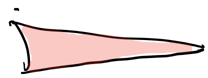
Lecture 2

Last time we considered several ways of finding a compact space closely related to

$M =$  $= \mathbb{H}/SL_2\mathbb{Z}$, namely

* add a point $\overline{M}^S$ (Satake compactification)
* add a circle $\overline{M}^{BS}$ (Borel-Serre compactification)
* cut off the cusp M^ε (Grayson)
* retract to a spine T (Ash's well-rounded retract)

I claimed M parameterizes lots of things:

lattices, flat tori, hyperbolic structures on a punctured torus, elliptic curves, positive definite quadratic forms, metric graphs with fundamental group F_2, weighted sphere systems, ...

We started with lattices, spanned by $u, v \in \mathbb{R}^2$

$$\mathcal{L} = GL_2\mathbb{R} / GL_2\mathbb{Z}$$

$$\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v \mapsto \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix} \cdot GL_2\mathbb{Z}$$

If we don't want to distinguish lattices Λ which are rotations, reflections or multiples of each other, i.e. lattices of the same "shape", the paramater space is

$$H \cdot O(2) \backslash GL_2\mathbb{R} / GL_2\mathbb{Z}$$

where $H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \middle| \lambda > 0 \right\}$

If we want to keep the choice of basis but not distinguish lattices w/ the same shape, get

$$X = O(2) \cdot H \backslash GL_2\mathbb{R}$$ = moduli space of marked lattices L

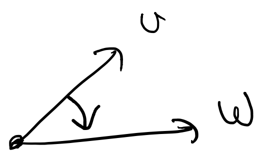
$\approx i$ (marking = choice of basis)

$$SO(2) \backslash SL_2\mathbb{R}$$

Identification of $X = H \cdot O(2) \backslash GL_2\mathbb{R}$ with upper half-plane $\mathbb{H}$:

<u>Remember</u>: We have $SL_2\mathbb{Z}$ acting on the <u>right</u>.

$$z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+c}{bz+d}$$



$\rightsquigarrow \quad L = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix}$ (<u>note</u> $\det L < 0$)

rotate until w is on the positive x-axis

$$r_\theta \begin{pmatrix} v, w \end{pmatrix} = \begin{pmatrix} \bullet & |w| \\ \bullet & 0 \end{pmatrix} = \begin{pmatrix} \frac{v \cdot w}{|w|} & |w| \\ -\frac{\det}{|w|} & 0 \end{pmatrix}$$

(det hasn't changed so $\bullet = -\frac{\det}{|w|}$

dot product hasn't changed, so $\bullet = \frac{v \cdot w}{|w|}$)

now normalize so $|w| = 1$:

$$\begin{pmatrix} \frac{1}{|w|} & 0 \\ 0 & \frac{1}{|w|} \end{pmatrix} \begin{pmatrix} \frac{v \cdot w}{|w|} & |w| \\ -\frac{\det}{|w|} & 0 \end{pmatrix} = \begin{pmatrix} \frac{v \cdot w}{w \cdot w} & 1 \\ -\frac{\det}{w \cdot w} & 0 \end{pmatrix} := \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$$

so $KH\, L = KH \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$ $\quad (K = SO(2))$

identify this coset with $x + iy \in \mathbb{H}$.

Note $GL_2\mathbb{Z}$ also acts on $KH\backslash GL_2\mathbb{R}$

What is the action on $\mathbb{H}$? suppose $\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=-1$

$$z\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix} = (x+iy)\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$$\updownarrow$$

$$\overbrace{KH\begin{pmatrix}x&1\\y&0\end{pmatrix}}\begin{pmatrix}a&b\\c&d\end{pmatrix} = KH\cdot\begin{pmatrix}ax+c & bx+d\\ ay & by\end{pmatrix}$$

negative

$$= KH\begin{pmatrix}\dfrac{(ax+c)(bx+d)+aby^2}{(bx+d)^2+b^2y^2} & 1\\ \dfrac{-y}{(bx+d)^2+b^2y^2} & 0\end{pmatrix}$$

Exercise
$$= KH\begin{pmatrix}\mathrm{Re}\left(\dfrac{az+c}{bz+a}\right) & 1\\ \mathrm{Im}\left(\dfrac{az+c}{bz+d}\right) & 0\end{pmatrix}$$

So define $z\cdot\begin{pmatrix}a&b\\c&a\end{pmatrix} = \dfrac{a\bar{z}+b}{c\bar{z}+d}$ if $ad-bc=-1$

Why am I opting for a right action of $SL(2,\mathbb{Z})$?

