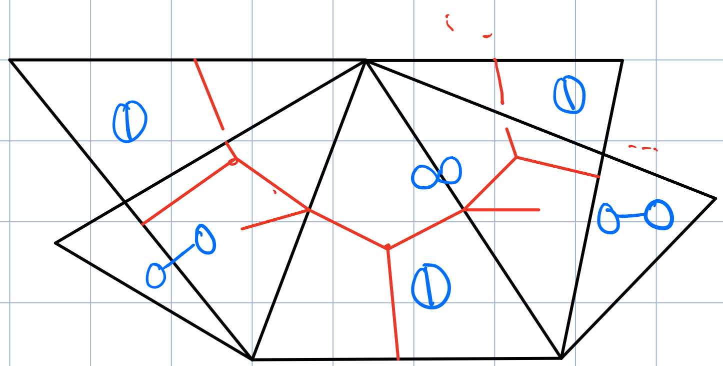
Lecture 6 Graph homology

We are trying to relate Kontsevich's graph complexes to moduli spaces of graphs. Specifically, we are now comparing Kontsevich's Lie graph homology with the cohomology of Out(Fn) via Outer space CVn.

We've defined the spine Kn of CVn



$K_n \subset CV_n$

= geometric realization of the poset of simplices $\sigma(G, g)$

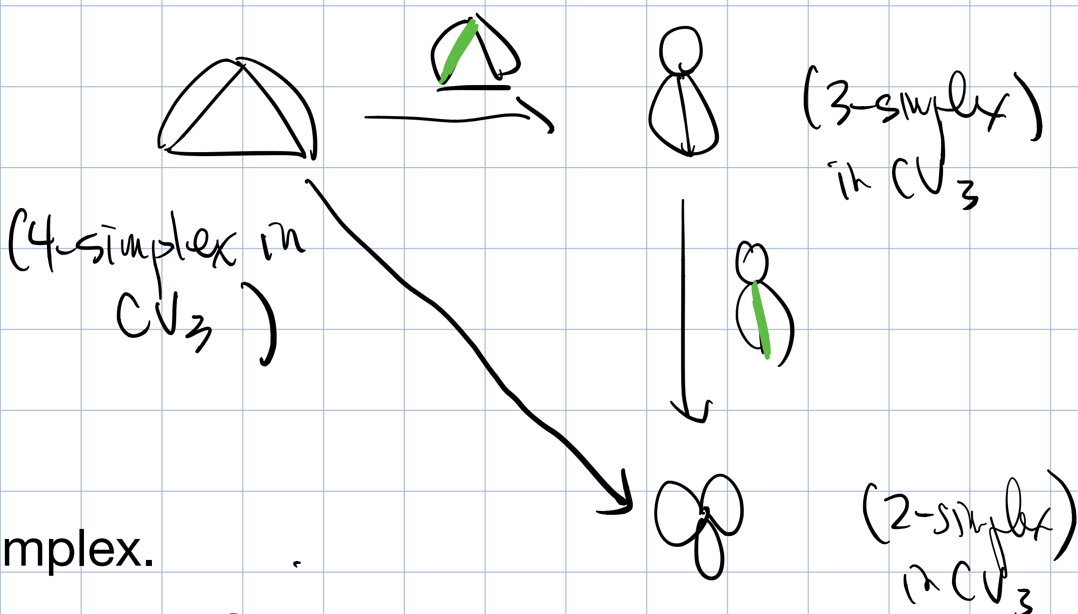
Kn is a deformation refract of CVn, so is contractible.

