Tokyo July 30, 2014

A review of Outer Space for F_n

What it is

What it's good for

Recent discoveries

F_n = free of rank $n < \infty$ = $F\langle a_1, \dots, a_n \rangle$

Want to study its automorphisms.

Eg ρ_{ij} : $a_i \longmapsto a_i a_j$

$a_k \longmapsto a_k \quad k \neq i$

ε_i : $a_i \longmapsto a_i^{-1}$

$\mathrm{Aut}(F_n)$

$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n) / \mathrm{Inn}(F_n)$

<u>Thm (Nielsen)</u>

$\mathrm{Aut}(F_n)$ is generated by the ρ_{ij} and ε_1

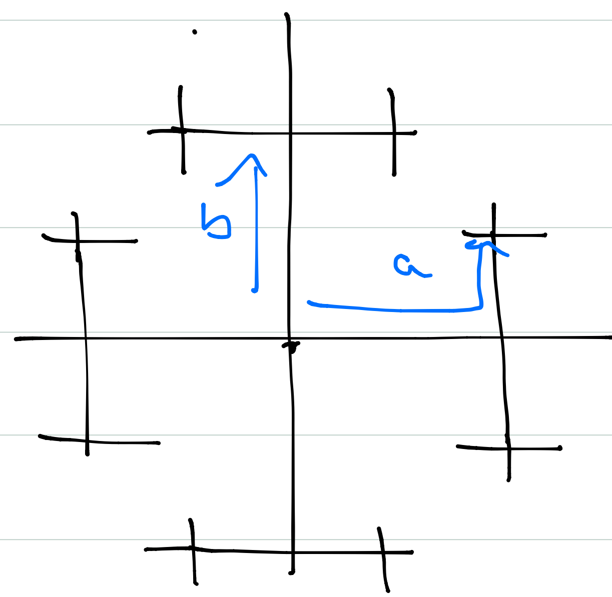
Outer space = geometric model for $\mathrm{Out}(F_n)$

$\longrightarrow$ nice space with proper action

Most succinct definition:

Space of free minimal actions of F_n
by isometries of metric simplicial trees

e.g. $F_2 = F\langle a,b \rangle$
acts on



edges have lengths in $\mathbb{R}_{>0}$

minimal = no invariant subtrees

F_2 also acts in other ways. Given any automorphism $\alpha : F_2 \to F_2$, modify the action by α

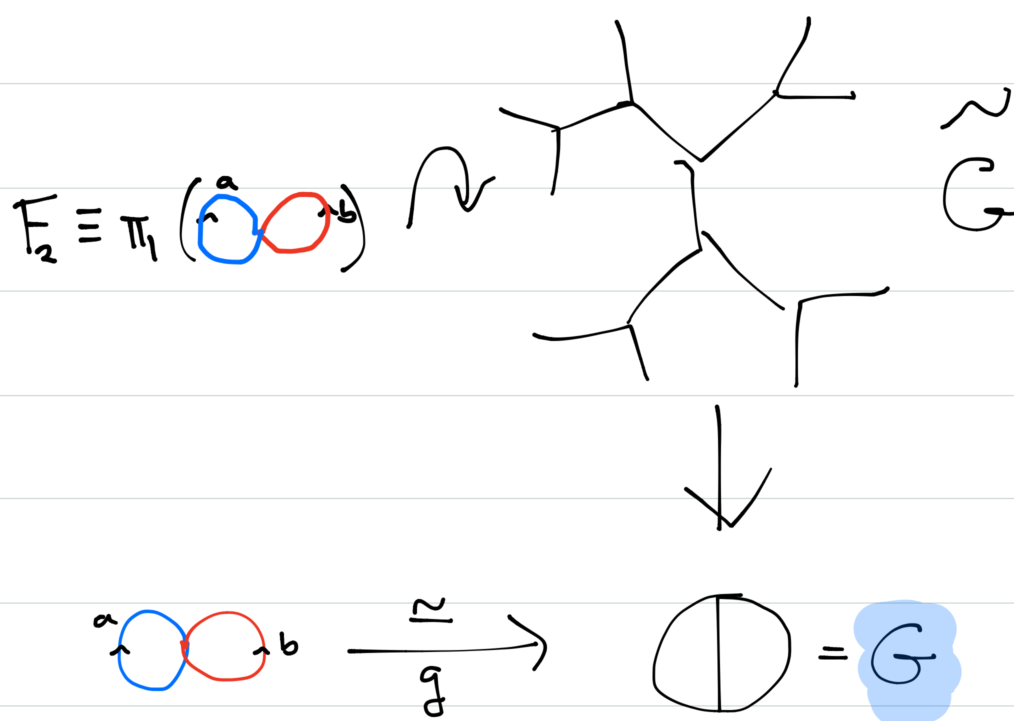
eg $\alpha = \rho_{ab} : a \longmapsto ab$

