Outer space and Automorphisms of free groups

LECTURE 5

Last time we defined the spine of Outer space and used it to get algebraic info about Out F_n (cohomology)

Today – take a closer look at the spine

K_n = spine of outer space
= subcomplex of barycentric subdivision $S'(M_n)$ of the sphere complex spanned by simple systems.

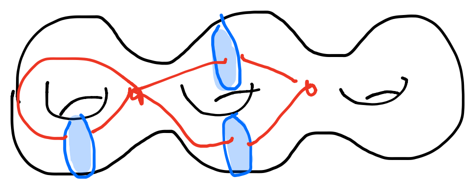
- contractible
- $\dim = 2n-3$
- $Out(F_n)$ acts properly, simplicially
- Quotient is compact

In particular, $Out(F_n)$ is quasi-isometric to K_n

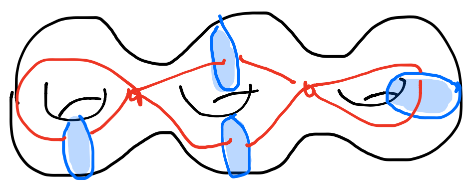
So q.i. invariants of K_n are q.i. invariants of $Out(F_n)$

Description of K_n in terms of marked graphs

sphere system $\mathcal{S} = \{s_0, \dots, s_k\} \longleftrightarrow$ dual graph G



$\mathcal{S}$ is simple $\iff$ $G \hookrightarrow M$ induces $\pi_1 G \xrightarrow{\cong} \pi_1 M$



So if we identify $\pi_1 M \equiv F_n$ the embedding provides a <u>marking</u>, ie $\mathcal{S} \leftrightarrow (g, G)$ = marked graph

(Can also make the marking go the other direction)

Edges in K_n are inclusions $\delta_0 \subseteq \delta_1$

Digression: poset terminology

X a simplicial complex X' its barycentric subdivision is an example of a geometric realization of a partially ordered set ($\equiv$ poset)

P a poset $\rightsquigarrow$ $|P|$ is a simplicial complex

with vertices = elements x of P

k-simplices = chains $x_0 \leq x_1 \leq \dots \leq x_k$

X' = realization of poset of simplices ordered by inclusion.

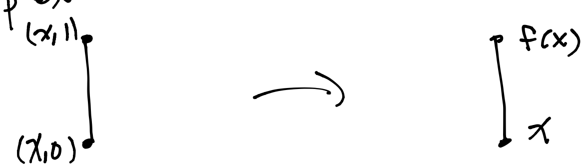
A poset map $f: X \to Y$ is a map respecting the partial ordering, ie $x \leq x' \Rightarrow f(x) \leq f(x')$

Useful Lemma $f: X \to X$ a poset map with $f(x) \geq x$ for all x. Then $|X| \simeq |f(X)|$.

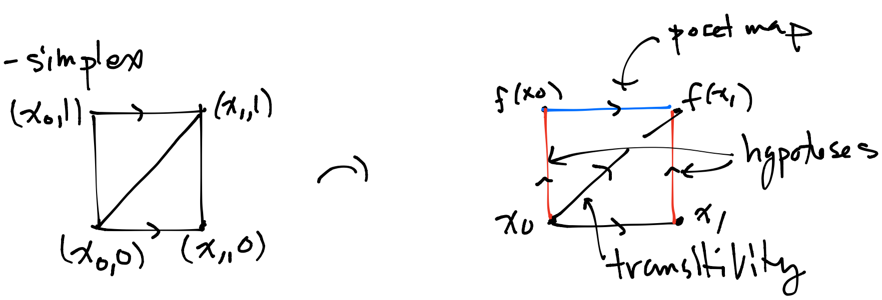
Proof: Want $H: |X| \times I \to |X|$ with $H(x,0) = x$, $H(x,1) = f(x)$.