$$\Rightarrow H^*(\mathrm{Out}\,F_n) = H^*(K_n/\mathrm{Out}\,F_n)$$

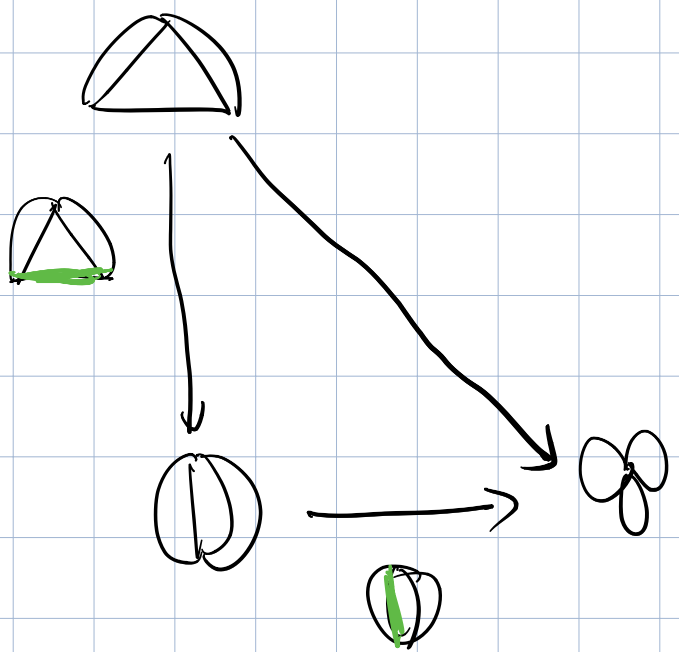
Kn is a simplicial complex.
Maximal simplex = chain of edge-collapses.

(n=2 picture is not enlightening here!
Try an n =3 picture.)

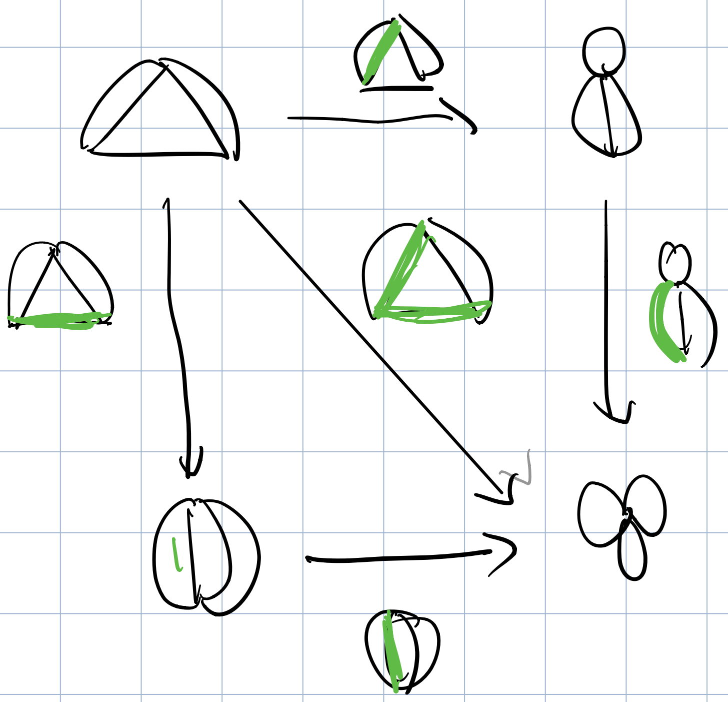
Simplex in Kn:



Another simplex.



In fact the simplices arrange themselves into cubes:

