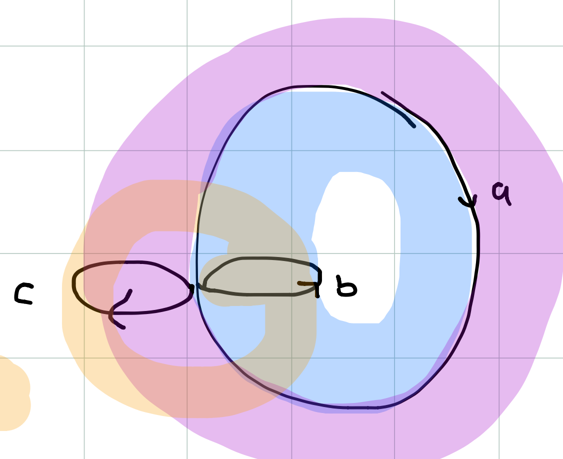


We want to shear it to achieve a Dehn twist

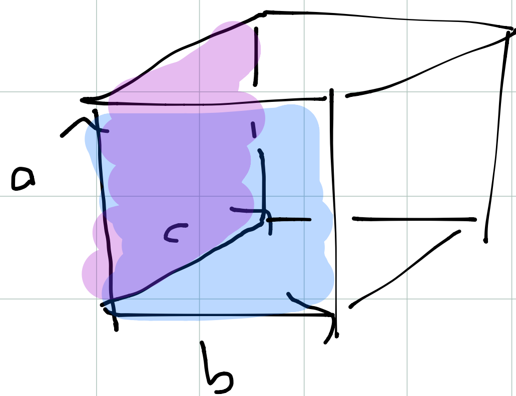


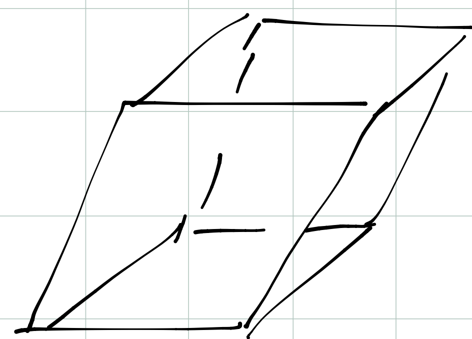
But this torus is attached to others

Can I twist it?



$[c,a]=1 \Rightarrow [c,b]=1$





So can extend an ab-shear over the whole cube

Get a space of Salvettis with (marked) sheared metrics — contractible with proper $T(A_\Gamma)$-action?

Need to understand $T(A_\Gamma)$ better

Abelianization $A_\Gamma \to \mathbb{Z}^n$ induces a natural map

$$\mathrm{Out}(A_\Gamma) \longrightarrow \mathrm{Out}(\mathbb{Z}^n) = GL_n\mathbb{Z}.$$

$$\rho_{ij} \longmapsto E_{ij}$$

<u>Lemma</u> $T(A_\Gamma) \hookrightarrow GL_n\mathbb{Z}$

In fact, we can identify the image:

In Γ, write $v < w$ if $\mathrm{lk}\, v \subseteq \mathrm{st}\, w$

ie $\mathrm{lk}\, v \subset \mathrm{lk}\, w$ or $\mathrm{st}\, v \subseteq \mathrm{st}\, w$

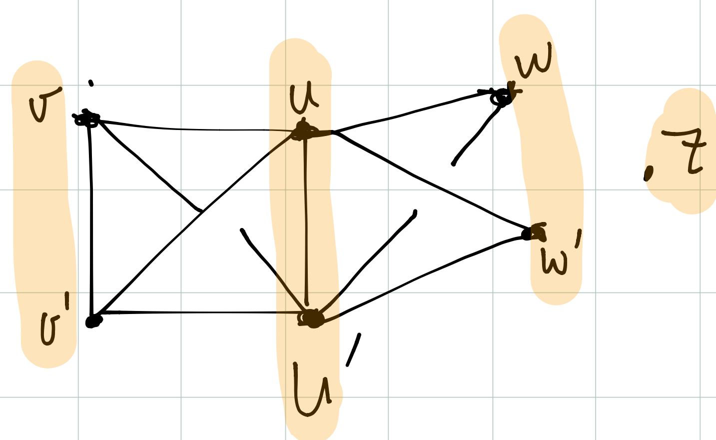
and $v \sim w$ if $v < w$ and $w < v$

ie if $\mathrm{lk}\, v = \mathrm{lk}\, w$ or $\mathrm{st}\, v = \mathrm{st}\, w$

Exercise: $\sim$ is an equivalence relation and $\prec$ makes the set of equivalence classes into a poset.

