Lecture 3 — Graph homology

Last time: Given a cyclic operad $\mathcal{O}$, defined "$\mathcal{O}$-graph" and the (odd) graph complex $fC\mathcal{O}_*$ generated by all oriented $\mathcal{O}$-graphs with no leaves

$$fC\mathcal{O}_* \supset C\mathcal{O}_* \quad : \text{connected graphs } G$$
all vertices at least trivalent

$$\begin{matrix} \cup & & \cup \\ fC\mathcal{O}^{(r)}_* & \supset & C\mathcal{O}^{(r)}_* \end{matrix} \quad : \chi(G) = 1-r$$

Given a cyclic operad $\mathcal{O}$ and a symplectic vector space $V \cong \mathbb{R}^{2k}$ defined a Lie algebra h_k generated by "$\mathcal{O}$-spiders", $h_\infty = \varinjlim h_k$.

Showed 2-legged spiders $h^{(2)}_k$ form a subalgebra isomorphic to sp_k

Thm (Kontsevich)

(1) $$H_d^{CE}(h_\infty) \cong H_d(fg\mathcal{O}_*)$$

This is a Hopf algebra, whose primitive part is

$$PH_d(sp_\infty) \oplus \bigoplus_{r\geq 2} H_d(C\mathcal{O}_*^{(r)})$$

(2) For $h_\infty = l_\infty$, $r \geq 2$

$$H_d(C\mathcal{O}_*^{(r)}) \cong H^{2r-2-d}(\mathrm{Out}\, F_n)$$

For $h_\infty = a_\infty$, $r \geq 2$

$$H_d(C\mathcal{O}_*^{(r)}) \cong \bigoplus_{\substack{\chi(S_{g,s}) = 1-r \\ s \geq 0}} H^{2r-2-d}(\mathrm{Mod}(S_{g,s}))$$

(3) For $h_\infty = c_\infty$, $r \geq 2$

$H_d(C\mathcal{O}_*^{(r)})$ contains invariants of manifolds M with

$$H^*(M;\mathbb{Q}) = H^*(S^d;\mathbb{Q}),\ d \text{ odd}.$$

(odd-dimensional "rational homology spheres")

$Out(F_n)$ and $Mod(S_{g,s})$ are important groups in geometric group theory and low-dimensional topology.

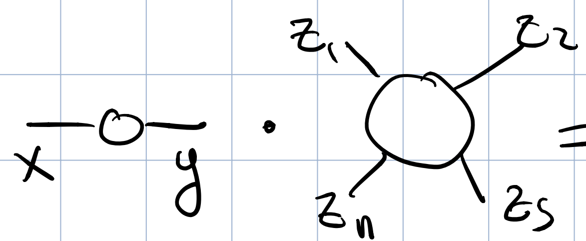
There are many other connections of graph homology to the rest of mathematics, coming from other variants of the graph complex (other cyclic operads, even orientation, allow univalent vertices, ...)

We will see some of these in the 2nd half of the course

Kontsevich's proof of part I exploits the action of $h_\infty^{(2)} \cong sp_\infty$ on h_∞:

We'll stick to $h_k^{(2)} = sp_k \curvearrowright h_k$ take limits later

<u>Action</u>



$$= \sum_i \begin{pmatrix} \bar{x} \mapsto y \\ \bar{y} \mapsto x \\ w \mapsto 0 \end{pmatrix} \cdot z_i$$

$x, y \in B$

$$= [A, S]$$

ie: $A \in sp$, S a spider

$$A \cdot S = \sum_{\substack{\text{legs } \lambda \\ \text{of } S}} (\text{change } x_\lambda \text{ to } A x_\lambda)$$

(This is just the usual action of $A \in sp$ on the labels on legs of S by multiplication.)