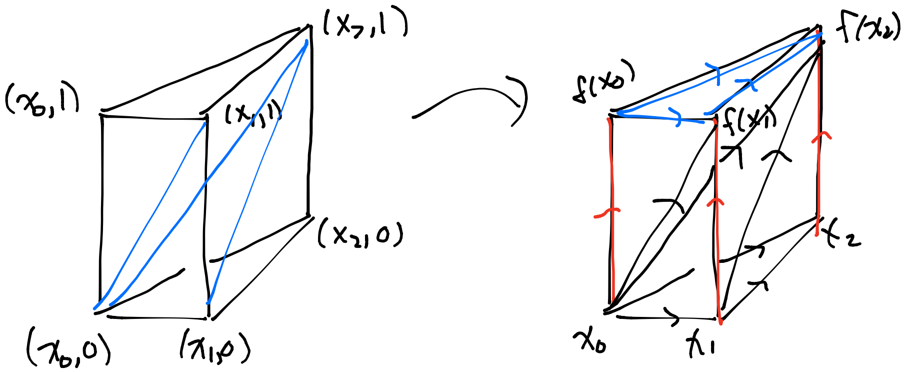
Use "prism operator" to triangulate $\sigma \times I$, σ a simplex

0-simplex



1-simplex



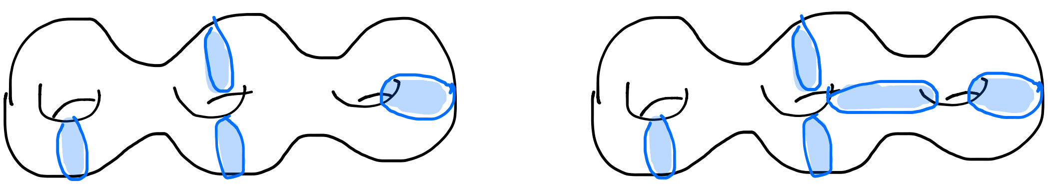


Note you can replace $f(x) \geq x$ with $f(x) \leq x$

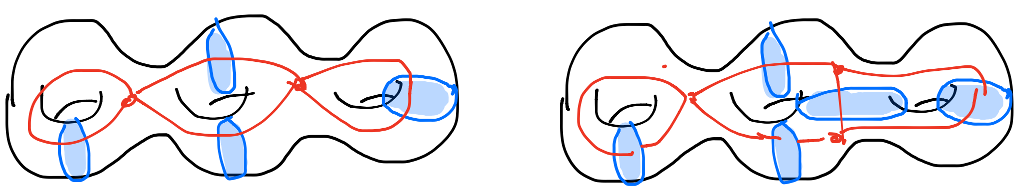
(but the picture is upside down...)

Back to spheres and graphs

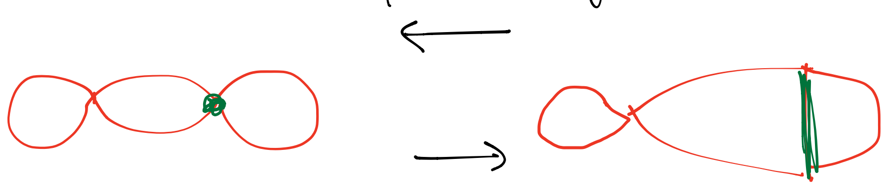
edge in K_n is inclusion $\mathcal{S}_0 \subset \mathcal{S}_1$



effect on dual graphs:



collapse an edge



expand a vertex into 2 vertices + edge

you can collapse many edges as long as they don't contain a cycle

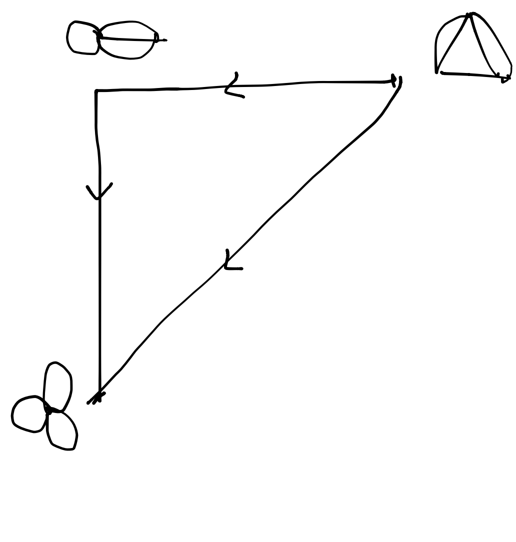
Def a _forest_ is a subgraph with no cycles