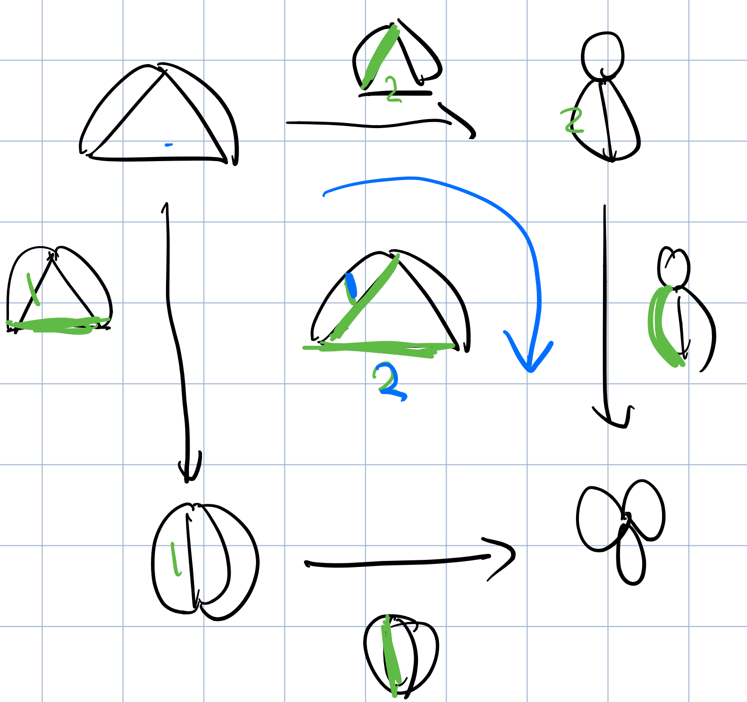


one cube for
each triple

$$(G, \Phi, g)$$

graph forest marking

The cube is oriented by choosing an ordering on the edges E(G), and has dimension =#E(G)



collapse 1 first

collapse 2 second.

What is the boundary operator of this cube complex?

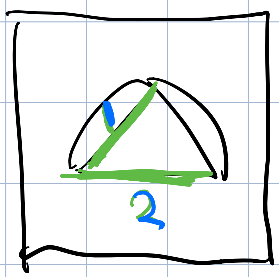
Collapsing an edge e pulls the marking along:

$$\begin{tikzcd} (G,\Phi) \arrow[r, "/e"] & (G_e,\Phi_e) \\ R_n \arrow[u, "g"] \arrow[ur, "g_e"'] & \end{tikzcd}$$

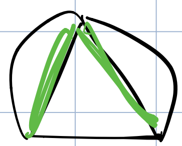
$$\partial(G,\Phi,g) = \sum_{e\in\Phi} (G,\Phi-e,g) - (G_e,\Phi_e,g_e)$$

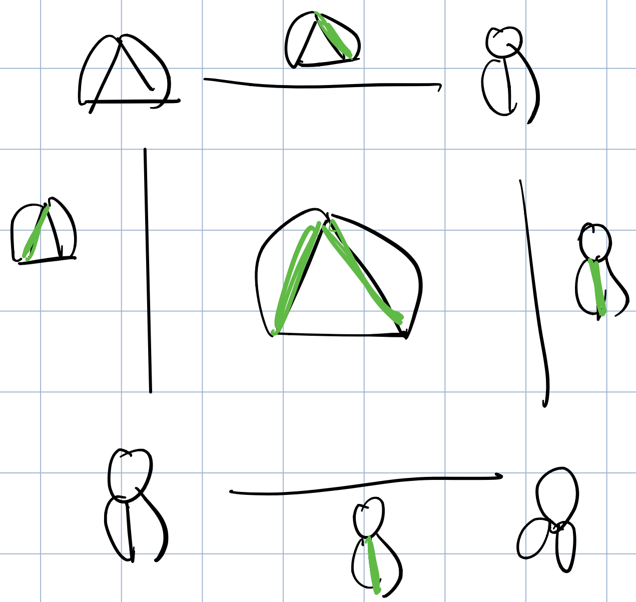
$$= \sum_e \underbrace{\partial_r}_{\text{remove an edge}}(G,\Phi,g) - \underbrace{\partial_c}_{\text{collapse an edge}}(G,\Phi,g)$$

$\mathrm{Out}(F_n)$ acts by changing the marking. It doesn't change the $\cong$ type of the graph.

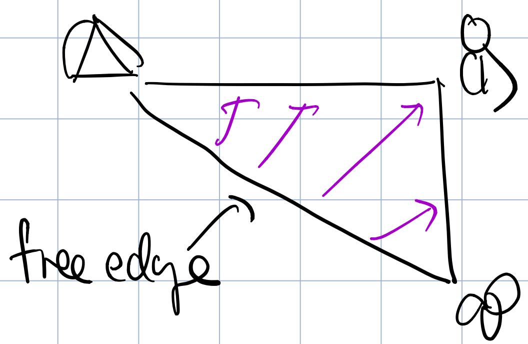
So the square 

embeds in $\overline{K_3} := K_3/\mathrm{Aut}\,F_3$

However, the square  does not embed in the quotient:



Opposite corners are identified:

$\rightsquigarrow$ 

so the quotient of (G, Φ, g) can be eliminated from $C_2(\overline{K}_3)$ without changing $H_2(\overline{K}_3)$.

