Introduction to graph homology

= homology of a "graph complex"

What *is* a graph complex?
Short answer: Chain complex with a simple combinatorial description in terms of finite graphs

Why study them?
Short answer: Many mathematical objects have some structure that can be described in terms of graphs, so many problems can be reduced to questions about a graph complex

Now for some slightly longer answers:

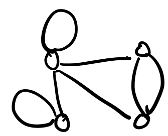
Graph homology was introduced by Kontsevich in 2 papers

1. Formal (non-)commutative symplectic geometry (1993)
2. Feynman diagrams and low-dimensional topology (1994)

The idea is very simple.

Graph = finite 1-dimensional CW complex G

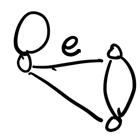
G^0 = vertices G^1 = edges

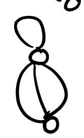


Want to make a chain complex with one generator for each graph G (up to isomorphism)

C_k gen by graphs w/ $k+1$ vertices, of valence ≥ 3

To define $d: C_k \to C_{k-1}$:

Given $e \in G^1$ not a loop $\rightsquigarrow$  G

form G_e by collapsing e $\rightsquigarrow$  G_e

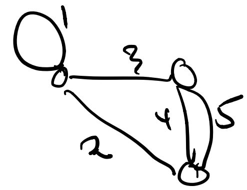
Want $d(G) = \sum_{\substack{e \in G^1 \\ \text{straight}}} G_e$

Problem: in a chain complex, need $d^2 = 0$.

So need a method of orienting graphs, so terms in d^2 cancel in pairs.

Two possibilities:

(1) order the edges of G



("even")

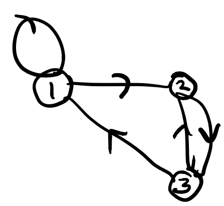
An orientation or is an equivalence class of orderings: two are the same if they differ by an even permutations

Another description: or is a unit vector in the (1-dimensional) vector space

$$\Lambda^{e(G)} \mathbb{R}^{e(G)} := \det \mathbb{R}(G^1) \quad \left(e(G) = |G^1| = \#\text{edges of } G\right)$$

(2) order the vertices of G and orient the edges

("odd")



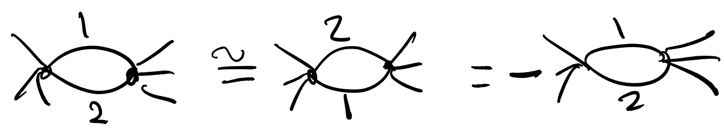
Two orientations are equivalent if they differ by an even number of edge flips and transpositions of vertex labels

ie or is a unit vector in

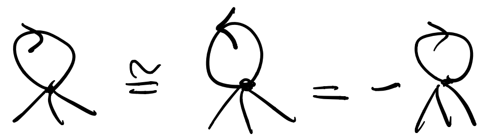
$$\det \mathbb{R}(G^0) \otimes \bigotimes_{e \in G^1} \det \mathbb{R}(H(e)) \quad , \ H(e) = \text{half-edges of } e.$$

In either case C_k is generated by pairs (G, or) modulo the relation $(G, or) = -(G, -or)$

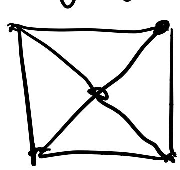
Observations:

even case (1): 

$\Rightarrow$ any graph with a multiple edge $= 0$

odd case (2):  $\Rightarrow$

$\Rightarrow$ any graph w/ a loop $= 0$

Exercise:  $= 0$ for either notion of orientation

To get $d^2 = 0$, need to know how to induce an orientation on G_e, given an orientation on G.