A _Forest collapse_ collapses each edge in the forest to a point (= vertex)

Graphical def'n of K_n: Geometric realization of poset of marked graphs (G,g), where $(G,g) \geq (G',g')$ if $\exists$ forest collapse $G \to G'$ making diagram commute:

$$\begin{tikzcd} G \arrow[r] & G' \\ R_n \arrow[u, "g"] \arrow[ur, "c \circ g"'] & \end{tikzcd}$$

Example of a 2-simplex



Local structure of K_n:

<u>Link of a vertex δ</u> :

Two types of edges connected to δ

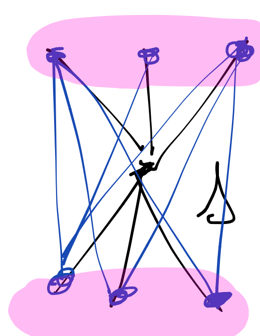
$\delta' < \delta$ $\qquad\qquad$ $\delta'' \supset \delta$

"lower link" $\qquad$ "upper link"

Everything in lower link is connected to everything in upper link, so

The link is the <u>simplicial join</u> of lower and upper links

$$lk(\delta) = lk_{<\delta} * lk_{>\delta}$$

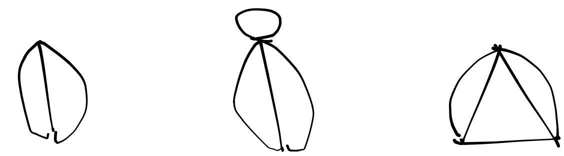


$lk_{>\delta}$

$lk_{<\delta}$

$\underline{\mathrm{lk}_{<\delta}}$. Use graph description of K_n

<u>examples</u>



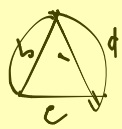
$G =$ 

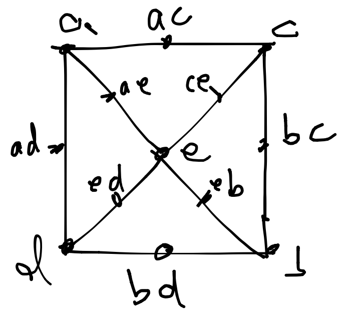
$\mathrm{lk}_{<G} =$ 

$\simeq$ a b c

$G =$ 

$\mathrm{lk}_{<G} =$ a b c

$G =$ 

$\mathrm{lk}_{<G} =$ 

In general, $\mathrm{lk}_{<\delta} = \begin{cases} \text{geom. realization of poset} \\ F(G) \text{ of forests in} \\ \text{the dual graph } G. \end{cases}$

Thm If G is connected then $|F(G)|$ is either contractible or homotopy equivalent to a wedge of spheres $\bigvee S^{V-2}$, where $V = \#$ vertices of G. $|F(G)|$ is contractible iff G has a separating edge.

Proof. Induct on $V+E$. (E = # edges of G)

Let e be an edge of G

If e is a loop, then $F(G) = F(G-e)$, so done by induction

If e is a bridge (= separating edge) then you can add e to any forest Φ, so

$\phi \mapsto \Phi \cup e \to e$ are poset maps

contracting $|F(G)|$ to $|e|$

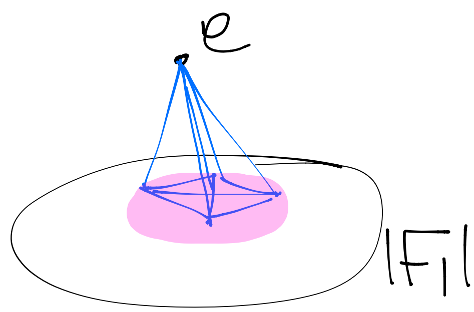
(So we may assume G has no bridges)

Then $G-e$ is connected so Thm is true for $G-e$.

Let $F_1 \subset F(G)$ be all forests <u>except</u> $\{e\}$

$F_1 \longrightarrow F(G-e)$ is a poset map giving

$$\phi \longmapsto \begin{cases} \phi & e \notin \phi \\ \phi - e & e \in \phi \end{cases} \qquad F_1 \simeq F(G-e) \simeq \bigvee S^{V-2} \text{ or } *$$



e is the only vertex of $F(G)$ <u>not</u> in F_1

$lk(e)$ = forests containing $e \cong F(G_e)$

$\simeq \bigvee S^{V-3}$ by induction.