Prop: (G, Φ) contributes a generator to $C_*(\overline{K}_n)$ iff (G, Φ) has no orientation-reversing automorphism.

Pf K_n is a CW-cplex, where the cells are the cubes (G, Φ, g)
$Out(F_n)$ acts by changing g.
$stab(G, \Phi, g) \cong Aut(G, \Phi)$

"cells" of $K_n / Out(F_n)$ = quotients of cubes = cones on quotients of spheres

Lemma. The quotient of a sphere by a finite group is a rational homology sphere or ball, depending on whether the group has an element that reverses orientation.

Pf Equivariant homology spectral sequence!

heuristics: A reflection creates a free face in the quotient - further identifications don't change that.

A rotation doesn't give a free face, just torsion in H_1, which rational homology doesn't see

So the rational cellular chain complex for

$$K_n / \mathrm{Out}(F_n)$$

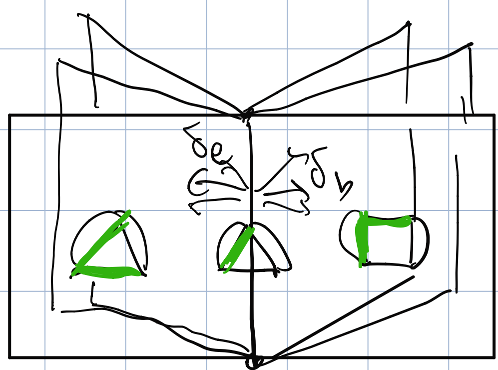
has one generator for each pair (G, Φ), modulo $(G, \Phi, or_\Phi) = -(G, \Phi, -or_\Phi)$

What is the co boundary operator in $C^{*}_{*}(X_n)$?
need a forest w/ one more edge

$$\delta(G, g, \Phi) = \delta_a + \delta_s$$

add an edge to ϕ split a vertex of G

$$= \sum_{\substack{\Phi \cup e \\ \text{a forest}}} (G, g, \Phi \cup e) + \sum_{v \in V(G)} \sum_{\substack{\pi = \text{partition} \\ \text{of } E(G)}} (G^{\pi}, g^{\pi}, \Phi^{\pi})$$



This looks similar to the chain cplex CG_*
except:

① In CG_* all graphs are trivalent.
② In CG_* gens defined up to $I(f)X$
(AS is taken care of by ordering edges)
③ (the ∂_s terms aren't there)

Claim: The two chain complexes have the same homology

Proof: $C_* =$ cellular chains for K_n

$$C_0 = \bigoplus_{p \geq 1} C_{p,p}$$

(p vertices, p trees

$\Rightarrow$ each vertex is a "tree"

$\Rightarrow$ only one forest Φ with no edges

$$C_1 = \bigoplus_{p \geq 2} C_{p,p-1}$$

(p vertices, $p-1$ trees

$\Rightarrow$ 1 edge in Φ

$$C_k = \bigoplus_{p-q=k} C_{p,q}$$

$\left\{ \begin{array}{l} p \text{ vertices}, \\ q \text{ trees} \\ \Rightarrow p-q \text{ edges in } \Phi \end{array} \right.$

Form a double complex

C_{k+2}

$C_{k+1} = C_{p-q+1}$

$$\begin{tikzcd}
C_{p+1,q-1} & C_{p+1,q} \arrow[l, "\delta_a"'] \\
C_{p,q-1} \arrow[u, "\delta_s"] & C_{p,q} \arrow[l, "\delta_a"] \arrow[u, "\delta_s"'] \arrow[ul, phantom, "\oplus"]
\end{tikzcd}$$

