Outer space and Automorphisms of free groups

LECTURE 4

Last time we defined Outer space CV_n in three different ways - as a space of

1. actions on trees
2. Marked graphs
3. Sphere systems in $M_n = \#_n S^1 \times S^2$

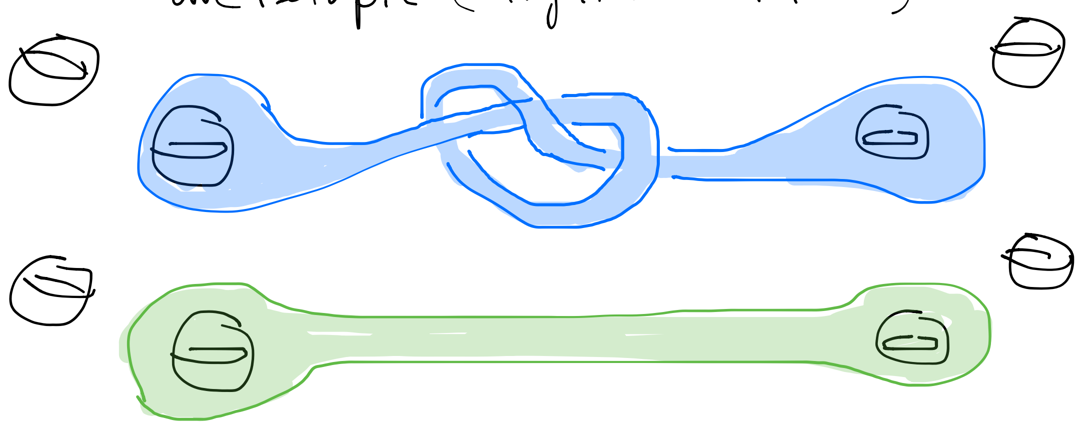
Today will prove CV_n is contractible.
There are proofs using all three models.

All proofs have a common idea: fix a point $X_0 \in CV_n$ then retract all of CV_n to X_0 by following paths which reduce some measure of complexity.

One of the most natural - due to Hatcher - uses the sphere system model

This requires some non-trivial results from 3-manifold theory, based ultimately on Laudenbach's theorem that in $M_n = \#_n S^1 \times S^2$ homotopic sets of embedded 2-spheres are isotopic

Eg: the green and blue spheres below are isotopic ("light bulb trick")

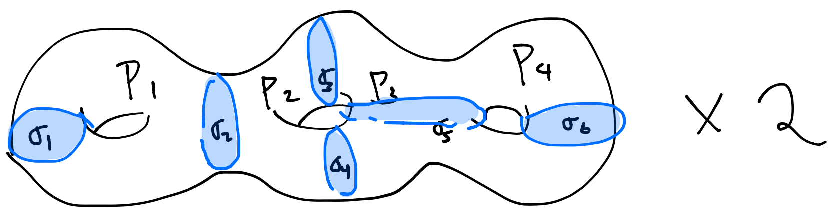


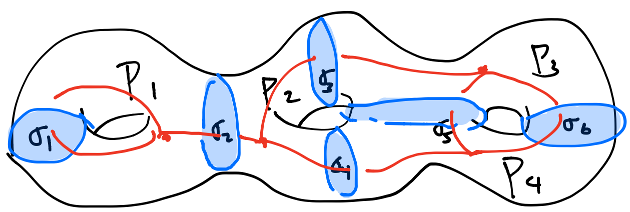
It's not so obvious when an entire sphere system is knotted and linked!

Fix $\Sigma = \{\sigma_1, \dots, \sigma_{3n-3}\}$ a <u>maximal</u> sphere system in M_n.