So $\mathrm{Aut}(F_n)$ acts on the set of actions!.

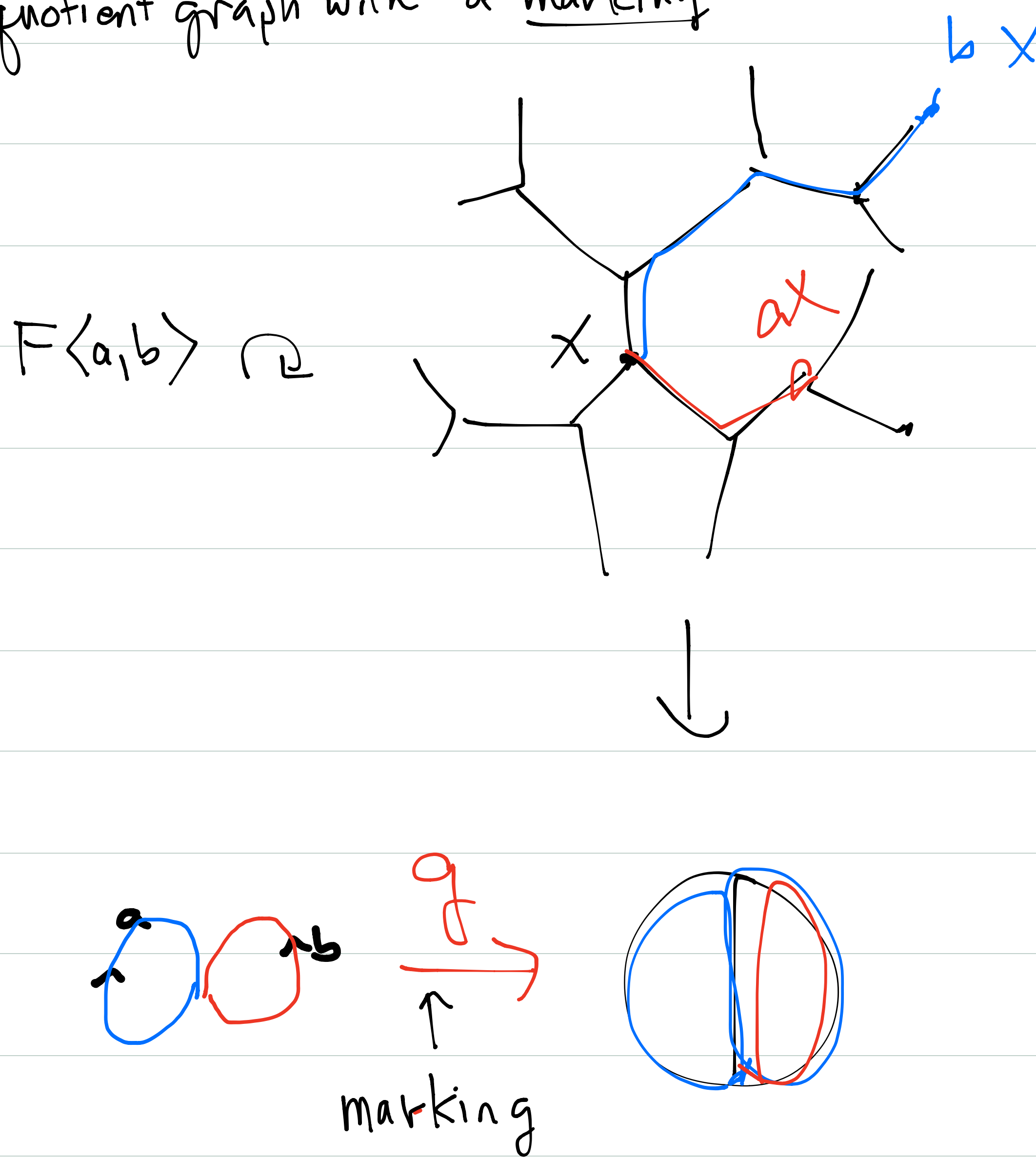
$\mathrm{Inn}(F_n)$ acts trivially

There are also other trees w/ an F_2-action: G any graph with $\pi_1(G) \cong F_2$

Then F_2 acts on $T = \tilde{G}$ by deck transformations



In fact, given any free minimal action of F_n on a metric simplicial tree, get a quotient graph with a <u>marking</u>



free guarantees $\pi_1(T/F_n) \cong F_n$

metric on T induces metric on quotient

minimal $\Rightarrow$ quotient graph has no univalent (or bivalent) vertices

Second definition of Outer space:

$$\boxed{\text{Space of marked metric graphs } (g,G) \,/\!\sim}$$

