Lecture 8 Graph homology

We've related associative graph complexes to mapping class groups of punctured surfaces $S_{g,s}$, $s>0$

There's also a relation of commutative graph complexes with mapping class groups of closed surfaces S_g

<u>Reference</u>: Chan-Galatius-Payne
Tropical curves, graph complexes and top weight cohomology of M_g
arXiv 1805.10186

Recall commutative graph complex $CG_*^{(n)}$, even orientation is generated by connected graphs G with all

- $\pi_1 G \cong F_n$
- all vertices at least trivalent
- no odd symmetries (symmetries giving odd permutation of edges)

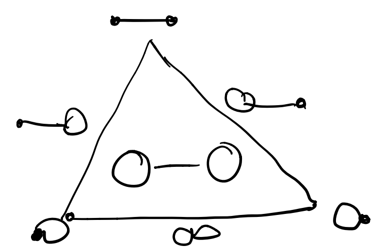
We saw

$$H_*(CG_*^{(n)}) \cong H_*(CV_n^*/\mathrm{Out}F_n,\ \partial CV_n^*/\mathrm{Out}F_n)$$

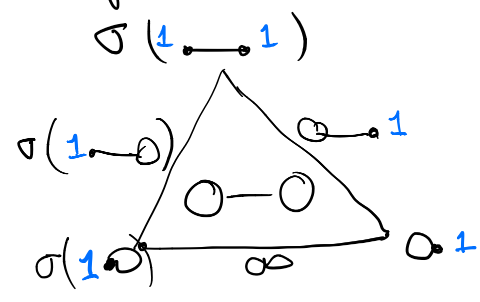
CGP call $CV_n^*/\mathrm{Out}F_n$ "the moduli space of tropical curves of genus n" and denote it Δ_n.

Points in $\partial CV_n^* = CV_n^* \setminus CV_n$ are the faces of simplices

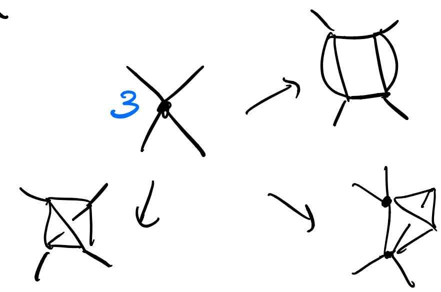
$\sigma(G,g)$ where some subgraph has "shrunk to 0"



We denote this by labeling the image of the subgraph with the ranks of the shrunken components:



We can expand a vertex with a positive weight m by inserting a rank m graph into G — in many ways:



ie if $G \in \partial CV_n^*$, $\sigma(G)$ is a face of many different simplices of CV_n

Remarks:

The idea of "inserting one graph into a vertex of another" makes the set of graphs with numbered vertices into something like a cyclic operad:

$O(n)$ = graphs w/ n vertices

Σ_n acts by permuting the vertices

$G_1 \circ_i G_2$ plugs G_2 into the ith vertex of G_1

The fact that there are many ways to plug G_2 into a vertex of G_1 makes it not quite fit our definition, but this can be fudged... see Willwacher.

Chan-Galatius-Payne observe

$C_* \Delta_n = C_* A_n \oplus C_* B_n$, where

$C_* B_n = \langle G \mid$ some vertex of G has label > 0 or G has a loop $\rangle$

$C_* A_n = \langle G \mid$ no positive labels, no loops $\rangle$

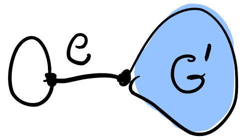
It's clear $C_* B_n$ is a subcomplex:
since collapsing an edge e either creates a positive label (if e is a loop) or preserves loops (if e is not a loop)

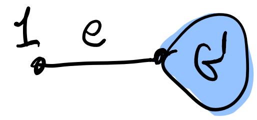
$C_* A_n$ is also a subcomplex, because:
No generator G of $C_* \Delta_n$ has a double edge, so you can't create a loop by collapsing an edge, and you can't create a positive label since $G \in C_* A_n$ has no loops.

