Lecture 4 Graph homology

Last time: Sketched proof of Kontsevich thm

$$H_d(\Lambda^* h_\infty) \cong H_d(\Lambda^* h_\infty)^{sp_\infty} \cong H_d(fC_*g)$$

$$\left(h_\infty^{(2)} \cong sp_\infty \text{ acts on } h_\infty \right)$$

Feedback: people got lost

Recap: Defined a Lie algebra h_k (in terms of spiders)

needed
- a cyclic operad $\mathcal{O}$ and
- a symplectic vector space $V^k = \mathbb{R}^{2k}$

For $\mathcal{O} =$ Comm (cyclic version) –
Gives a pictorial description of the Poisson algebra of polynomial functions on V_k with no constant or linear terms

– a key object in mathematical physics

Poisson bracket with a fixed function gives
a derivation, and
We also saw how spiders can be viewed
as derivations

We then observed that the pictorial definitions
make sense with Θ-spiders
(instead of just commutative spiders)

and that for any Θ (with $\Theta[1] = \{1\}$)
h_k contains $\mathfrak{h}_k^{(2)} \cong sp_k$, which acts on h_k

Finally, we sketched proofs that
(1) $H_*(\Lambda^* h_k, \partial_{CE}) \cong H_*((\Lambda^* h_k)^{sp}, \partial_{CE})$

<u>and</u>, letting $k \to \infty$
(2) $(\Lambda^* h_\infty)^{sp} \cong \Theta$-graph complex $\{C_* G\}^{odd}$

Upshot: We have a combinatorial
way to compute $H_*^{CE}(h_\infty)$ using graphs

The second part of Kontsevich's theorem
says:

- $H_*(h_\infty)$ is a Hopf algebra, w/ primitive part

$$PH_*(h_\infty) \cong PH_*(h_\infty^{(2)}) \oplus \bigoplus H_*(C_*g)$$

cont:

$$H_*(C_*g) \cong \bigoplus H^*(Out(F_n)) \quad (\mathcal{O} = Lie)$$

$$H_*(C_*g) \cong \bigoplus H^*(Mod(S_{g,s})) \quad (\mathcal{O} = Ass)$$

($H_*(C_*g)$ $\mathcal{O} = Comm$ related to invariants ~~of~~ rat'l H_* spheres.

Why is this good?

Relates $H^*(Out F_n)$, $H^*(Mod(S_{g,s}))$ to other areas of mathematics, potentially giving new tools for understanding $Out(F_n)$, $Mod(S_{g,s})$ and the spaces they act on.

eg h_∞ for $\mathcal{O} = Lie$ was studied previously by Morita, who used his previous results to find potential homology classes in $H_*(Out F_n)$ ("Morita cycles") eg Kontsevich constructed cycles and cocycles for $\mathrm{Mod}(S_{g,s})$ that pair non-trivially, giving classes in the stable homology of $\mathrm{Mod}(S_{g,s})$.

eg Willwacher identified H^0 of even commutative graph homology with $\mathfrak{grt}_1$ ("Grothendieck-Teichmüller")
Brown showed $\mathfrak{grt}_1$ contains a free Lie algebra.

Chan-Galatius-Payne related this to cohomology of $\overline{M}_g$ = Deligne-Mumford compactification of the moduli space of genus g Riemann surfaces.

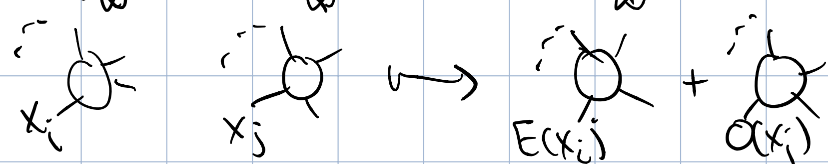
How is $H_*(\mathfrak{h}_\infty)$ a Hopf algebra?

1) What is the product $H_*(\mathfrak{h}_\infty) \otimes H_*(\mathfrak{h}_\infty) \to H_*(\mathfrak{h}_\infty)$?