G = metric graph, all vertices of valence ≥ 3

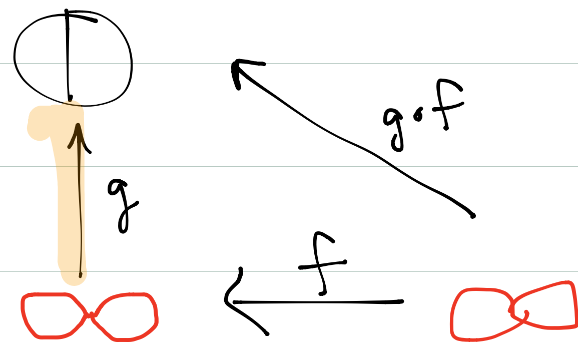
$g : R_n \longrightarrow G$ a homotopy equivalence

$(G,g) \sim (G',g')$ if $\exists$ isometry $G \xrightarrow{h} G'$

making
$$\begin{tikzcd} G \arrow[rr, "h"] & & G' \\ & R_n \arrow[ul, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$$
commute. (up to homotopy)

Action of $\varphi \in \mathrm{Out}(F_n)$: Realize φ by

$f: R_n \longrightarrow R_n$ a homotopy equivalence



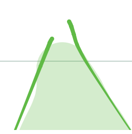
$(G, g)\cdot\varphi = (G, g \circ f)$

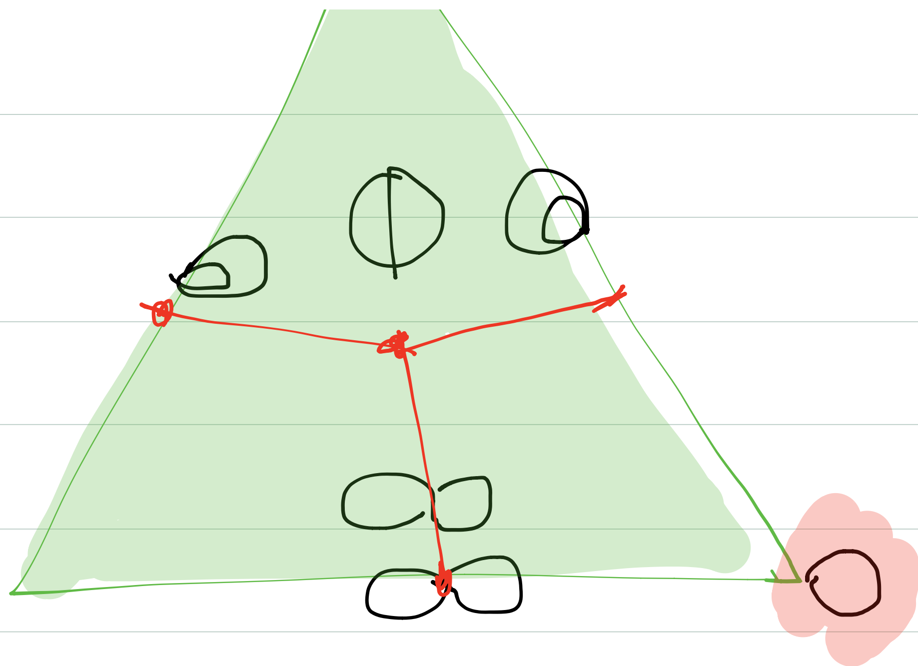
Action changes the marking, not the metric graph