$C_k = C_{p-q}$

Consider the vertical chain complexes (with differential δ_s):

$$
\begin{tikzcd}
C_{2n-2,1} & C_{2n-2,q-1} \arrow[l] & C_{2n-2,q} \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
C_{p+1,1} \arrow[u] & C_{p+1,q-1} \arrow[u] & C_{p+1,q} \arrow[l, "\delta_a"'] \arrow[u] \\
C_{p,1} \arrow[u, "\delta_s"] & C_{p,q-1} \arrow[u, "\delta_s"] & C_{p,q} \arrow[l, "\delta_a"'] \arrow[u, "\delta_s"] \\
\vdots \arrow[u] & C_{q,q-1} \arrow[u] & C_{q,q} \arrow[l, "\delta_a"'] \arrow[u] \\
C_{1,1} \arrow[u] & C_{q-1,q-1} \arrow[u, "\delta_s"] &
\end{tikzcd}
$$

$C_{2n-2,q}$ ← trivalent graphs w/ q trees ($2n-2-q$ edges)

$C_{q,q}$ ← q vertices, q trees (each tree is a point)

$C_{q-1,q-1}$ ← $q-1$ vertices, $q-1$ trees

Proposition. The homology in the vertical direction is zero, except in the top dimension.

Assuming this, take δ_s homology first. All terms vanish except top row, which becomes

(*)

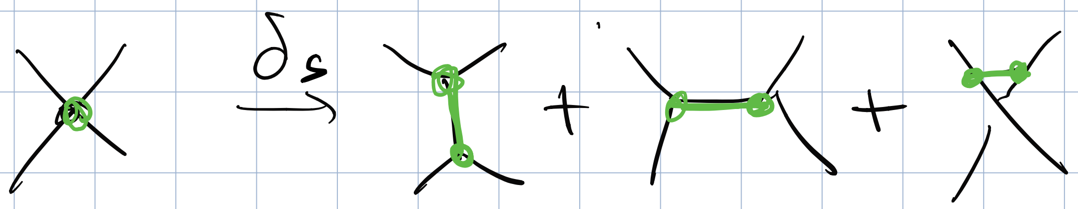
$$\frac{C_{2n-2,1}}{\operatorname{im}\delta_s} \leftarrow \cdots \leftarrow \frac{C_{2n-2,q-1}}{\operatorname{im}\delta_s} \xleftarrow{\delta_a} \frac{C_{2n-2,q}}{\operatorname{im}\delta_s} \leftarrow \cdots \leftarrow C_{2n-2,2n-2}$$

$$C_{2n-3,q} \xrightarrow{\delta_s} C_{2n-2,q}.$$ Image?

$C_{2n-2,q}$ is generated by trivalent graphs with q trees

$C_{2n-3,q}$ is generated by graphs with one 4-valent vertex, the rest trivalent

There are 3 ways to split the 4-valent vertex



ie the image of δ_s is the subspace spanned by the IHX relation.

So $(*)$ is exactly the Lie graph complex (reinterpreted as the forested graph complex.)

Proof of proposition: write the vertical chain complex horizontally (so it fits in these notes)
Look in CV_n first, then take quotient by $\mathrm{Out}\,F_n$

$$0 \to \hat{C}_{q,q} \to \cdots \longrightarrow \hat{C}_{2n-2,q} \longrightarrow \hat{C}_{2n-3,q} \xrightarrow{\delta_s} \hat{C}_{2n-2,q} \to 0$$

$(**)$

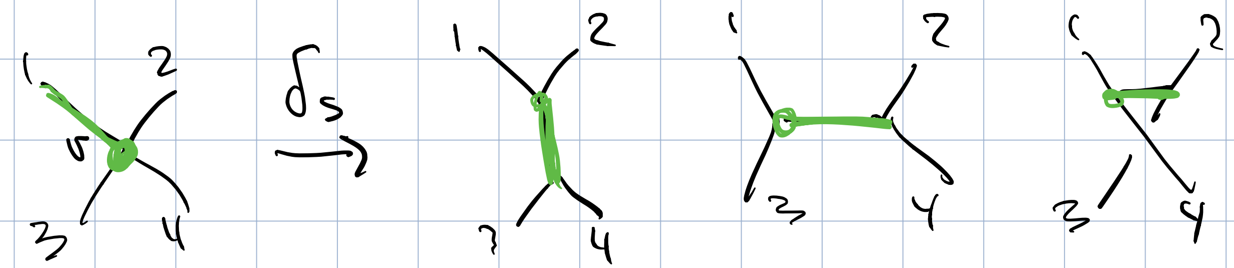
$\hat{C}_{g,q}$ has one generator for each (G,g)
$\#v(G) = q$

$(**)$ breaks up into a direct sum of chain complexes, one for each (G,g) s.t. G has q vertices (so $(G, v(G), g) \in \tilde{C}_{g,q}$)
Look at each of these separately.