Recall $B_k = \{p_1, \dots, p_k, q_1, \dots, q_k\}$ = basis for V_k

$x_i \in B_\infty$ Define $E(x_i) = x_{2i}$ $O(x_i) = x_{2i-1}$

This induces $\mathfrak{h}_\infty \times \mathfrak{h}_\infty \longrightarrow \mathfrak{h}_\infty$



which in turn induces (Künneth)

$$H_*(\mathfrak{h}_\infty) \otimes H_*(\mathfrak{h}_\infty) \longrightarrow H_*(\mathfrak{h}_\infty)$$

on the chain level:

$$\sum_{k+l=r} \Lambda^k(\mathfrak{h}_\infty) \otimes \Lambda^l(\mathfrak{h}_\infty) \longmapsto \Lambda^r \mathfrak{h}_\infty$$

$$(s_1 \wedge \dots \wedge s_k) \otimes (s_1' \wedge \dots \wedge s_l') \longmapsto E(s_1) \wedge \dots \wedge E(s_k) \wedge O(s_1') \wedge \dots \wedge O(s_l')$$

$$\Big\downarrow \psi_{k+l}$$

In terms of graphs?

$$\sum (G \sqcup G')$$

(even edges never get identified w/ odd edges)

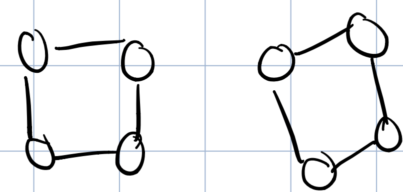
So the elements of $H_*(h_\infty) = H_*(fC_*g)$ that are <u>not</u> products are represented by cycles using only connected graphs — ie these are the <u>primitive</u> elements

$$PH_*(h_\infty) = H_*(cC_*g)$$

connected, valence ≥ 2

recall $h_\infty^{(2)}$: generated by spiders w/ 2 legs $\cong \mathfrak{sp}_\infty$

$\varphi_k(\Lambda^* h_\infty^{(2)})$ = union of polygons



$$fC_*g = fC_*g^{\chi=0} \oplus fC_*g^{\chi<0}$$

(χ is preserved by d; no univ vertices $\Rightarrow \chi \leq 0$)

Prim elts are connected graphs

Primitive elts are polygons

Prim. elts are connected w/ $\chi = 1-r < 0$ ie $r = \mathrm{rk}\,\pi_1 \geq 2$

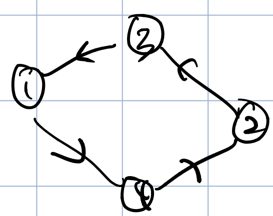
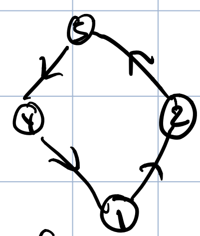
$$cC_*g = \bigoplus_{r\geq 1} cC_*g^{(r)}$$

connected, $\pi_1 G \cong F_r$

$$PH_*(h_\infty) = PH_*(\Lambda^* h^{(2)}) \oplus \bigoplus_{r\geq 2} H_*(cC_*g^{(r)}))$$

exercise: Compute $PH_*(\Lambda^* h^{(2)})$

Hint: Show a generator of $cC_*\mathcal{G}^{(2)}$ with k vertices is zero unless $k \equiv 3 \pmod 4$

eg  $= -$  $\Rightarrow (G, or) = 0$

$$(G, or) = -(G, or)$$

Next claim $r \geq 2 \Rightarrow H_*(cC_*\mathcal{G}^{(r)}) = H_*(C_*\mathcal{G}^{(r)})$

connected, all vertices have valence ≥ 3

(ie can ignore bivalent vertices)

Proof is an induction on the # of bivalent vertices.

Graph homology and spaces of graphs:

$C_* \mathcal{G}^{(r)}$: graphs are connected with $\pi_1 \cong F_r$
and no univalent or bivalent vertices

This is a combinatorially-defined chain complex. Is it related to the homology of some topological space?

We'll stick to $\mathcal{O}$ = Comm, Ass, Lie.

Moduli spaces of graphs.