(G_e has no bridges since G has no bridges.)

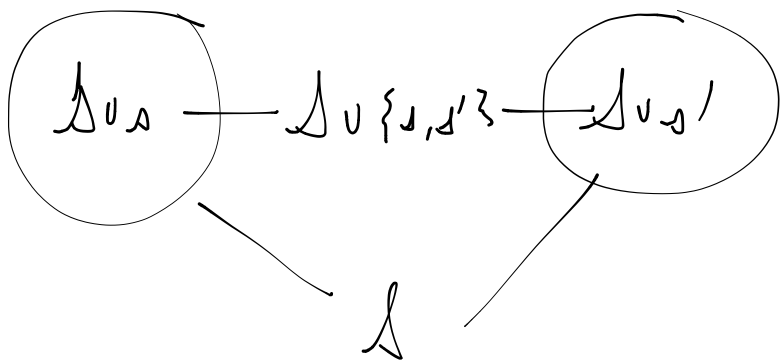
So $|F(G)| \simeq c(\bigvee S^{V-3}) \underset{\bigvee S^{V-3}}{\cup} (\bigvee S^{V-2} \text{ or } *)$

$\simeq \bigvee S^{V-2}$ ✓

What about $lk_{>\sigma}$?

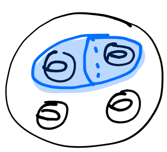
You can add spheres to any component of $M \setminus \sigma$ independently, you still get a sphere system

so $lk_{>\sigma} = * S'(P)$, P = component of $M \setminus \sigma$



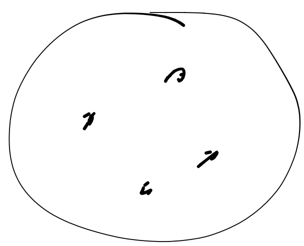
Complex of sphere systems in P

$$= S'(P) \cong S(P)$$



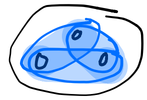
P

new sphere is determined up to isotopy by how it separates the old spheres

Draw spheres as dots 

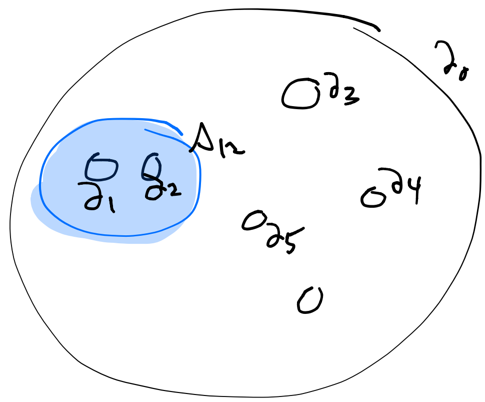
Prop $S(P) \simeq \bigvee S^{B-4}$ $\quad B = \#$ bdry components of P

pf $B = 3 \Rightarrow$ can't add spheres $\Rightarrow S(P) = \emptyset$

$B = 4 \Rightarrow$  3 spheres, none compatible

$$\Rightarrow S(P) \simeq \bigvee S^0$$

$B>4$. Choose Δ_{12} cutting off 2 spheres ∂_1 and ∂_2



$S_1 \subset S(P) = \langle$spheres compatible with $\Delta_0\rangle$

$\equiv$ cone on Δ_{12}

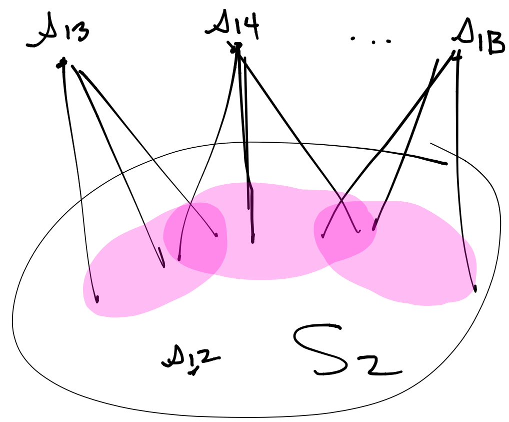
(Not in S_1: spheres separating ∂_1 from ∂_2)