Lemma (CGP) $C_* B_n$ is acyclic
(ie has no <u>reduced</u> homology)

Proof: Let G be a generator of $C_* B_n$

An edge of G is a <u>stem</u> if

$G =$ 

or $G =$ 

Let $C_{t,s} \subseteq C_* B_n$ be generated by graphs with s stems and t non-stem edges

Then $\partial : C_k B_n \longrightarrow C_{k-1} B_n$

$$\partial = \partial_s + \partial_t$$

where ∂_s collapses stems and ∂_t collapses non-stem edges

Then $C_k B_n = \bigoplus C_{s,t}$ forms a double complex

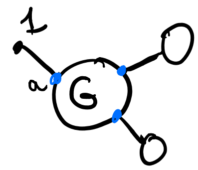
C_k $\qquad$ $C_{k-1} B$

$$\begin{tikzcd}
C_{s,t} \arrow[r, "\partial_t"] \arrow[d, "\partial_s"] & C_{s,t-1} \arrow[r] \arrow[d, "\partial_s"] & {} \\
C_{s-1,t} \arrow[r, "\partial_t"] & C_{s-1,t-1} \arrow[r] & {}
\end{tikzcd}$$

$\oplus$ $\qquad$ $\oplus$

To compute its homology, first compute the vertical homology: $C_{s,t} = \bigoplus_G R^G$

$\downarrow$

$C_{s,t}$ 

(max no of stems is n-genus(G))

$\downarrow \qquad\qquad \downarrow \partial_s$

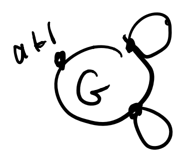
$C_{s-1,t}$

R^G = graphs obtainable from G by growing stems.

$\downarrow \qquad\qquad \downarrow$

$\vdots$

$\downarrow$

$C_{0,t} \ni$ 

$\leftarrow$ no stems.

Grow one stem from G and set $\tilde{G} =$ 

Graphs in R^G are all compatible w/ $\tilde{G}$: either they already contain s, or you could add s to them.

R^G is the augmented chain complex of a poset, where the poset relation is stem collapse.

$G' \to G' \cup s \to \tilde{G}$ is a poset map, giving a retraction $R^G \simeq pt$.

This works unless $t=0$

because $G = \overset{1}{\circ}\!\text{———}\!\overset{n-1}{\circ}$, is the only graph w/ $t=0$, and $\tilde{G} = \overset{1}{\circ}\!\text{———}\!\underset{n-2}{\circ}\!\text{———}\!\overset{1}{\circ} = 0$

so the last column gives $\mathbb{Q} \to \mathbb{R} \to 0$

$H_0 = \mathbb{R}$. But we will take <u>reduced</u> homology of Δ_n by adding the graph $\overset{n}{\bullet}$

<u>Remark</u>

We have implicitly assumed the graphs in R^G have no symmetries, but this can be taken care of by an equiv. homology spectral sequence, since R^G is the quotient of an acyclic cplex by a finite group.

<u>Cor</u>:
$$H_*(C_*A_n) \cong \tilde{H}_*(C_*\Delta_n) = \tilde{H}_*(\Delta_n)$$
$$\cong H_*(C_*\Delta_n, C_*B_n)$$

Exercise A similar proof shows $C_* \Delta_n^\infty$ is acyclic, so by les of the triple $(\Delta_n, B_n, \Delta_n^\infty)$ we get

$$\tilde{H}_*(C_*\Delta_n) = H_*(C_*\Delta_n, C_*\Delta_n^\infty)$$

(which we previously identified with Kontsevich's commutative graph homology $H_*(\mathcal{CG}_*^{(n)})$)

$\Longrightarrow$ end of Part I

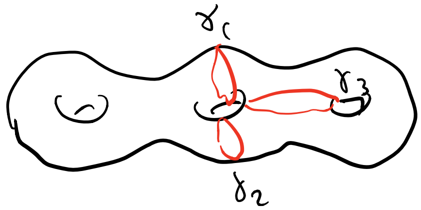
Relation with $\mathrm{Mod}(S_n)$?