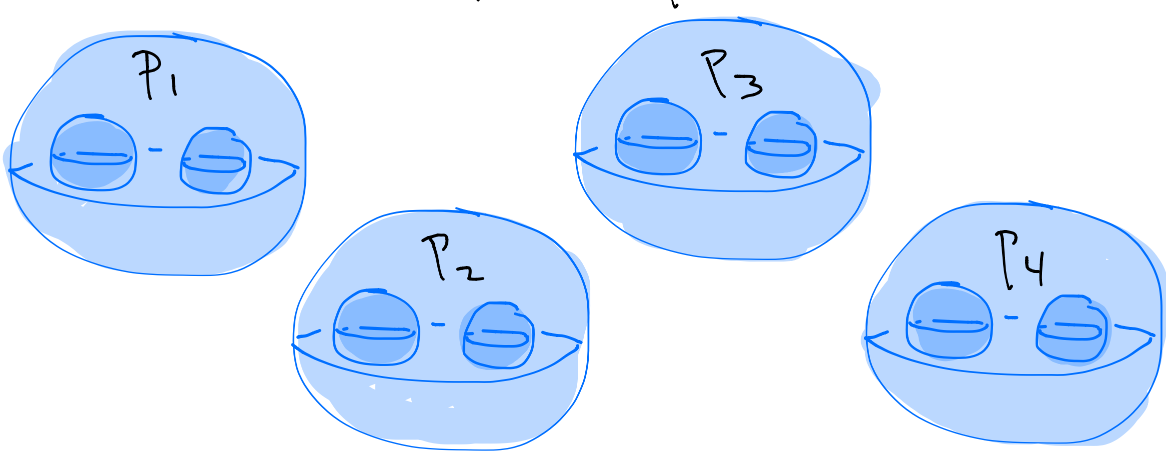
($\leftrightarrow$ trivalent marked graph (g, G))

<u>eg</u>:





Σ cuts M_n into 3-punctured spheres: $P_i = S^3 \setminus (B_1 \cup B_2 \cup B_3)$



So Σ is <u>simple</u>, gives simplex $|\Sigma|$ in CV_n

(Recall open <u>simplex</u> in $CV_n \leftrightarrow$ simple sphere system

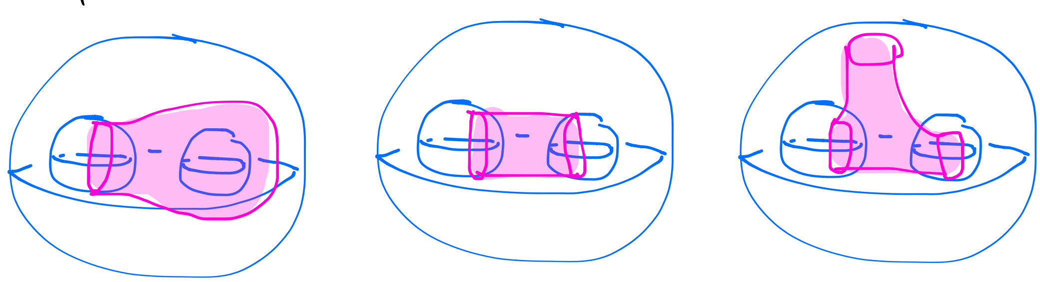
face $\leftrightarrow$ simple sub-system)