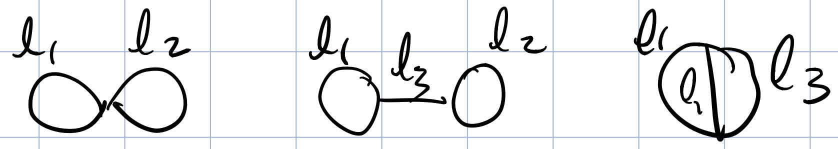
$\mathcal{MG}_r$ $(r \geq 2)$: points are connected metric graphs with $\pi_1 \cong F_r$, all vertices have valence ≥ 3.

metric: $l_e \in \mathbb{R}_{>0}$ on each edge e
ie each edge is isometric to an interval $(0, l_e) \subseteq \mathbb{R}$

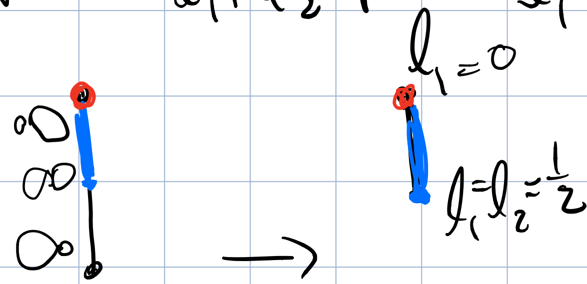
$\Rightarrow$ G is a metric space w/ path metric

Normalize: $\sum_{e \in G} l_e = 1$

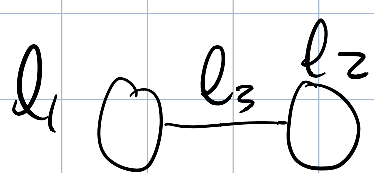
eg $\mathcal{MG}_2$:

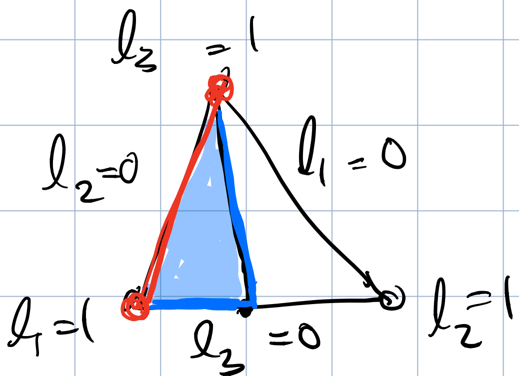


$l_1+l_2=1$ $\qquad$ $l_1+l_2+l_3=0$

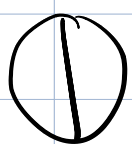


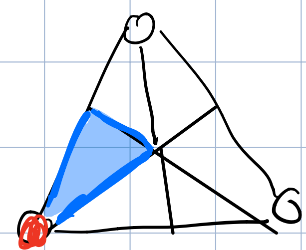
(half-open interval)





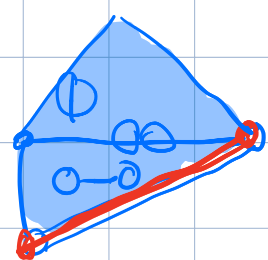
(triangle minus edge)





(triangle minus vertex)

$\mathcal{MG}_2$:



Better way to think of $\mathcal{MG}_n$:

As quotient of CV_n by $Out(F_n)$

Add markings $g: R_n \xrightarrow{\simeq} G$ to metric graphs

Define $F_n := \pi_1 R_n$

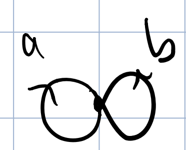
Then $g_*: \pi_1 R_n \xrightarrow{\cong} \pi_1 G$

identifies $\pi_1 G$ with F_n

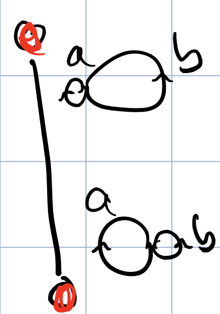
(Notice I didn't give G or R_n base points)

Point in $CV_n = (G, g)/\sim$

$$\begin{tikzcd} G \arrow[rr, "\text{isometry}"] & & G' \\ & R_n \arrow[ul, "g"] \arrow[ur, "g'"'] \arrow[u, phantom, "\simeq"] & \end{tikzcd}$$