CGP observe $\Delta_n = CV_n^* / \mathrm{Out} F_n \cong \mathcal{C}(S_n) / \mathrm{Mod}(S_n)$

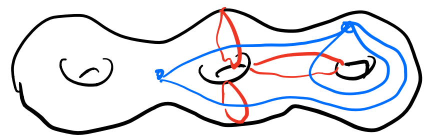
where $\mathcal{C}(S_n)$ is the <u>curve complex</u> of S_n

To see this,

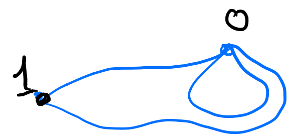
Let $\bar{\gamma} = \{\gamma_0, \ldots, \gamma_k\}$ be a curve system on $S = S_n$



There is a dual graph



- Label each vertex with the genus of the corresp. surface



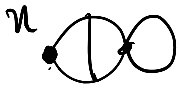
A simplex in $\mathcal{C}(S)$ is given by weighting each γ_i so that $\sum wt(\gamma_i) = 1$

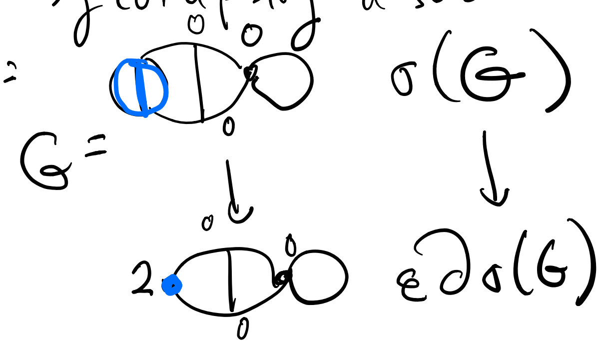
We can think of these weights as lengths on the edges of G.

The action of Mod(S) doesn't change the homeomorphism type of the complementary surfaces, which is determined by the genus and # of ∂ components, by the classification of surfaces

So this labeled graph G is well-defined modulo Mod(S).

It also gives a unique point in $CV_n^*/\mathrm{Out}F_n$

You get  by collapsing a sub-graph of rank n:



This is a codim 3 face of $\sigma(G)$

It's a face of <u>every</u> $\sigma(G')$ where G' is obtained by inserting a rank 2 graph into the vertex labeled "2"

(Modulo $\mathrm{Out}(F_n)$, the marking is irrelevant.)

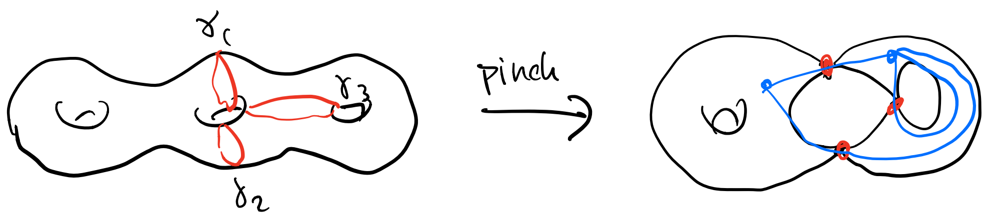
To make the connection with $\mathrm{Mod}(S_g)$, recall $\mathrm{Mod}(S_g)$ acts properly on Teichmüller space $\mathcal{T}_g$, and the quotient $\mathcal{M}_g$ is called the moduli space of Riemann surfaces.

Since $\mathcal{T}_g$ is contractible and the action is proper,

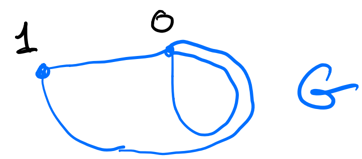
$$H_*(\mathcal{M}_g;\mathbb{Q}) = H_*(\mathrm{Mod}\, S_g;\mathbb{Q})$$

$\mathcal{M}_g$ is a complex variety but is non-compact

Pinching some simple closed curves on S_g gives a ray going off to infinity (leaving every compact set):



Label each vertex of the dual graph G by the genus of the corresponding pinched surface

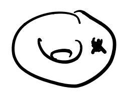


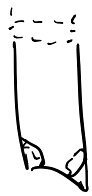
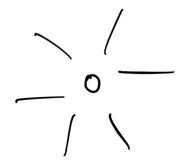
If each component of $S \setminus \bar{\gamma}$ is a sphere with holes then $\operatorname{rank}(G) = \operatorname{genus}(S)$

In general $\operatorname{genus}(S) = \operatorname{rank} G + \sum(\text{vertex labels})$

Pinching the γ_i at different rates gives a simplex of inequivalent rays.