Exercise:

$$A \cdot [S_1, S_2] = [A \cdot S_1, S_2] + [S_1, A \cdot S_2]$$

This action extends to an action on $\Lambda^* h_k$:

$$A \cdot (S_1 \wedge \dots \wedge S_n) = \sum_i S_1 \wedge \dots \wedge A \cdot S_i \wedge \dots \wedge S_n$$

The invariants of the action are the elements of $\Lambda^* \mathfrak{h}_k$ killed by every $A \in \mathfrak{sp}_k$

$$(\Lambda^* \mathfrak{h}_k)^{\mathfrak{sp}} = \{ x : \mathfrak{sp} \cdot x = 0 \}$$

Exercise: $A \cdot d^{CE}(x) = d^{CE}(A \cdot x)$

Therefore the subspace of invariants is a subcomplex

$$(\Lambda^* \mathfrak{h}_k)^{\mathfrak{sp}} \xhookrightarrow{i} \Lambda^* \mathfrak{h}_k$$

Lemma i induces an isomorphism

$$H_k((\Lambda^* \mathfrak{h}_k)^{\mathfrak{sp}}) \to H_k(\Lambda^* \mathfrak{h}_k)$$

Proof $\mathfrak{sp} = \mathfrak{sp}_k$ is reductive so any $\mathfrak{sp}$-module E splits as

$$\ker \oplus \text{image} = E^{\mathfrak{sp}} \oplus \mathfrak{sp} \cdot E$$

in particular $Z_d =$ cycles in $\Lambda^d \mathfrak{h}_{\mathbb{F}}$

$B_d =$ boundaries in $\Lambda^d \mathfrak{h}_{\mathbb{F}}$

$$Z_d = Z_d^{\mathfrak{sp}} \oplus \mathfrak{sp} \cdot Z_d$$

$$B_d = B_d^{\mathfrak{sp}} \oplus \mathfrak{sp} \cdot B_d$$

$$\frac{Z_d}{B_d} = \frac{Z_d^{\mathfrak{sp}}}{B_d^{\mathfrak{sp}}} \oplus \frac{\mathfrak{sp} \cdot Z_d}{\mathfrak{sp} \cdot B_d}$$

$$H_d(\Lambda^* \mathfrak{h}_{\mathbb{F}}) = H_d((\Lambda^* \mathfrak{h}_{\mathbb{F}})^{\mathfrak{sp}}) \oplus$$

claim $\mathfrak{sp} \cdot Z_d \subset B_d$

pf: $\xi = \overset{x}{-}\!\circ\!\overset{y}{-} \in \mathfrak{sp}$

Then $\xi \cdot a = d(a \wedge \xi) + (da \wedge \xi)$

So a a cycle $\Rightarrow \xi \cdot a = d(a \wedge \xi)$

Then $[\mathfrak{sp}, \mathfrak{sp}] Z_d \subset \mathfrak{sp} \cdot B_d$

$[\mathfrak{sp}, \mathfrak{sp}] = \mathfrak{sp}$. ($\mathfrak{sp}$ is simple, not abelian)

Main point of Kontsevich's proof of part (1): is to identify sp-invariants with Θ-graphs

Define two maps:

$$\phi_k : \mathcal{O}g \longrightarrow \Lambda^* h_k$$

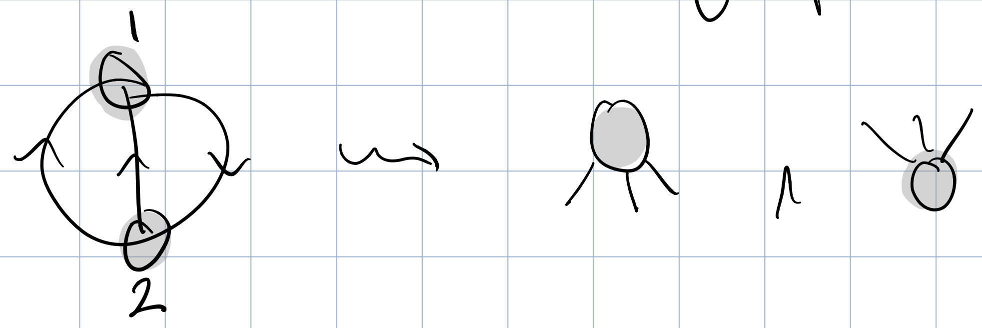
Θ-graph $\longmapsto$ wedge of spiders

$$\psi_k : \Lambda^* h_k \longmapsto \Theta\text{-graphs}$$

wedge of spiders $\longmapsto$ Θ-graph

$$\mathcal{O}g \xrightarrow{\phi_k} \Lambda^* h_k \xrightarrow{\psi_k} \mathcal{O}g$$

ϕ_k: Need to get a wedge of spiders from an (odd-oriented) Θ-graph



clear how to get a wedge of spiders - but what labels to put on the edges?

A ==state== of an Θ-graph labels each half-edge with an element of B_k, each edge with ± 1

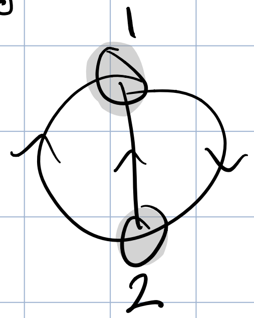
==Rules==: each edge $\xrightarrow{e}$ has matching labels p_i q_i and sign

$\underset{+}{\overset{p_i \quad e \quad q_i}{\longrightarrow}}$ or $\underset{-}{\overset{q_i \quad e \quad p_i}{\longrightarrow}}$

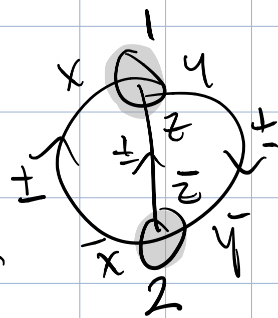
Each state on G gives a wedge of Θ-spiders and a total sign $\pm 1 = \prod_e \operatorname{sign}(e)$

$\phi_n : G \longmapsto \sum(\text{states of } G) \longmapsto \sum \pm$ wedge of Θ-spiders.

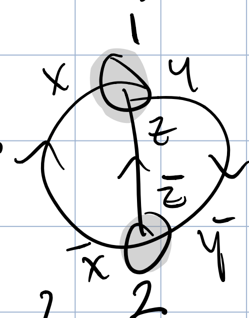
eg



$\rightsquigarrow \quad \sum_{x,y,z\,\in\, B_k}$



$\sum_{x,y,z\,\in\, B_k} \pm 1 \cdot$



$\Big\}$

$\sum_{x,y,z\,\in\, B} \pm \Big($

$\wedge$

$\Big)$

Claim: This is an sp_k-invariant

Proof (deferred)

so $\quad \mathcal{O}g \xrightarrow{\phi_k} (\Lambda^* h_k)^{sp} \subset \Lambda^* h_k$

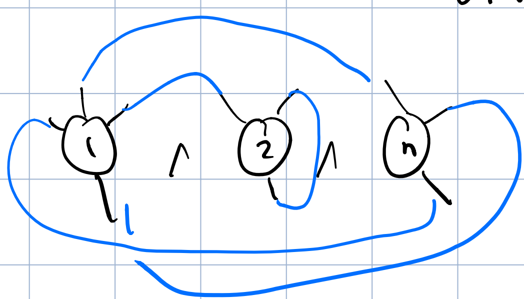
Now want to get Θ-graphs from a wedge of Θ-spiders.

$$\psi_k(S_1 \wedge \dots \wedge S_n)$$

$= 0$ unless the total number of legs is even

If the total number is even, pair them with a pairing π.