(1) An ordering of edges on G induces an ordering on the edges of G_e, giving
$$(G, or) \longrightarrow (G_e, \overline{or})$$

Then $$d(G,or) = \sum_{\substack{e_i \text{ not} \\ \text{a loop}}}' (-1)^i (G_{e_i}, \overline{or})$$

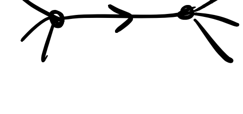
An equivalent description:

Choose a representative of or on G with e the first edge, let $\overline{or}$ be the induced ordering of G_e

Then
$$d(G,or) = \sum_{\substack{e \text{ not a} \\ \text{loop}}} (G_e, \overline{or})$$

(2) This works for the orientation (2) as well:

Given $e \in G$, orient G_e by:

choose rep. of orientation w/ 

(you can do this since G has ≥ 3 edges)
orientation $\overline{or}$ on G_e is given by:
The new vertex is labeled 1
The other vertices have labels reduced by 1. Edges retain their arrows.

Then $$d(G, or) = \sum_{e \text{ not a loop}} (G_e, \overline{or})$$

Lemma: $d^2 = 0$ for either of these definitions
(Exercise for orientation (1))

Some observations:

* If G has no univalent vertices, then vertices in G_e have valence $\geq$ vertices in G

* G connected $\Rightarrow$ G_e connected

* $\chi(G) = \chi(G_e) = v - e$

in particular:

· G connected $\Rightarrow \pi_1 G \cong \pi_1 G_e$ is a free group of rank $1-\chi(G)$

Define G to be admissible if connected and all vertices at least trivalent

Exercise: $\exists$ only fin. many (connected) graphs w/ $\chi = 1-n$, all vertices at least trivalent

$A_* =$ subcplex of C_* gen by admissible graphs

Then $A_* = \bigoplus_{2k=n} A^n_*$, each A^n_k is finite-dimensional

Kontsevich's motivation for studying graph complexes came from physics, specifically deformation quantization.

Digression: Deformation quantization

In classical mechanics, study the phase space M of a system (pts have position and velocity coords)

M is a symplectic manifold

Interested in <u>functions</u> on M (eg energy, aka the Hamiltonian of a system)

The algebra $C^\infty(M)$ of smooth functions $M \xrightarrow{f} \mathbb{R}$ is a commutative algebra with unit ($f(x) \equiv 1$)

It has more structure: Poisson bracket

$$\{f,g\} = \sum_i \partial f/\partial p_i \, \partial g/\partial q_i - \partial f/\partial q_i \, \partial g/\partial p_i$$

satisfies

$$\{f,g\} = -\{g,f\} \quad \text{(anti-symmetry)}$$

and $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$ (Jacobi)

ie it is a Lie algebra

<u>and</u> $D_f = \{f, -\}$ is a derivation

$$D_f(gh) = D_f(g)\,h + g\,D_f(h)$$

ie $\{f, gh\} = \{fg, h\} + \{g, fh\}$

This structure is used to understand the geometry of M.

In quantum mechanics, M is replaced by a Hilbert space $\mathcal{H}$, functions on M by operators on $\mathcal{H}$. The algebra of operators is no longer commutative.

Nevertheless, want to imitate all the structure on M -- do "non-commutative geometry"

To quantize:
Replace $A = C^\infty(M)$ by $A((\hbar))$ = ring of formal power series in A

Define a new non-commutative (but still associative and unital) product $\star$ on $A((\hbar))$ s.t.

$$f \star g = fg + O(\hbar)$$

and bracket $[f,g] = f \star g - g \star f$ st

$$[f,g] = \{f,g\} + O(\hbar^2)$$

(plus interaction of derivations...)

Kontsevich used graph complexes and Feynman integrals to do this...

(End of digression - possibly more later in the course.)

Takeaway - want to define some sort of non-commutative analogs of the Lie algebra given by Poisson bracket

Kontsevich defines 3 "flavors" of ∞-dim'l Lie algebras: commutative c_∞, associative a_∞, Lie l_∞

Each is a limit of algebras c_n, a_n, l_n

To define $h_n = c_n, a_n$ or l_n you need

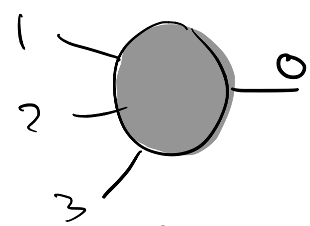
- a symplectic vector space V_n (2n real dims)
- the cyclic operad Comm, Ass or Lie

(but you can do it for <u>any</u> cyclic operad)

Comm, Ass, Lie operads.