I like to think of a marked lattice as a <u>function</u>

$$\mathbb{Z}^2 \xrightarrow{L} \mathbb{R}^2 \qquad (v = Le_1\ ,\ w = Le_2)$$

$A \in GL_2\mathbb{Z}$ acts on the right by pre-composing

$$\begin{tikzcd} \mathbb{Z}^2 \arrow[r, "L"] & \mathbb{R}^2 \\ \mathbb{Z}^2 \arrow[u, "A"] \arrow[ur, "LA"'] & \end{tikzcd}$$

(doesn't change image of L, just its <u>marking</u>, ie its basis's)

Rotation/homothety acts on the <u>left</u>

(you can rotate $\mathbb{R}^2$ by θ but not $\mathbb{Z}^2$)

$$\begin{tikzcd} \mathbb{Z}^2 \arrow[r, "L"] \arrow[dr, "r_\theta \circ L"'] & \mathbb{R}^2 \arrow[d, "{r_\theta,\ h,\ \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)}"] \\ & \mathbb{R}^2 \end{tikzcd}$$

X as a moduli space of quadratic forms

Note: for any $M \in GL_2\mathbb{R}$, $Q = M^t M$ is symmetric, has $\det > 0$, and $Q_{11} > 0$; ie Q is the matrix of a positive definite quadratic form, namely $q(x) = x^t Q x$

$\langle x, y \rangle_Q = x^t Q y$ is the associated bilinear form (inner product)

Given $L = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$ set $Q = L^t L$

Rotating $L \rightsquigarrow r_\theta L$ (or reflecting)

has no effect on Q: $(r_\theta L)^t (r_\theta L)$

$= L^t r_\theta^t r_\theta L = L^t L = Q$

So get a well-defined map

$$X \longrightarrow \text{pos. def. q. forms of det } 1$$

Changing basis changes Q:

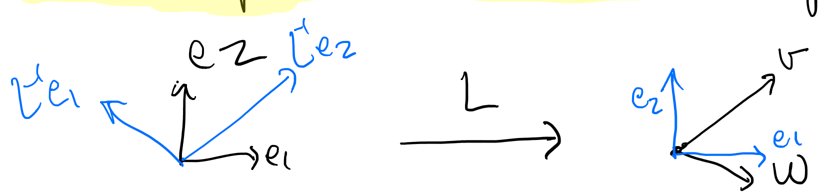
$$A \in GL_2\mathbb{Z} \Rightarrow (LA)^t(LA) = A^t L^t L A$$
$$= A^t Q A$$

$Q \longmapsto A^t Q A$ is a <u>right action</u> of $GL_n\mathbb{Z}$ on the space of quadratic forms.

If $Q' = Q \cdot A = A^t Q A$,

$$\langle x, y \rangle_{Q'} = x^t Q' y = x^t A^t Q A y$$
$$= \langle A x, A y \rangle_Q$$

relation of Q with v, w = columns of L?



$$\langle L^{-1}e_1, L^{-1}e_2 \rangle_Q = e_1^t L^{-1t} L^t L L^{-1} e_2 = e_1^t e_2 = 0$$

the columns of $\underline{\underline{L^{-1}}}$ are Q-orthogonal!

Why you might prefer a left action

It might seem more natural to associate to a marked Lattice $\mathbb{Z}v \oplus \mathbb{Z}w$ the quadratic form Q which makes v and w orthonormal:

$$v^t Q w = (Le_1)^t Q (Le_2) = e_1^t L^t Q L e_2$$

so take $Q = (L^t)^{-1} L^{-1}$

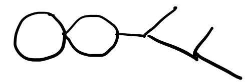
To make this invariant under rotations r_θ you need r_θ to act on the <u>right</u>, ie on row vectors $L \mapsto L.r_\theta$.

so $SL_2\mathbb{Z}$ acts on the <u>left</u>:

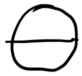
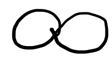
$$L \mapsto AL$$

sends $\quad Q \mapsto (A^{-1})^t Q A^{-1}$

X as space of marked metric graphs Γ with $\pi_1 \cong F_2$, volume 1, no separating edges

<u>no sep edges</u>: eliminates 

or 

all that's left are $\Gamma =$  and $\Gamma =$ 

<u>marking</u>: h. equiv $\quad R_0 = \overset{a}{\bigcirc}\!\!\overset{b}{\bigcirc} \xrightarrow{\ \gamma\ } \Gamma$

identifies $F\langle a,b\rangle$ with $\pi_1\Gamma$.