eg

$\Gamma =$



Now totally order the vertices subordinate to $\prec$

$v_i \leq v_j \Leftarrow v_i \prec v_j$ and write the elements of $T(A_\Gamma)$ as matrices in the basis $v_1, \dots v_n$

u u' v v' w w' z

$$\begin{array}{c} u \\ u' \\ v \\ v' \\ w \\ w' \\ z \end{array}
\left[\begin{array}{cc|cc|cc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{$SL_2\mathbb{Z}$}} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \multirow{2}{*}{0} \\
 & & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \\
\hline
\multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multicolumn{2}{c|}{\multirow{2}{*}{$SL_2\mathbb{Z}$}} & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multirow{2}{*}{0} \\
 & & & & & & \\
\hline
\multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 & \multirow{2}{*}{0} \\
 & & & & 0 & 1 & \\
\hline
\multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 1
\end{array}\right]$$

Image of $T(A_r)$ is block upper triangular

ie $\subseteq$ parabolic subgroup P of $SL_n\mathbb{R}$

Recall $SO_n \backslash SL_n\mathbb{R} = P \cap SO_n \backslash P$

Let $T_{\mathbb{R}}(A_r)$ be generated by the E_{ij}^r, $r \in \mathbb{R}$ for $E_{ij} \in T(A_r)$.

Set $\mathbb{D}_\Gamma = T_{\mathbb{R}}(A_\Gamma) \cap SO_n \backslash T_{\mathbb{R}}(A_\Gamma)$

<u>Proposition</u>
$\mathbb{D}_\Gamma$ is contractible, $T(A_\Gamma)$ acts properly

$\mathbb{D}_\Gamma \leftrightarrow$ marked metrics on S_Γ which can be obtained from the cube metric by shearing tori.

points = "twisted Salvettis"

Recap: We have Outer spaces of marked metric objects for $T(A_\Gamma)$ $\mathbb{D}_\Gamma$ and for $U(A_\Gamma)$ K_Γ

Would now like to form a hybrid space on which all of $Out(A_\Gamma)$ acts properly

For simplicity, ignore inversions and graph automorphisms

If twists always commute with folds and partial conjugations, can just take $K_\Gamma \times \mathbb{D}_\Gamma$ with the product action.

This is not always the case

$\tau : a \mapsto ab = ba$ a twist

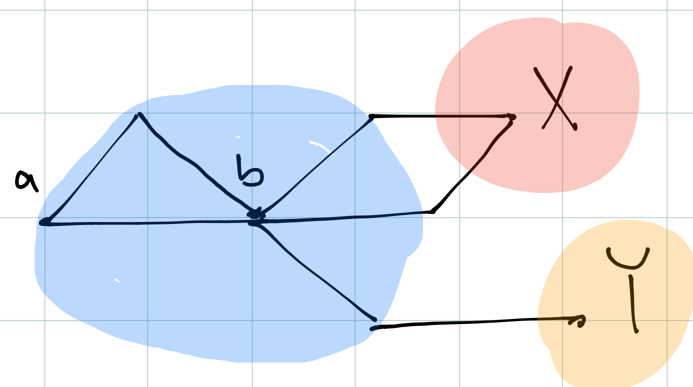
τ commutes with partial conjugations γ

<u>unless</u> $\gamma = \gamma_a : X \mapsto aXa^{-1}$

$$\begin{cases} a \xmapsto{\tau} ab \xmapsto{\gamma_a} ab \\ X \longmapsto X \longmapsto aXa^{-1} \end{cases}$$

$\gamma_a \tau = \tau \gamma_a \gamma_b$

$$\begin{cases} a \xmapsto{\gamma_a} a \xmapsto{\tau} ab \\ X \longmapsto aXa^{-1} \longmapsto abXb^{-1}a^{-1} \end{cases}$$



τ commutes with folds φ

unless $\varphi = \varphi_{ca} : c \mapsto ca \neq ac$

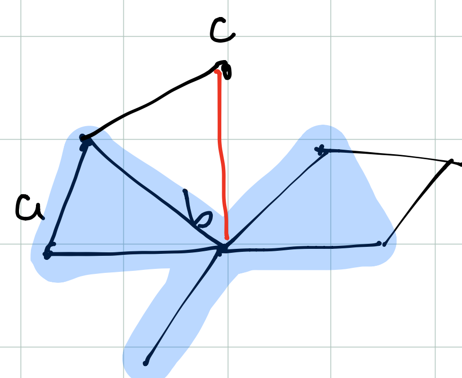
$$\begin{cases} a \xmapsto{\tau} ab \xmapsto{\varphi} ab \xrightarrow{\alpha} ab \\ c \mapsto c \mapsto ca \longrightarrow cba = cab \end{cases}$$