An operad tries to capture the essence of some algebraic structure, without specifying an algebraic object that has that structure (think of smile on Cheshire cat)

Think of an operad as a set of "black boxes" P_k that take k ordered inputs and generalizes for producing one output of the same type



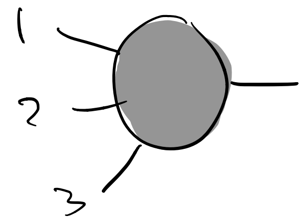
- but don't specify the nature of the inputs and output

I will illustrate by example: the Associative operad.

What is associativity? You have some operation $a, b \mapsto a \cdot b$ on a set S

satisfying $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

There is an operad element s.t. if you input $a \to 1$, $b \to 2$, $c \to 3$

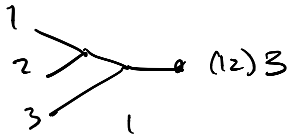


the result is $a \cdot b \cdot c$ – it doesn't matter

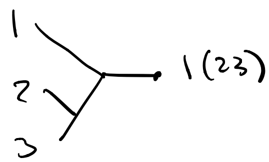
But if you put $b \to 1$, $a \to 2$, $c \to 3$ you might get a different answer

What's inside the black box?

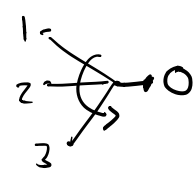
Can represent $(1 \cdot 2) \cdot 3$ by a tree:



Then $1 \cdot (2 \cdot 3) \Leftrightarrow$



They give the same answer, which we can represent by



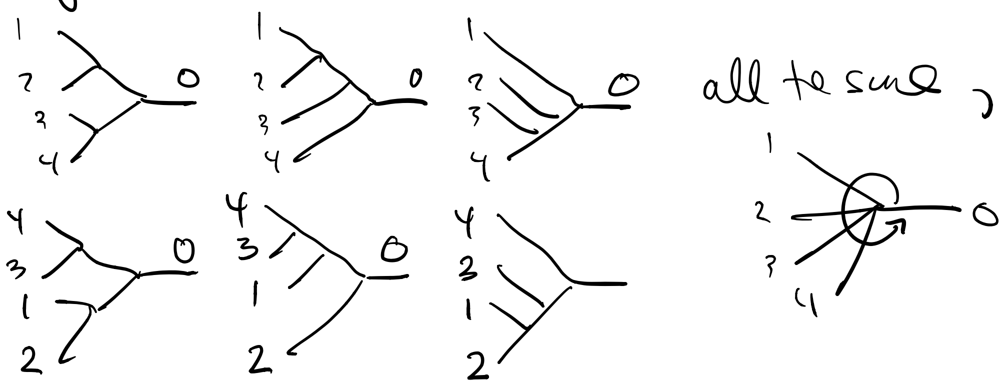
The order of the inputs matters, but not the specific tree structure