Advantages of the graph description of Outer space

- Description of point stabilizers ($\mathrm{Isom}(G)$)
- Decomposition into open simplices
- Cocompact deformation retract (spine)
- Natural (non-symmetric) metric

Assume $\sum(\text{edge lengths}) = 1$





Outer space CV_n =

Space of marked metric graphs (g, G) $/\sim$

Spine of Outer space K_n

= realization of p.o.set of open simplices

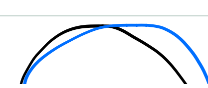
= realiz of poset of marked graphs (no metric) under forest collapse

There are other definitions of Outer space, its quotient and its spine

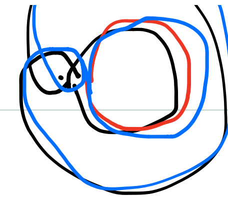
- sphere systems in $\#_n S^1 \times S^2$
- moduli space of tropical curves
- Certain Radon measures on $\partial^2 F_n$
- cube complex

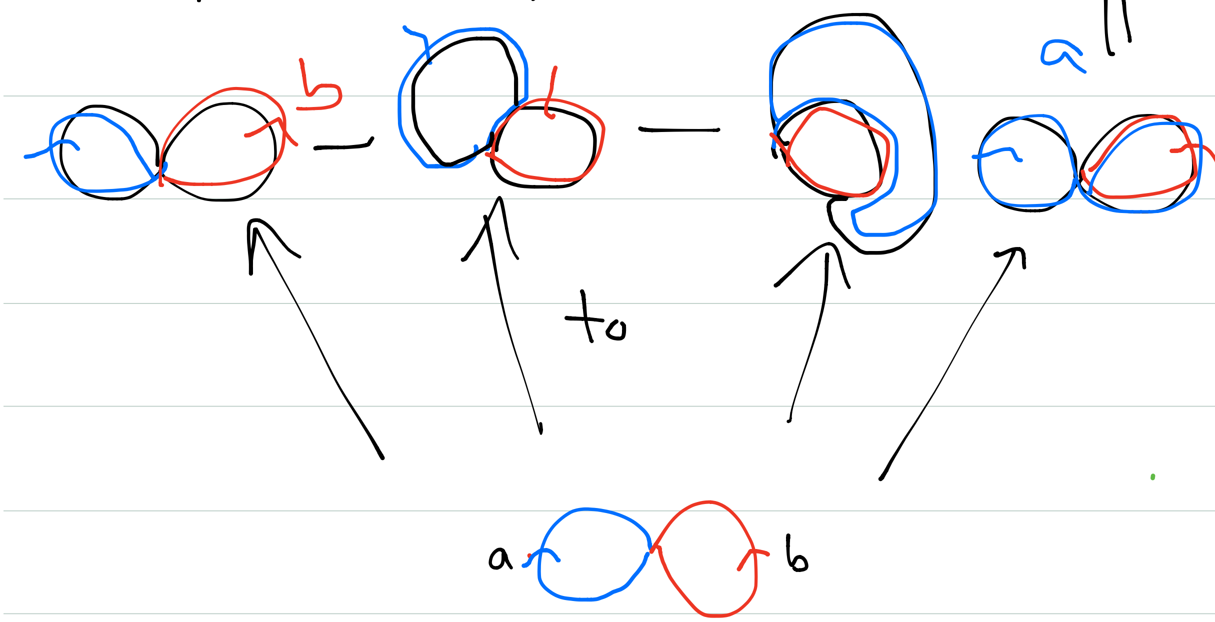
etc   Each with its own advantages

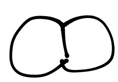
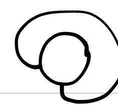
Exploring

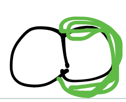
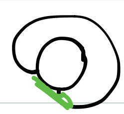