This gives instructions for forming a graph; orient the edges arbitrarily



$$:= (S_1 \wedge \dots \wedge S_n)^{\pi}$$

If π pairs legs λ and μ, with labels x_λ and x_μ, $x_\lambda \longrightarrow x_\mu$

define $\displaystyle w(\pi) = \prod_{\text{pairs } (\lambda,\mu)} \langle x_\lambda, x_\mu \rangle$

Note $w(\pi) = 0$ unless labels of pairs match

Then

$$S_1 \wedge \dots \wedge S_n \xrightarrow{\psi_n} \sum_{\pi} w(\pi) (S_1 \wedge \dots \wedge S_k)^{\pi}$$

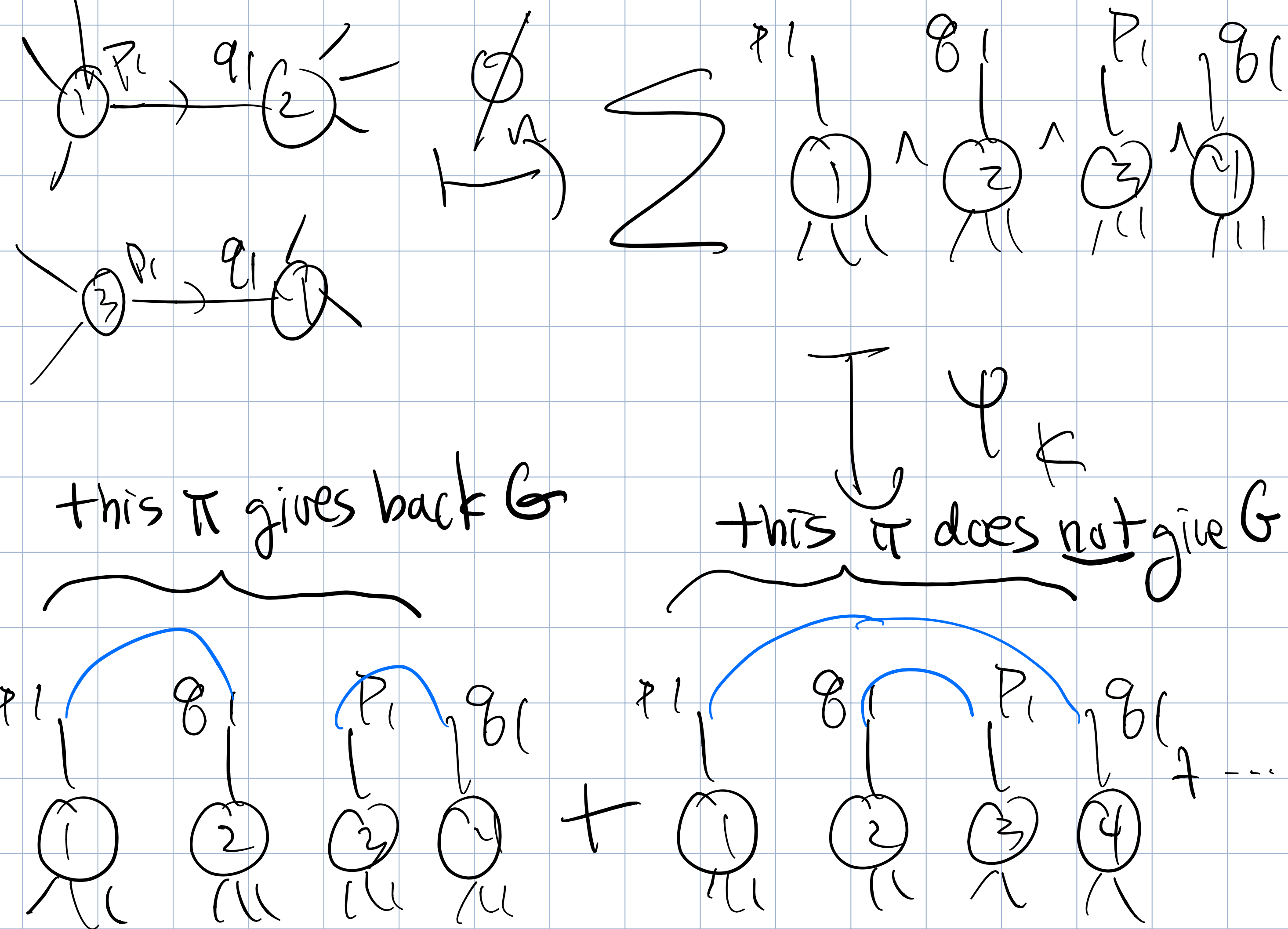
It would be nice if ψ and φ were inverses, but