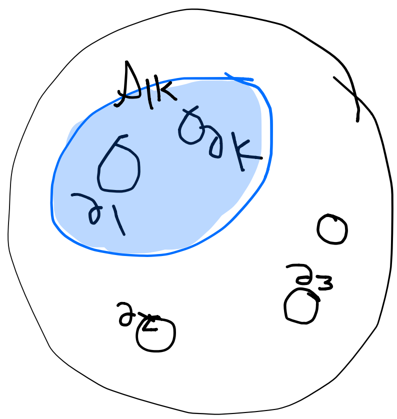
$S_2 =$ all spheres <u>except</u> $\Delta_{13}, \Delta_{14}, \ldots, \Delta_{1B}$

<u>Claim</u> S_2 deformation retracts to S_1 (so is contractible)

If we can prove that, we're done by induction:

$$S(P) \simeq \bigvee_{k\geq 3} \mathrm{Susp}(\mathrm{lk}\, \partial_1\partial_k)$$





$\mathrm{lk}(\Delta_{1k}) = S(P')$

P' has $B-1$ ∂-components

$\Rightarrow S(P') \simeq \bigvee S^{B-5}$

So $$S(P) \simeq \bigvee_{k\geq 3} \mathrm{Susp}(\mathrm{lk}\, \Delta_{1k}) \simeq \bigvee \mathrm{Susp}(\bigvee S^{B-5})$$
$$\simeq \bigvee S^{B-4} \checkmark$$

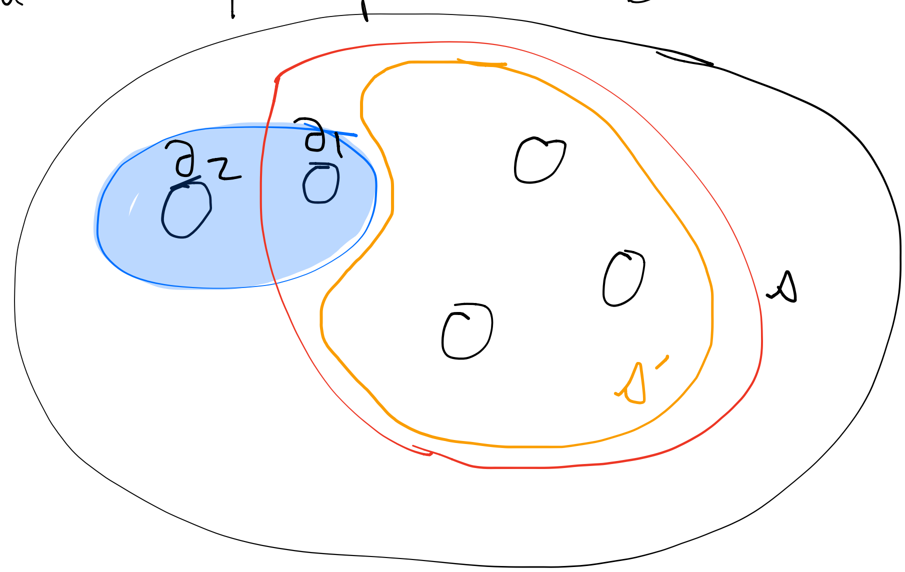
To contract $S_2 \searrow S_1$, define

<u>inside</u> of Δ = side containing ∂_1

<u>size</u> of Δ = # of ∂ components on inside

Add spheres to S_1 in order of decreasing size

Add spheres of size $B-2$



Push σ across ∂_1 to obtain σ'

Check: $lk\, \sigma \cap S_1$ = cone on $\sigma' \simeq pt$

Exercise: What happens next? (Finish the proof)

Note I "un-baricycentrically subdivided" $S'(P)$ in the above proof.

Theorem is also true if you don't allow spheres which separate M ($\leftrightarrow$ separating edges of G) but is harder to prove.

Application: $\varphi \in Out(F_n)$ gives a simplicial automorphism of K_n

Thm The map $Out(F_n) \longrightarrow Aut(K_n)$ is an isomorphism

The first step is showing any simplicial automorphism of K_n must send vertices (g, G) to (g', G') with $G' \cong G$