Suppose, for simplicity, that G has only one vertex v that is not trivalent. (this is not a serious restriction---!)

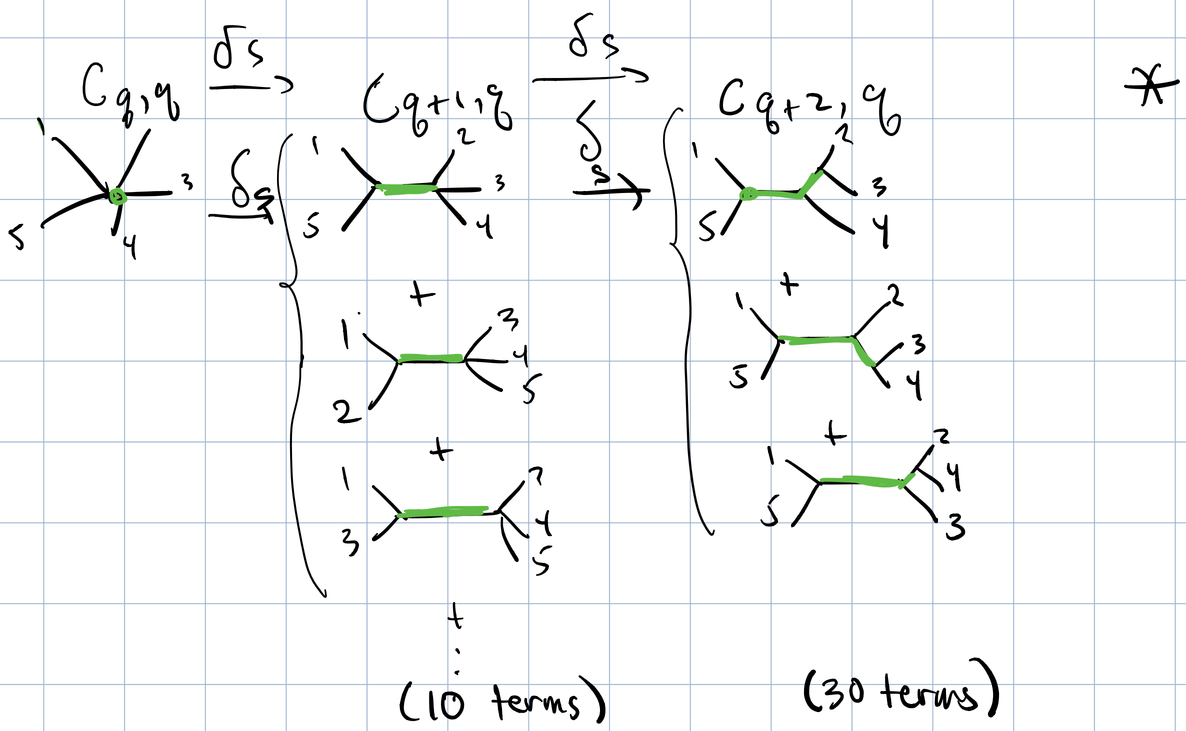
$\delta_s(G, v(G), g)$ splits v:

eg



The edges 1,2,...n are attached to the rest of the graph. The order matters... if you reattach the edges in a different order, you may get a non-isomorphic graph. You will definitely get a different marked graph, which is why we are working upstairs in $\hat{C}$ instead of. C.

If $|v| = 5$, you can split twice:



* is the augmented chain complex of a simplicial complex T_n with

- a 0-cell for each 1-edge tree with n labeled leaves
- a 1-cell for each 2-edge tree with n labeled leaves

⋮

- an (n-4) cell for each (n-3)-edge tree with n labeled leaves.