The **Deligne-Mumford compactification** $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ adds a product of moduli spaces of the components to $\mathcal{M}_g$, making it into a projective variety (i.e., it is compact)

eg $S =$  $\mathcal{M}_{g,1} = \mathbb{C}_{\mathrm{Re}\, z>0}/SL_2\mathbb{Z}$

$=$  $\leftarrow \infty$ cylinder . Nbhd of ∞ is conformally 

To get $\overline{\mathcal{M}}_{g,1}$ add the missing point ∞; then $\overline{\mathcal{M}}_{g,1}$ is compact

$\overline{\mathcal{M}}_g - \mathcal{M}_g$ has a dual complex, which is Δ_g
($\overline{\mathcal{T}}_g - \mathcal{T}_g$ covered by products of Teichmüller spaces for the components we pinched off
The (nerve of this cover/$\mathrm{Mod}(S_g)$) $= \Delta_g$.)

The quotient of $\sigma(G) \in C(S)$ modulo $\mathrm{Mod}(S)$ (or $\sigma(G) \in CV_n^*$ modulo $\mathrm{Out}(F_n)$) is a rational cell (cone on rat'l homology sphere) unless G has an automorphism inducing an odd permutation of its edges.

$H^*(\mathcal{M}_g)$ has a "weight-filtration"
this is related to Hodge theory of
the cohomology with complex coefficients.

CGP say
the "Top weight cohomology" of $\mathcal{M}_g$
$\cong \tilde{H}_*(\overline{\mathcal{M}}_g \setminus \mathcal{M}_g)$

specifically: $Gr^W_{2d}(H^{2d-k}(\mathcal{M}_g)) \cong \tilde{H}_{k-1}(\Delta_n; \mathbb{Q})$
($d = \dim \mathcal{M}_g$, with the appropriate grading of $\tilde{H}_*(\Delta_n)$)

By the discussion last hour, $\tilde{H}_*(\Delta_n; \mathbb{Q}) = H_*(C\mathcal{G}_*)$

By Willwacher's theorem, $H^0(C\mathcal{G}_*) \cong \mathfrak{grt}_1$
By Brown's theorem, $\mathfrak{grt}_1$ contains a free Lie algebra on generators $\sigma_3, \sigma_5, \ldots$
So we get lots of nontrivial classes in $H_*(C\mathcal{G}_*)$
CGP translate this into the statement that these give top weight classes in $H^*(\mathcal{M}_g)$

Why is $H^0(\mathcal{CG}_*)$ a Lie algebra?

This uses the operation of inserting graphs into vertices of other graphs:

Willwacher uses $\deg(G) = v(G) - rk(G) - 1$
$= e(G) - 2rk(G)$

Define $G_1 \cdot G_2 = \sum_{v \in G_1}$ (graphs obtainable by inserting G_2 into v)

(If G_i are commutative graphs, order the G_2-edges after the G_1-edges.)

exercise: $\deg(G_1 \cdot G_2) = \deg G_1 + \deg G_2$

Now define

$$[G_1, G_2] = G_1 \cdot G_2 - (-1)^{(\deg G_1)(\deg G_2)} G_2 \cdot G_1$$

In particular, this gives C_0 a Lie algebra structure.

Exercise $[\cdot,\cdot]\big|_{C_0}$ is anti-symmetric and satisfies the Jacobi identity

(in general it is graded anti-symmetric and satisfies the graded Jacobi identity)

Exercise: Show $\delta G = [\bullet\!-\!\bullet, G]$

Conclude $[\cdot,\cdot]$ induces a Lie algebra structure on $H^0(Cg_*) = \ker \delta_0$