$\varphi\tau = \tau\varphi\alpha$

$$\begin{cases} a \xmapsto{\varphi} a \xmapsto{\tau} ab \\ c \mapsto ca \mapsto cab \end{cases}$$

$\alpha : c \mapsto cb$

fold or twist



Proposed Outer space $\mathcal{O}_\Gamma$

Point in $\mathcal{O}_\Gamma = (X, g)$

X = flat Γ-complex

$g : S_\Gamma \to X$ a homotopy equivalence

Observe X contains an embedded torus for each torus in S_Γ. Metric on X induces metrics on these tori

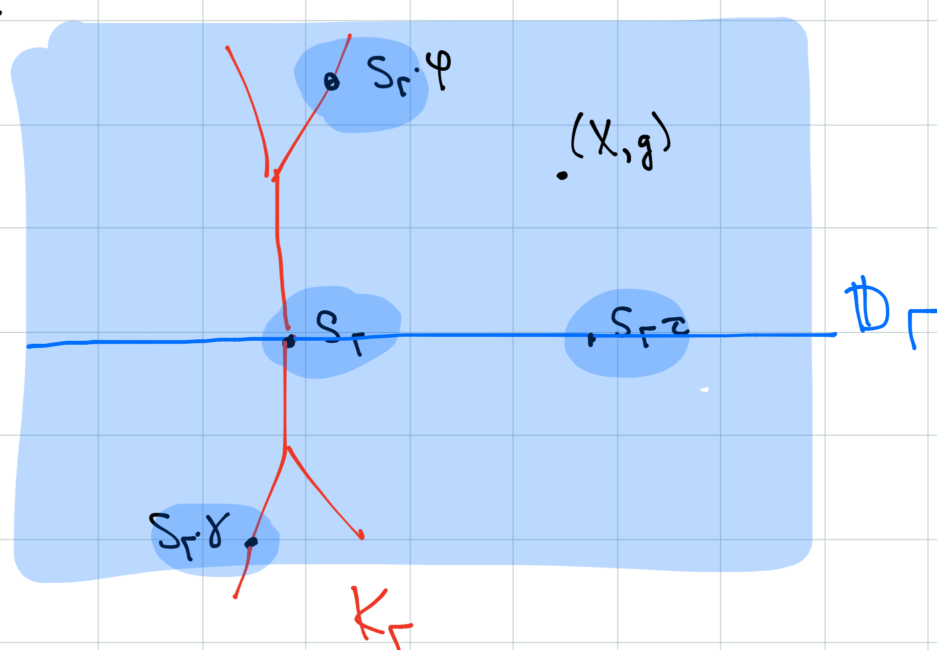
A metric on X is flat if these correspond to some twisted metric on S_Γ.

$\varphi \in \mathrm{Out}(A_\Gamma)$ <u>acts</u> by:

Realize φ by $f : S_\Gamma \to S_\Gamma$. Then

$$(X, g)\cdot\varphi = (X, g\circ f)$$

<u>Picture</u>



Prop There is a projection $\mathcal{O}_\Gamma \longrightarrow \mathbb{D}_\Gamma$

$g: S_\Gamma \longrightarrow X \quad \leftrightarrow$ action of $A_\Gamma = \pi_1 S_\Gamma$

on $\tilde{X} = CAT(0)$

Δ = clique $\Rightarrow A_\Delta \subseteq A_\Gamma$ is free abelian

has min set $M_\Delta \subseteq \tilde{X}$

$M_\Delta \cong \mathbb{R}^d \Rightarrow$ get marked flat torus

$(X, d) \longmapsto (S_\Gamma, d)$ = twisted S_Γ inducing same markings on the $S_\Delta \subset S_\Gamma$

Questions

1. Is $\mathcal{O}_\Gamma$ contractible?

2. Find a cocompact spine for $\mathcal{O}_\Gamma$

   idea: adapt Ash's well-rounded retract of $SO_n \backslash SL_n\mathbb{R}$

   Is the dimension $=$ VCD$(\mathrm{Out}(A_\Gamma))$?

3. Is the fixed point set of a finite subgroup contractible?

   Is it even non-empty?

4. Is $\mathrm{Out}(A_\Gamma)$ a virtual duality group?

   Is there a Borel-Serre / Bestvina-Feighn bordification?

5. Is there a good metric theory of $\mathcal{O}_\Gamma$ generalizing the Lipschitz metric on CV_n and classical metric on $SO\backslash SL$

6. Is the simplicial closure of the "fattened" K_Γ Gromov hyperbolic?
Is there a natural complex associated to all of $\mathcal{O}_\Gamma$ which is Gromov hyperbolic?