We will retract all of $S(M)$ to this simplex $|\Sigma|$

Let $\mathcal{S}$ be another sphere system $= \{s_1, \dots, s_k\}$

$\mathcal{S}$ is in <u>normal form</u> wrt Σ if

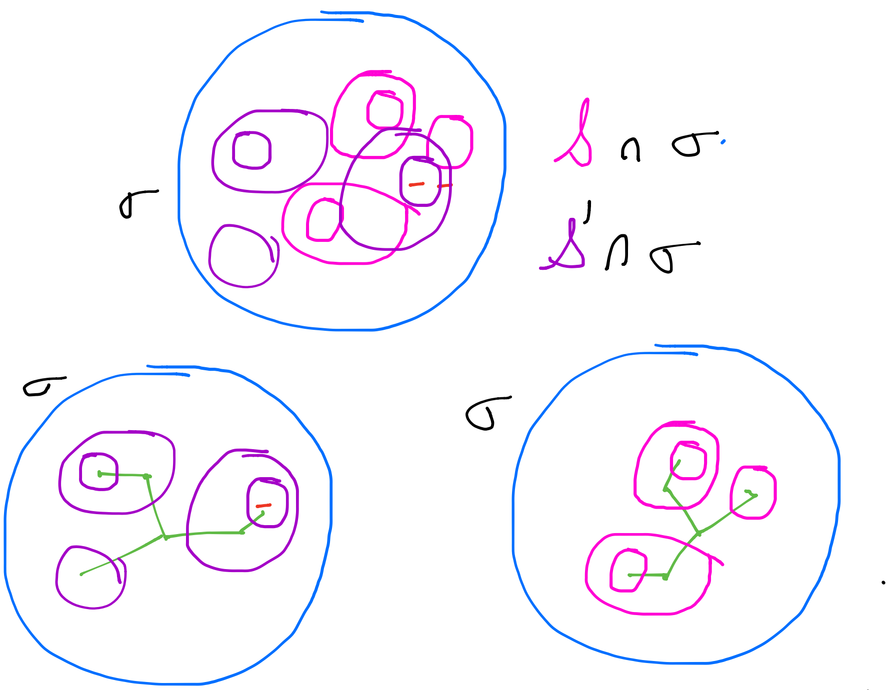
- Each $s \in \mathcal{S}$ intersects Σ transversely in a finite number of circles, which cut s into pieces
- Each piece is a disc, cylinder or pair of pants, w/ at most one ∂ circle on each component of ∂P_i:



Hatcher's normal form theorem

1. Every sphere system is isotopic to a sphere system in normal form with respect to Σ.
2. If $\mathcal{S}$ and $\mathcal{S}'$ are both in normal form and are isotopic, then they are isotopic by an

isotopy which preserves the pattern of intersection circles on each $\sigma \in \Sigma$.



(The pattern is encoded in the dual tree)

The proof relies heavily on Laudenbach's theorem

Prop $\mathcal{S}$ is in normal form wrt Σ if and only if $\mathcal{S}$ is transverse to Σ and the number of intersection circles $\mathcal{S} \cap \Sigma$ is minimal.

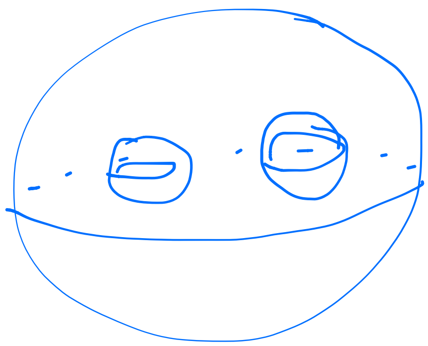
(If it's not in normal form, you can reduce the # of intersection circles.)

A point in $S(M_n)$ is given by barycentric coords = weights on the spheres $s_i \in \mathcal{S}$

We want a path from a weighted sphere system $\mathcal{S}$ to a point of $|\Sigma|$.

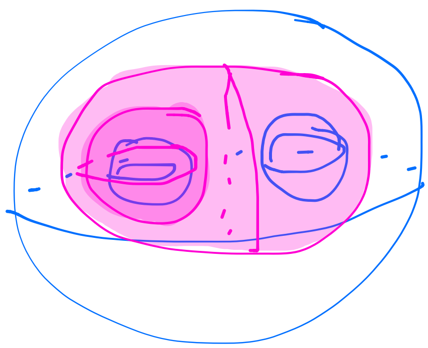
Idea: Do (weighted) surgery on $\mathcal{S}$ using innermost disks on spheres in Σ to eliminate intersection circles. First describe unweighted surgery:

If $\mathcal{S}'$ is disjoint from Σ then $\mathcal{S}'$ is a subset of Σ

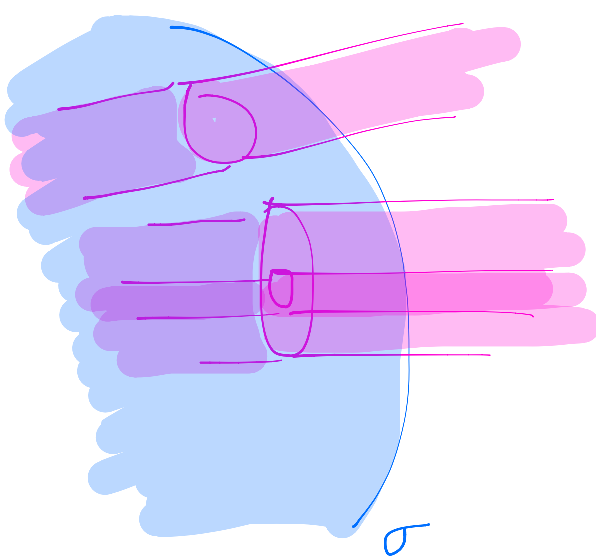


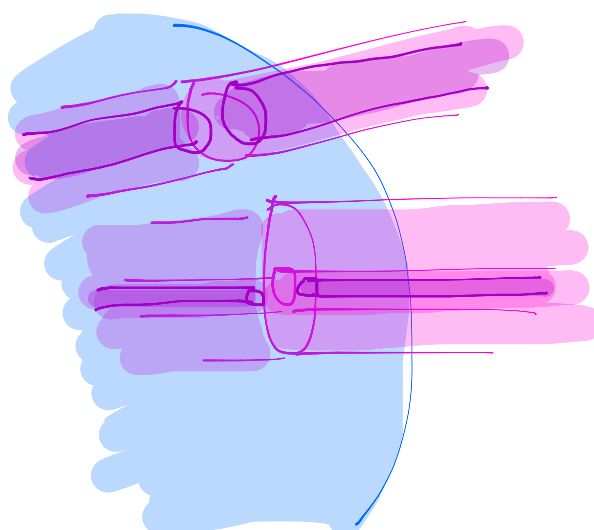
← any sphere in here

is parallel to one of the boundary spheres:

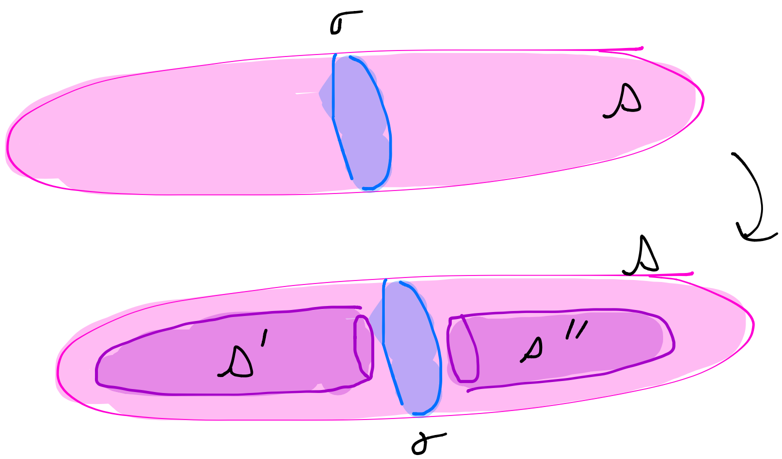


If $\mathcal{S}$ is <u>not</u> disjoint from Σ, do surgery

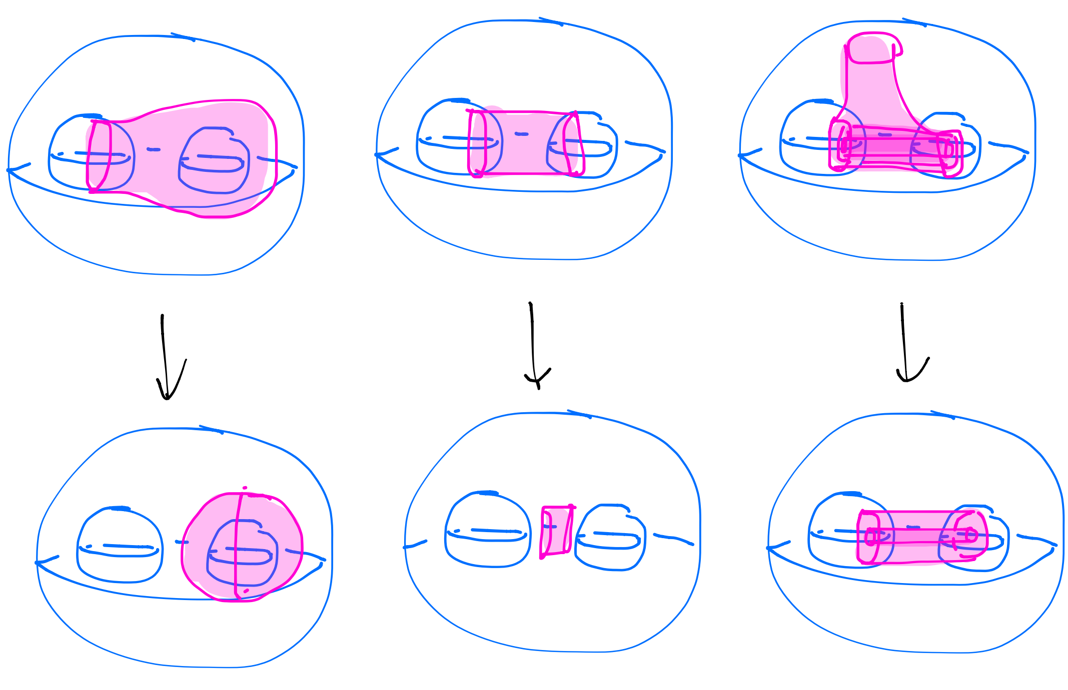




Another view of a single surgery:



What this does to the pieces of S:



Each disk piece becomes ∂-parallel
Each cylinder piece becomes trivial
A pants piece can become a cylinder or trivial.

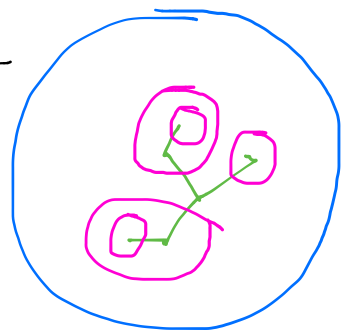
Discard trivial and boundary spheres. The resulting system is still in normal form!

To describe a path from $\mathcal{S}$ to $\mathcal{S}'$: transfer the weight of $\mathcal{S}$ to $\mathcal{S}'$

To get a path from $\mathcal{S}$ to $|\Sigma|$: do surgery on all innermost disks simultaneously, transfer weight to new spheres gradually, discard spheres when they have weight zero
If there's only one circle on σ, use both innermost disks (weight will $\to 0$ twice as fast).

Can see what is happening by looking at the dual trees in the σ_i:

σ



length of edge = weight of dual circle.

Innermost disks $\leftrightarrow$ leaves

As circle disappears, leaves grow shorter, then disappear. Process recommences with remaining (or new) leaves.

This is a nice intuitive picture, but there is work to be done: need to show deformation along path $\Delta \longrightarrow |\Sigma|$ is continuous.

Hatcher uses the dual trees to show that process gives a piecewise linear flow on $S(M_n)$

Contraction of CV_n:

We have $S(M_n)$ = sphere complex

= simplicial closure CV_n^*

simplex in $S(M) \leftarrow$ sphere system

point in $S(M_n)$ = sphere system w/ barycentric coords

= weighted sphere system

$CV_n \subset S(M)$ = weighted simple sphere systems

Exercise: If $\mathcal{S}$ is simple, then so is
$\mathcal{S}' = \mathcal{S} \setminus \{s\} \cup \{s', s''\}$ (Van Kampen's Thm)

So Hatcher's retraction restricts to a retraction of CV_n to a point.