$$\mathcal{O}g \xrightarrow{\phi_n} \Lambda^k h_k \xrightarrow{\psi_n} \mathcal{O}g$$

is not the identity, or even a multiple of the "identity"

Problem: "accidental" pairings

$$G \longrightarrow \phi_k G.$$



Prop $\Psi_k \circ \phi_k = M_k : \mathcal{O}g \to \mathcal{O}g$

where M_k is defined as follows

$G \in \mathcal{O}g$

Let π be a pairing of the half-edges of G

gluing them gives a graph G^π

Let σ be the pairing that pairs half-edges if they are in the same edge of G

(the "standard pairing")

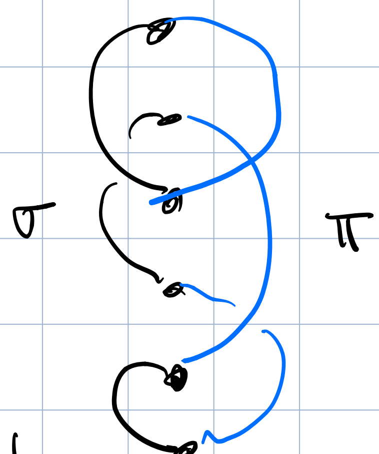
Then $G^\sigma = G$.

Form a graph by pairing half-edges using both σ and π :

<u>Vertices</u> = half-edges

<u>edges</u> $\leftrightarrow$ pairs in σ

$\cup$ pairs in π



Define $c(\pi)$ = # of components (= circles) in this graph

Then $c(\sigma)$ = # edges of G

and $c(\pi) < c(\sigma)$ if $\pi \neq \sigma$

Note $\mathcal{OG}$, $\Lambda^* h_k$ decompose into direct sums of finite-dimensional pieces

namely $\mathcal{OG}_{n,m}$ = Θ-graphs with n vertices, m edges

$\Lambda_{n,m} \subset \Lambda^n h_k$ = wedges of spiders with a total of $2m$ legs

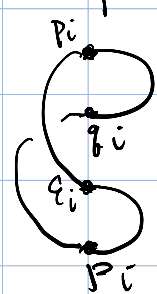
ϕ_k, ψ_k respect these pieces:

$$\mathcal{OG}_{n,m} \xrightarrow{\phi_k} (\Lambda_{n,m})^{Sp} \xrightarrow{\psi_k} \mathcal{OG}_{n,m}$$

so $\psi_k \circ \phi_k$ has a matrix $M_{n,m}$

<u>Claim</u> $G^\pi - G$ entry is $(2k)^{c(\pi)}$

pf There are $(2k)^{c(\pi)}$ ways to put $p_i - q_i$ labels on each circle



$i = 1, \dots, k$

$p_i \leftrightarrow q_i$

So matrix is
$$\begin{array}{c|c}
 & \begin{matrix} G_1 & G_2 & \cdots & & G & & \end{matrix} \\
\begin{matrix} G_1 \\ G_2 \\ \vdots \\ G \\ \\ G^{\pi} \\ \\ \end{matrix} & \begin{bmatrix} \ddots & & & \vdots & & \\ & \ddots & & \vdots & & \\ & & \ddots & & & \\ & & & (2k)^{c(\sigma)} & & \\ & & & \vdots & \ddots & \\ & & & (2k)^{c(\pi)} & & \ddots \\ & & & \vdots & & \ddots \end{bmatrix}
\end{array}$$