There are $3!$ possible outcomes

S_3 acts on the inputs, so are the possible opera

If you input more elements there are more ways to associate them,

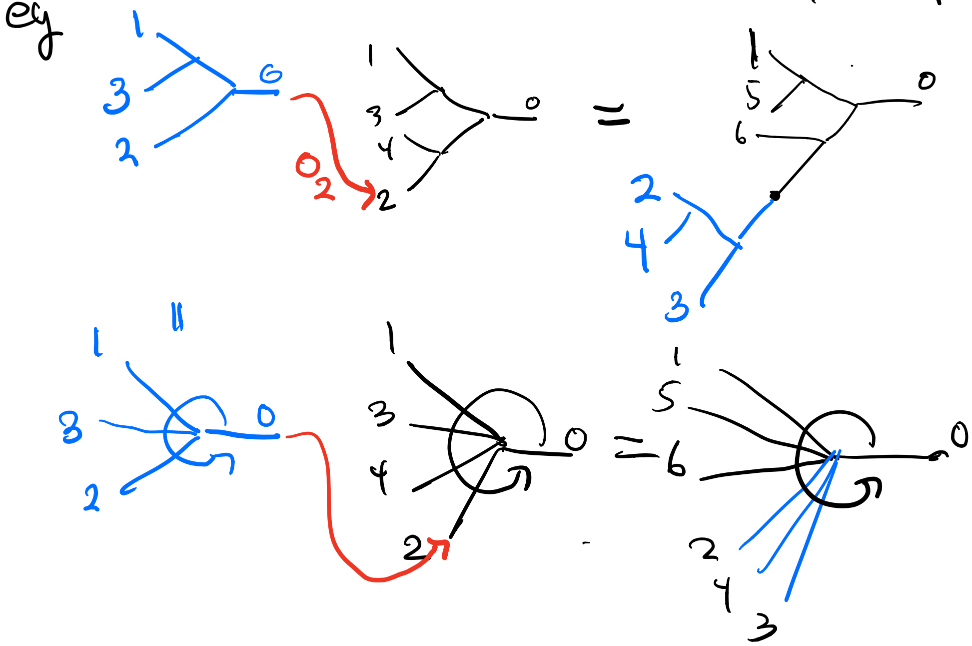
eg



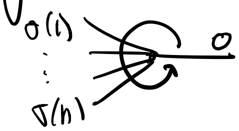
S_n acts on the possibilities $\exists$ $n!$ possible outcomes

There's a composition $\circ_i$: plug output of one association into the i^{th} input of another

eg



The operad Ass has one generator for each elt of S_n, which we picture as



Here's a more formal definition of operad

Operad = sets $\{P_n\}_{n\in\mathbb{N}}$, each w/ action of S_n and composition rules.

$$: P_k \times P_{i_1} \times \dots \times P_{i_k} \longrightarrow P_{i_1+\dots+i_k}$$
$$(\theta, \theta_1, \dots, \theta_k) \longmapsto \theta \circ (\theta_1, \dots, \theta_k)$$

w/ a <u>unit</u> $1 \in P_1$ satisfying

unit • $\theta \circ (1, \dots, 1) = 1 \circ \theta = \theta$

associativity • $\theta \circ (\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,i_1}), \theta_2 \circ (\theta_{2,1}, \dots, \theta_{2,i_2}), \dots, \theta_k \circ (\theta_{k,1}, \dots, \theta_{k,i_k}))$

$= (\theta \circ (\theta_1, \dots, \theta_k)) \circ (\theta_{1,1}, \dots, \theta_{1,i_1}, \theta_{2,1}, \dots, \theta_{2,i_2}, \dots, \theta_{k,1}, \dots, \theta_{k,i_k})$

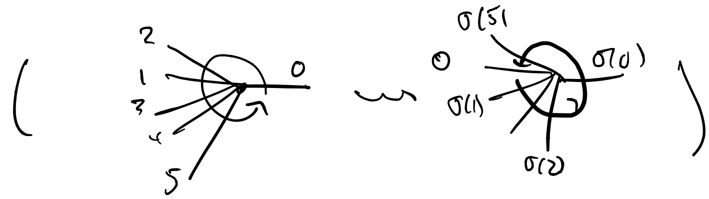
(easy to understand, but complicated to write down!!)

equivariance • $(\theta \cdot t) \circ (\theta_1, \dots, \theta_n) = (\theta \circ (\theta_1, \dots, \theta_n)) * t$

$t \in S_k$, $s_j \in S_{i_j}$ • $\theta \circ (\theta_1 s_1, \dots, \theta_n s_n) = (\theta \circ (\theta_1, \dots, \theta_n)) * (s_1, \dots, s_n)$

(think of trees ...)

In Ass, the action of S_n extends to an action of S_{n+1}



A cyclic operad is the result of extending the action of S_n on P_n to an action of S_{n+1} on P_n for all n

ie any input slot can also serve as the output slot - so we call it an i/o slot.

Commutativity operad?

Add commutativity to associativity
There's only one picture for each k

The output doesn't depend on the ordering of the inputs, there is no cyclic ordering of the edges around a vertex.