There is a "Jacobian map":

J: Marked metric graphs $\longrightarrow$ pos. def. quadratic forms

Def of J

$$R_0 \xrightarrow{\gamma} \Gamma$$

induces
$$\mathbb{R}^2 \equiv H_1(R_0;\mathbb{R}) \xrightarrow{\cong} H_1(\Gamma;\mathbb{R})$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad \ker\left(\mathbb{R}^{E(\Gamma)} \xrightarrow{\partial} \mathbb{R}^{V(\Gamma)}\right)$$

Equip $\mathbb{R}^{E(\Gamma)}$ with the pos. def form $\begin{pmatrix} l_1 & & 0 \\ & l_2 & \\ 0 & & \ddots\ l_e \end{pmatrix}$

where l_i = length of i^{th} edge.

Restrict this to $H_1(\Gamma;\mathbb{R})$ to get a pos. def Q. form on $H_1(\Gamma;\mathbb{R}) \subset \mathbb{R}^{E(\Gamma)}$

Prop: J is bijective for $n=2$

$\alpha \in \mathrm{Out}(F_2)$ acts on marked metric graphs

$\alpha\cdot(\Gamma,\gamma) = (\Gamma, \gamma\circ\alpha)$ (on the <u>right</u>)

$$\begin{tikzcd}
R_0 \arrow[r, "\gamma"] & \Gamma \\
R_0 \arrow[u, "\alpha"] \arrow[ur, "\gamma\circ\alpha"'] &
\end{tikzcd}$$

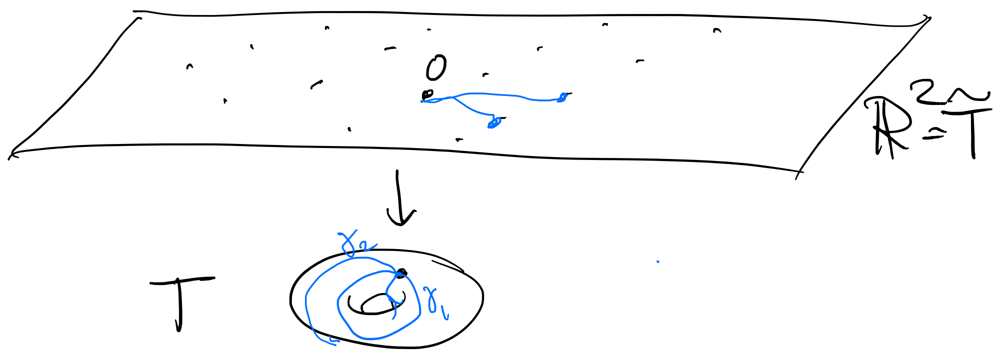
Prop: $\mathrm{Out}(F_2) \longrightarrow \mathrm{Out}\,\mathbb{Z}^2 = GL_2\mathbb{Z}$ is an isomorphism.
$\qquad\qquad\quad \alpha \longmapsto A$

Prop J commutes with the $GL_2\mathbb{Z}$-action

$$\begin{array}{ccc} \alpha \in \mathrm{Out}\,F_2 \\ \downarrow \quad \parallel 2 \\ A \in GL_2\mathbb{Z} \end{array} \qquad \begin{tikzcd} \text{Graphs} \arrow[r, "J"] \arrow[d, "\alpha"] & \text{Forms} \arrow[d, "A"'] \\ \text{Graphs} \arrow[r, "J"] & \text{Forms} \end{tikzcd}$$

X is also a moduli space of flat tori of area 1.

A flat torus is a metric torus T^2 whose universal cover is $\mathbb{R}^2$



Lift a point to determine the origin, other lifts give a lattice only defined up to rotation, reflection (and homothety — area is 1)

- choosing a basis for $\pi_1(T) \cong \mathbb{Z}^2$: Lift the loops to $\tilde{T}$ ⇝ vectors v, w in $\mathbb{R}^2$.

X is a moduli space of hyperbolic structures on a once-punctured torus

A flat metric on a torus gives a flat metric on the torus minus a point ($T_{1,1}$), and this gives a way of measuring angles — a conformal structure.

The universal cover of $T_{1,1}$ is conformally equivalent to the hyperbolic plane, by the uniformization theorem.