Next: The spine of CV_n

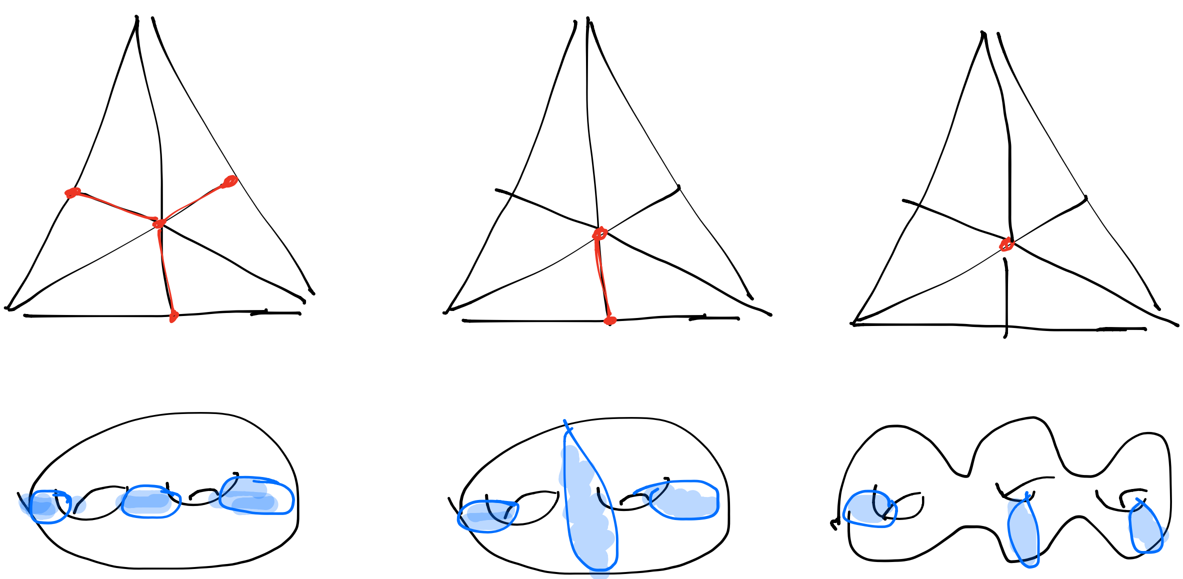
$S'(M_n)$ = barycentric subdivision of $S(M_n)$

vertices = sphere systems

k-simplex = chain $\delta_0 \subset \delta_1 \subset \cdots \subset \delta_k$

Definition $K_n \subset S'(M_n)$ is the subcomplex spanned by simple sphere systems.

= spine of Outer space



Out(F_n) acts on K_n, since diffeomorphisms preserve the property of being simple

Furthermore, CV_n retracts onto K_n:

$S = \{s_0, \dots, s_k\}$ a simple sphere system

$x = a_0 s_0 + \dots + a_k s_k$ = point in CV_n

is in $\{s_0\} \subset \{s_0 s_1\} \subset \dots \subset \{s_0 \dots s_k\} = S$ some simplex of $S'(M_n)$

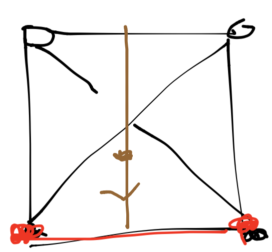
$$S_0 \subset S_1 \subset \dots S_i \subset \dots \subset S_k = S$$

$$x = b_0 S_0 + b_1 S_1 + \dots + b_k S_k$$

If S_i is the first simple system, move x to the point

$b_i S_i + \dots + b_k S_k$ (normalized so $\sum b_j = 1$)

by uniformly shrinking $b_1, \dots, b_{i-1}$ and expanding $b_i, \dots, b_k$



Exercise: The retractions on an open simplex in CV_n extends continuously to a face which is in CV_n.