Now 

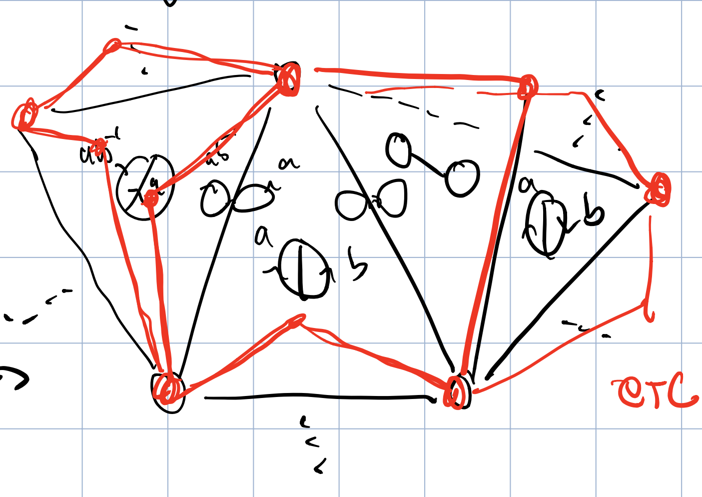
$F_2 = \langle a, b\rangle$



these are not equal as marked graphs,

so edge is not folded...

CV_2 looks like



$Out(F_n)$ acts on marked graphs

by changing the marking

$\varphi : F_n \xrightarrow{\cong} F_n$ can be realized by

$f : R_n \xrightarrow{\simeq} R_n$

Then

$$\begin{tikzcd} R_n \arrow[r, "g", "\simeq"'] & G \\ R_n \arrow[u, "f"] \arrow[ur, "g \circ f"'] & \end{tikzcd}$$

$$(G, g) \cdot \varphi = (G, g \circ f)$$

(exercise: this action is well-defined)

CV_n is <u>not</u> a simplicial complex -

each simplex $\sigma(G, g)$ has missing faces (e.g. <u>all</u> vertices are missing.)

$CV_n^* =$ simplicial completion

The action of $Out(F_n)$ extends continuously to CV_n^*

Quotient $\mathcal{MG}_n^*$ is the "Moduli space of tropical curves" if you are a tropical geometer

<u>Thm</u> (Culler-V 86) CV_n is contractible, the action of $Out(F_n)$ is proper

By Hurewicz, this implies

$$H^*(Out F_n; \mathbb{Q}) \cong H^*(CV_n; \mathbb{Q})$$

<u>Thm</u> (Hatcher 90) CV_n^* is contractible

$\partial CV_n^* = CV_n^* \setminus CV_n$ is a

subcomplex of CV_n^*

The relative chain complex

$$C_k(CV_n^*, \partial CV_n^*) = C_k(CV_n) / C_k(\partial CV_n)$$

has one generator for each marked graph (G, g), $g: R_n \xrightarrow{\simeq} G$

The simplex $\sigma(G,g)$ is oriented by ordering the edges of G

The faces of $\sigma(G,g)$ are obtained by collapsing one edge of G.

A face is a relative generator iff the edge is not a loop.

Out F_n acts transitively on markings,

The stabilizer of a point (G,g)

is isomorphic to $\mathrm{Isom}(G)$

$$\begin{tikzcd} G \arrow[rr, "i"] & & G \\ & R_n \arrow[ul, "g"] \arrow[ur, "g"'] & \end{tikzcd}$$

If an isometry induces an odd permutation of the edges, $\sigma(G,g)/\mathrm{Isom}(G)$

If no isometry of G gives an odd permutation of $E(G)$

then $\sigma(G,g)/\mathrm{Isom}(G)$ is

a cone on a homology sphere.

so gives a non-trivial generator of

$$C_*(CV_n^*/\mathrm{Out}, \partial CV_n^*/\mathrm{Out})$$

$$= C_*(\mathcal{MG}_n^*/\partial\mathcal{MG}_n^*)$$

The boundary map in the commutative graph complex, contracts non-loop edges, so preserves the rank of $\pi_1 G$.

ie $$\mathcal{g} = \bigoplus_{\substack{rk=n \\ \geq 2}} \mathcal{g}^{(n)}$$

The generators of $\mathcal{g}^{(n)}$ are (unmarked) graphs, all vertices $\geq$ trivalent, rank $= n$.

With the even orientation, we therefore get

$$H_*(\mathcal{g}^{(n)}_*) = H^{Out}_*(CV^*_n, \partial CV^*_n)$$

even commutative graph homology

$$= H_*(CV^*_n/_{Out}, \partial CV^*_n/_{Out})$$

$$= H_*(\mathcal{MG}^*_n, \partial \mathcal{MG}^*_n)$$

Exercise: Find a graph with no odd symmetries, describe $\sigma(G,g)/_{Isom(G)}$