So the flat metric on $T_{1,1}$ gives rise to a hyperbolic structure on $T_{1,1}$

X as space of elliptic curves

$$\Lambda \rightsquigarrow \wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

Weierstrass p-fcn

$\wp_\Lambda$ is doubly periodic, poles at lattice pts

satisfies $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$

where g_2, g_3 depend on Λ

Ie the pts $(\wp(z), \wp'(z))$ lie on the

elliptic curve $y^2 = 4x^3 - g_2 x - g_3$

Borel-Serre

Works for G any non-compact semisimple algebraic group defined over $\mathbb{Q}$,

K = maximal compact subgroup
Γ = arithmetic subgp

We'll use $G = SL_n\mathbb{R}$, $K = SO(n)$,
$\Gamma = SL_n(\mathbb{Z})$

Symmetric space $X = K\backslash G$

$$= SO(n)\backslash SL_n(\mathbb{R})$$

We'll often care more about GL_n, so note

$$X = O(n)\cdot H\backslash GL_n\mathbb{R}$$

$$\left(H = \left\{ \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \middle| \lambda > 0 \right\} \right)$$

Prop X is homeomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}-1}$

Pf This is easiest to see using the description of X as the space of (homothety classes of) positive definite quadratic forms q.

Given $q_1, q_2 \in X$, then for $0 \leq t \leq 1$

$t q_1 + (1-t) q_2$ is also positive definite! So is λq, for $\lambda > 0$

So X is a (slice of a) convex cone in $\mathbb{R}^{\frac{n(n+1)}{2}}$ (q is given by a symmetric $n \times n$ matrix)

$\blacksquare$

G acts on the right on X,
and the quotient by $\Gamma < G$ is not compact

The Borel-Serre bordification $\overline{X}^{BS} = \overline{X}$
is an enlargement of X, satisfying

* $\overline{X}$ is contractible,
* The action ~~of Γ~~ extends continuously
* stabilizers are finite, and
* $\overline{X}/\Gamma$ is compact.

For $n=2$ $\overline{X}$ is obtained by adding a line at ∞, and at every rational point

For $n>2$, we will add a Euclidean space for every rational parabolic subgp

How to get from points to subgps?
look at stabilizers

$$\text{stab}(\infty) = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\} = \text{stab}(1,0) \text{ under}$$

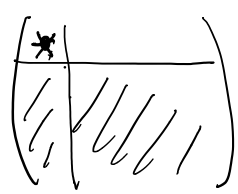
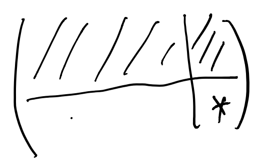
$\left(= \text{stab} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right.$ under left action!$\Big)$ right action of $GL_n\mathbb{R}$

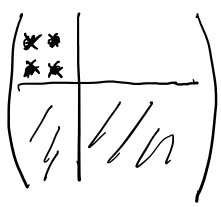
Def: A parabolic subgroup (of $GL(n,\mathbb{R})$) is the stabilizer of a subspace V or of a chain of subspaces $V_1 \subset \dots \subset V_k$

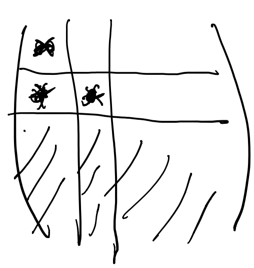
A subspace is rational if it is the solution space of a set of equations with rational coefficients

A rational parabolic subgroup is the stabilizer of a (chain of) rational subspaces

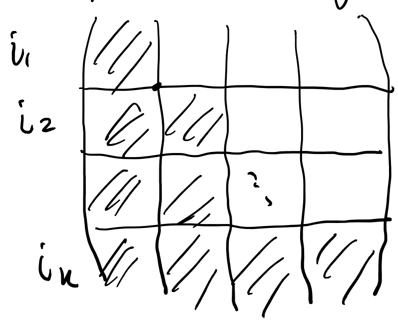
eg $\{e_i\}$ = standard basis elts for $\mathbb{R}^n$

$P_1 = \mathrm{stab}\langle e_1\rangle =$  $\mathrm{stab}\langle e_n\rangle =$ 

$P_2 = \mathrm{stab}\langle e_1, e_2\rangle =$  (zeroes in blank areas)

$\mathrm{stab}\ \langle e_1\rangle \subset \langle e_1, e_2\rangle =$  $= P_1 \cap P_2$

The standard parabolics are the block lower triangular subgroups



$= \operatorname{stab}\ E_1 \subset E_2 \subset \cdots \subset E_{k-1}$

$\qquad \langle e_1,\dots,e_{i_1}\rangle \subseteq \langle E_1, e_{i_1+1},\dots,e_{i_1+i_2}\rangle \quad \cdots$

Every parabolic is conjugate to one of these

$P = \operatorname{stab}\ V_1 \subset \cdots \subset V_{k-1}$

$g \in GL_n$ sending $e_1,\dots,e_{i_1}$ to a basis for V_1, extended by $e_{i_1+1}\dots e_{i_1+i_2}$ to a basis for V_2, etc

then gPg^{-1} is standard.