This may not be invertible! But for k <u>sufficiently large</u> the diagonal entries $\left((2k)^{c(\sigma)}\right)$ dominate the off-diagonal entries $\left((2k)^{c(\pi)} \text{ or } 0\right)$ and $M_{n,m}$ <u>is</u> invertible

so $\psi_k \circ \phi_k$ is an isomorphism for k suff. large.

$$\begin{tikzcd}
Og_{n,m} \arrow[r, "\phi_k"] \arrow[rr, bend right, "\cong"'] & \Lambda_{n,m} \arrow[r, "\psi_k"] & Og_{n,m}
\end{tikzcd}$$

So ϕ_k is injective for k suff. large

By our previous claim, $\mathrm{Im}\, \phi_k = (\Lambda_{n,m})^{ep}$

$$\begin{tikzcd} \text{so} \quad OG_{n,m} \arrow[r, "\phi_k", "\cong"'] \arrow[rr, bend right, "\cong"'] & (\Lambda_{n,m})^{sp} \arrow[r, "\Psi_k"] & OG_{n,m} \end{tikzcd}$$

We would like to conclude that ϕ_k induces an $\cong$ on homology, but

Exercise ϕ_k is not a chain map!

Luckily, Ψ_k is a chain map.

Since $\Psi_k = 0$ if spider legs don't match in pairs, $\operatorname{Im} \Psi_k \subseteq \operatorname{Im} \Psi_k \big|_{(\Lambda_{n,m})^{sp}}$

$$\therefore \Psi_k : (\Lambda_{n,m})^{sp} \longrightarrow OG_{n,m}$$

is an $\cong$ for k sufficiently large.

So Ψ_∞ induces an isomorphism

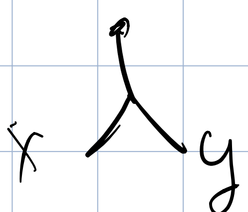
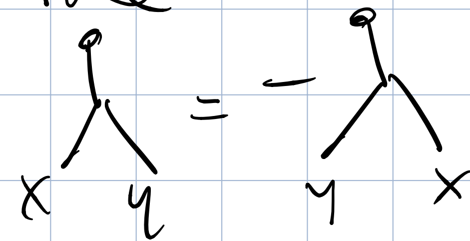
$$H_*(\Lambda^* h_\infty) \longrightarrow H_*(OG) \qquad \blacksquare$$

We still need to check $\operatorname{Im}\phi_K \subset (\Lambda^k \mathfrak{h}_k)^{sp}$

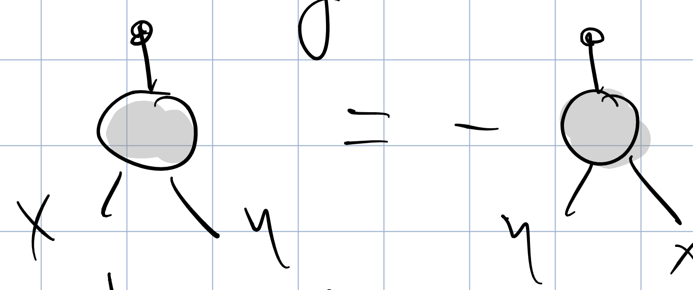
This starts from the fact that

on $V \wedge V$, $\sum p_i \wedge q_i$
is an sp-invariant.