A path in Outer space from





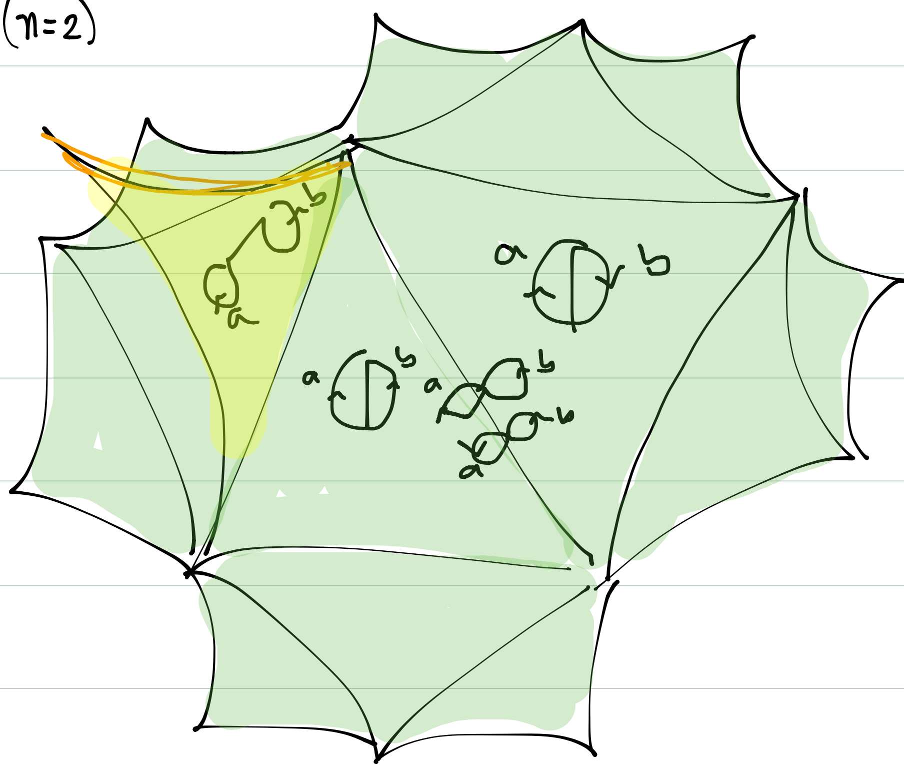
The intermediate graphs  and 
can be collapsed to either rose by
collapsing the appropriate maximal tree

 and   and 

So Outer space is connected...

Thm (Culler-V 86): Outer space is contractible, the action is proper, the spine is a cocompact deformation retract of dim $2n-3$.

Picture $(n=2)$



There are now several proofs of contractibility

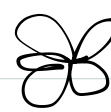
At least one using each description –

(my favorite uses sphere complexes but that won't help with tomorrow's lecture)

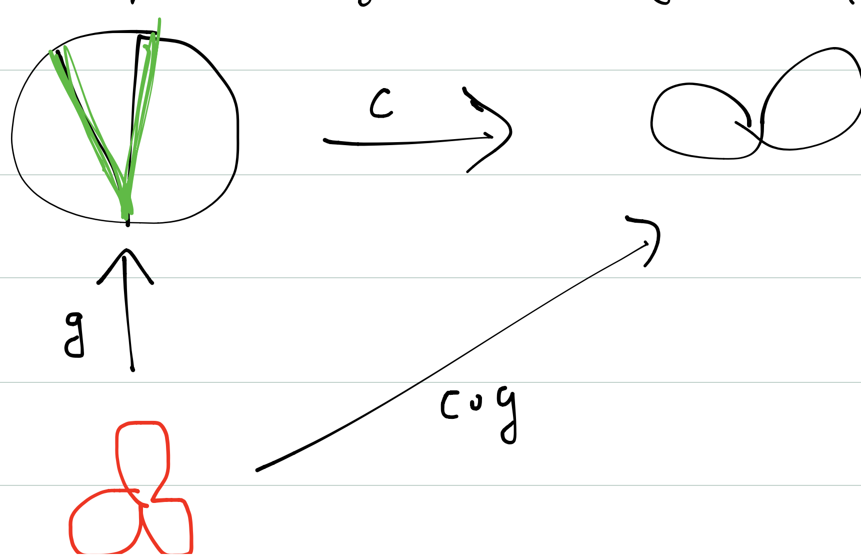
Here's a very rough idea of the original proof of contractibility:

(1) Retract CV_n onto K_n (the spine)

(2) Notice $K_n = \bigcup st(p)$ $\quad p = (g, R)$

$R =$  a rose

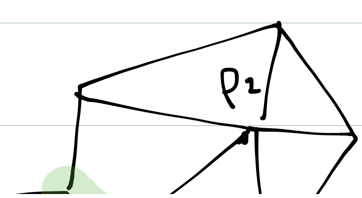
because collapsing any maximal tree in Γ gives an edge $(\Gamma, g) — (R, c \circ g)$ in K_n

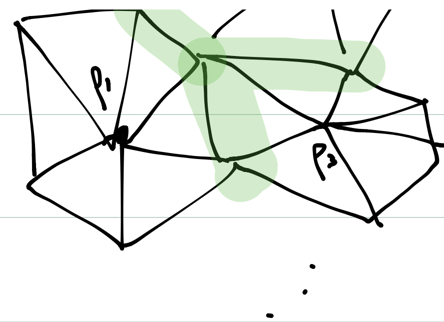


(3) Order the roses (g,R) in some natural way (order conjugacy classes w in F_n, measure the length of each $g(w)$ in G, order the roses lexicographically using this list of lengths)

(4) Prove this is a well-ordering

(5) Build K_n by gluing the stars of roses on in the order you have. Prove that at each stage you are gluing along something contractible.





This involves understanding how elementary mores (eg folds) affect lengths of conjugacy classes.

What Outer space is good for:

(1) G acts freely on contractible space $X \Rightarrow$ homotopy type of X/G depends only on G (Hurewicz)

(2) G acts properly cocompactly on contractible $X \Rightarrow G$ is quasi-isometric to X. (Svarc-Milnor)

Sample applications

(1) — Any tffi subgp of $\mathrm{Out}(F_n)$ acts freely on K_n so has cohomological dimension $\leq \dim K_n$

— Combinatorial methods can be used to compute $H^*(\mathrm{Out}\, F_n; \mathbb{Q})$ for n small

- $H_c^*(CV_n)$ is concentrated in one dimension $\Rightarrow \exists$ duality between H^* and H_* (Bestvina-Feighn)

(2) - Ends (K_n) = Ends $(Out(F_n))$ $(= 1 \quad n \geq 3)$

- Dehn function $(Out(F_n))$ = Isoperimetric fcn (K_n)

exponential : f $n \geq 3$

- Can play ping-pong on Outer space - one ingredient in proof of Tits' alternative for $Out(F_n)$

Bestvina, Feighn, Handel

More recently:

- Adding the missing faces to each simplex gives a simplicial complex Σ_n called the

* simplicial closure of Outer space

or * Sphere complex for F_n

or * Free splitting complex for F_n

The action of $Out(F_n)$ on CV_n extends to a (non-proper) action on Σ_n

$\underline{\text{Thm}}$ (Handel-Mosher 2012) The complex Σ_n is Gromov hyperbolic

- Bestvina, Algom-Kfir, Feighn, ... studied Lipschitz metric on Outer space
  → new proof of "train track" theorem of BH
  (tt = especially nice homotopy equivalence realizing an automorphism on a graph — critical ingredient in much work on $\mathrm{Out}(F_n)$, especially subgroup structure, study of mapping torus of an automorphism, fixed point set of an automorphism, etc.
  Also an ingredient in proof that

  The free factor complex is Gromov hyperbolic (Bestvina-Feighn 2012)

Much work on $Out(F_n)$ is inspired by studying analogy with $Mod(S_g)$ via its action on Teichmüller space $\mathcal{T}_g$

Eg Masur-Minsky's study of the relation between the geometry of $\mathcal{T}_g$ and the curve complexes of subsurfaces (via "subsurface projections")

Recently, Bestvina-Feighn have defined "subfactor projections" relating the geometry of CV_n and free factor and free splitting complexes.

One goal: determine the asymptotic dimension of $Out(F_n)$ (or at least whether it is finite).