Corollary: K_n is contractible

Proposition: $\dim(K_n) = 2n-3$

pf: A simple sphere system needs at least n spheres and can have at most $3n-3$

So the longest chain has length $2n-2$

So the largest simplex in K_n has dimension $2n-3$.

The stabilizer of a (weighted) sphere system & permutes the spheres. Since an orientation-preserving diffeomorphism of a punctured sphere which fixes the boundary is isotopic to the identity, the stabilizer is finite. ▲

The quotient is compact:

Up to diffeomorphism there are only finitely many sphere systems in M_n (there are only finitely many ways to glue together punctured balls to obtain M_n, by X.

We now have a contractible, $(2n-3)$-dimensional simplicial complex K_n on which $Out(F_n)$ acts with finite stabilizers.

Algebraic Consequences:

Thm (Hurewicz) X = CW-complex, $G = \pi_1 X$.
If $\tilde{X}$ is contractible, then $H^*(X)$ is an invariant of G.

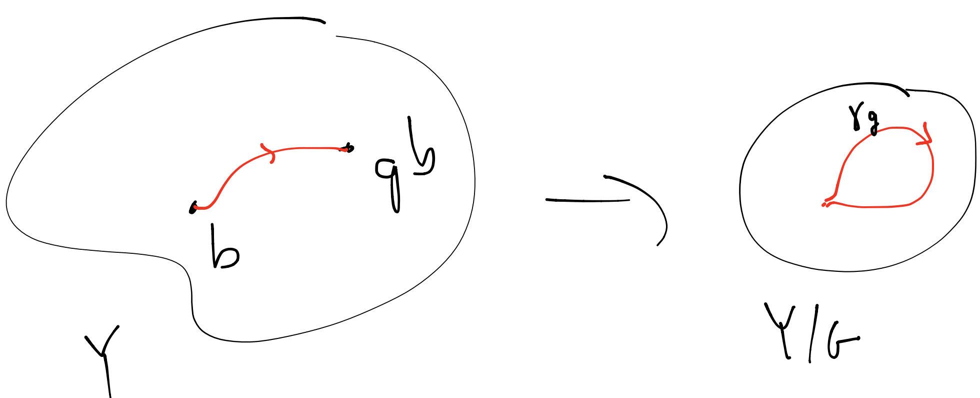
This is one way to define the cohomology of a group. X is called a $K(G,1)$.

There are also purely algebraic definitions – see Ken Brown's excellent book, Cohomology of Groups.

How to find such an X?

Note $\pi_1 X$ acts freely on $\tilde{X}$

Conversely, if G acts freely on a contractible Y, then the map $G \longrightarrow \pi_1(Y/G)$



is an isomorphism. (Algebraic topology)

We have K_n contractible, $\mathrm{Out}(F_n)$ acting,
but the action is not free, stabilizers are finite

Thm: (Baumslag-Taylor) $\mathrm{Out}(F_n)$ has finite-index subgroups with no torsion

Let $\Gamma < \mathrm{Out}(F_n)$ be such a subgroup
Γ acts on K_n, too, and contains no stabilizers,
so Γ acts freely!

so $H^*(K_n/\Gamma) = H^*(\Gamma)$

We now get:

Thm $H^i(\Gamma) = 0$ for $i > 2n-3$ $(= \dim K_n)$

Turns out the choice of Γ is irrelevant for this:

Thm (Serre) Γ, Γ' torsion-free finite index subgroups of $G \Rightarrow$ cohomological dimension of Γ = cohomological dimension of Γ'.

Called the <u>virtual</u> <u>cohomological</u> <u>dimension</u> of G

Also, $K_n / \mathrm{Out}(F_n)$ is a finite CW complex (finitely many cells) so K_n/Γ has the same property (Γ is finite index) and we get

Thm $H^i(\Gamma)$ is finitely generated for all i.