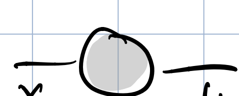
Pictorially: represent a generator
$x \wedge y$ of $V \wedge V$ $(x, y \in B)$
by a rooted planar tree

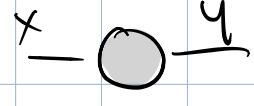
 modulo 

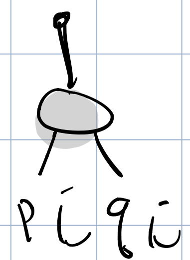
We can decorate the internal
vertex by an elt of Θ

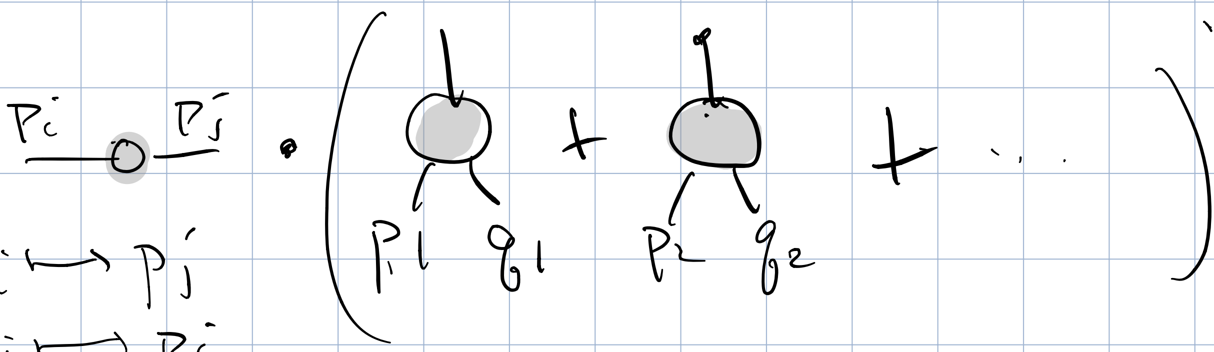


The action of sp is determined

by the Θ-spiders  $x, y \in B$

Note that every  kills

$\omega_\theta = \sum_i$  $\in V \wedge V$



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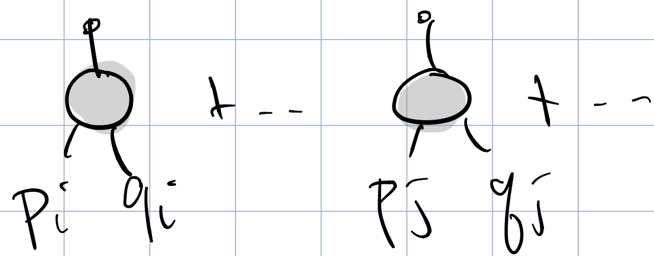
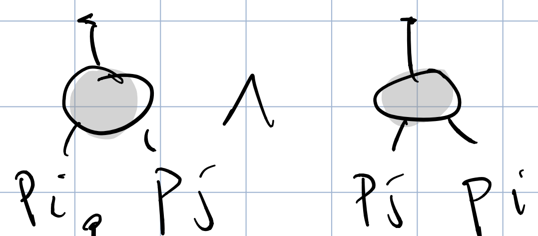
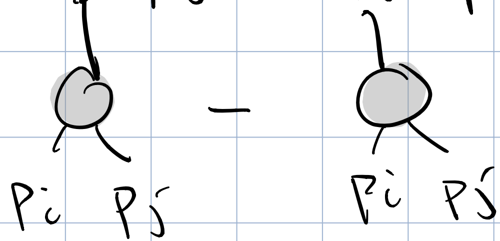
$q_i \longmapsto p_j$

$q_j \longmapsto p_i$

$= 0 + 0$ 

$=$ 

$=$  $= 0$

etc.

Next let's find invariants in $\Lambda^n V_k$:

If n is even, pair the terms, by π

$$V_k \wedge \cdots V_k \wedge \cdots V_k \wedge V_k$$

permute until π looks like

$$= \pm V_k \wedge V_k \wedge V_k \wedge V_k \wedge \cdots \wedge V_k \wedge V_k$$

Then $\omega = \pm \omega_k \wedge \omega_k \wedge \cdots \wedge \omega_k$

is an sp_k-invariant

Weyl: These span the space $(\Lambda^n V_k)^{sp}$ of all invariants.