Scientific Research

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Most of my research has been in the general area of ergodic theory and dynamical systems, with a particular emphasis on the interaction with geometry, analysis, number theory and mathematical physics. My most referenced works in the mathematics literature seem to be related to dynamical zeta functions. On the other hand, my most cited papers in the physics literature are those on decay of correlations and resonances. However, these are basically "two sides of the same coin".

1 Closed orbits and Zeta functions.

1.1 Zeta functions for flows

My early and, ironically, my later research has been bound up with dynamical zeta functions.

The best known zeta function is the Riemann zeta function for primes numbers $(p_n) = 2, 3, 5, 7, 11, 13.17, \cdots$ defined by

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}, s \in \mathbb{C}.$$

In 1956, Selberg introduced his famous zeta function for closed geodesics on surfaces of constant negative curvature and showed using trace formulae for $SL(2, \mathbb{R})$ that it had an extension to the entire complex plane.

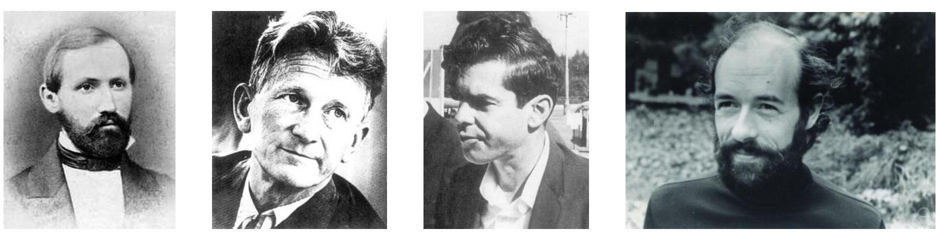


Figure 1: Riemann, Selberg, Smale and Ruelle

In 1967 Smale proposed extending the definition to a dynamical setting and, following Ruelle's formulation, and by analogy with the Riemann zeta function, let

$$\zeta(s) = \prod_\tau \left(1 - e^{-sh\lambda(\tau)}\right)^{-1}, s \in \mathbb{C}.$$

where τ denotes a closed orbit for an Anosov flow of least period $\lambda(\tau)$ then the zeta function can defined by where $h > 0$ denotes the entropy of the flow. This converges in the half-plane $Re(s) > 1$.

The key connection between the geometry and the dynamics is the following.

Proposition 1.1. *In the particular case of negatively curved manifolds, the associated geodesic flow is an example of an Anosov flow and there is a bijection between closed geodesics and closed orbits for the flow.*

Following even earlier work of Ruelle, and Parry and mine, a result of mine from 1985 showed it has a meromorphic extension to a strip $1 - \delta < Re(s) \leq 1$, say [1]. Finally, developping a very different method the result of Selberg was eventually extended as follows:

Theorem 1.2 (Giulietti-Liverani-Pollicott)**.** *For the geodesic flow on any C^∞ manifolds with negative sectional curvatures (or, more generally, Anosov flows) the zeta function $\zeta(s)$ has an extension to the entire complex plane.*

1.2 Closed orbits and closed geodesics

As an application of the original Selberg zeta function, Huber showed there is a "Prime Geodesic Theorem" (analogous to the Prime Number Theorem) giving a simple and elegant asymptotic formula for the number of closed orbits $N(T)$ for the geodesic flow whose period is at most T. This was subsequently extended to compact manifolds with variable negative curvature by Margulis in his thesis (but without the more delicate error term established by Huber in the constant curvature case).

More generally, let $N_\alpha(T)$ be the number of closed geodesics which are additionally restricted to fixed homology class α, then Phillips and Sarnak had an asymptotic formula for $N_\alpha(T)$ in the case of constant curvature.

Figure 2: Margulis, Sunada, Sarnak and Parry

Theorem 1.3. *We have the following counting function asymptotics.*

[1] This was already useful in studying the dependence of the entropy h on the flow. or the Riemannian metric in the case of geodesics flows, used in the modern theory of linear response

1. *For weak-mixing Anosov flows*

$$N(T) \sim e^{hT}/hT \text{ as } T \to \infty.$$

(Parry-Pollicott)

2. *For geodesic flows on surfaces or manifolds with* $\frac{1}{9}$*-pinched negative sectional curvatures there exists* $h > 0$ *and* $\epsilon > 0$ *such that*

$$N(T) = Li(e^{hT})(1 + O(e^{-\epsilon T})) \text{ as } T \to \infty$$

where $Li(x) = \int_2^x \frac{du}{\log u}$*. (Pollicott-Sharp, Giulietti-Liverani-Pollicott)*

3. *For manifolds with (Variable) negative curvature, then there exist constants* $c_0, c_1, c_2 \cdots$ *such that*

$$N_\alpha(T) = \frac{e^{hT}}{T^{g+1}} \left(c_0 + \frac{c_1}{T} + \frac{c_1}{T^2} + \cdots \right) \text{ as } T \to \infty.$$

(Katsuda-Sunada, Lalley, Pollicott (first term); Pollicott-Sharp, N. Ananrathaman (full expansion))

The statement, and the proof of this theorem is analogous to the classical results in number theory. The basic principle is that closed orbits τ correspond to prime numbers p, where we substitute p in the classical formulae by $e^{-h\lambda(\tau)}$.

1. The Prime Number Theorem *(Hadamard, de la Vallee Poussin)*:

$$\text{Card}\{\text{primes } p \leq x\} \sim \frac{x}{\log x}, \text{ as } x \to +\infty$$

2. The Riemann Hypothesis *(still to be proved)*

$$\text{Card}\{\text{primes } p \leq x\} = \text{Li}(x) + O(x^{1/2+\epsilon}) \text{ as } x \to +\infty$$

3. Sums of squares *(Landau, Ramanujan)*:

$$\text{Card}\{\text{primes } n^2 + m^2 \leq x\} = \frac{x}{\sqrt{\log x}} \left(C_0 + \frac{C_1}{\log x} + \frac{C_2}{(\log x)^2} + \cdots \right) \text{ as } x \to +\infty$$

1.3 generalizations

1. There is a result for Poincaré series and orbital counting functions (e.g., the orbit of a point in the universal cover of a negatively curved manifold under the action of the fundamental group). This extends results of Patterson, etc. from constant curvature to variable curvature.

2. In this setting additional progress was made using ideas from geometric group theory. The orbital counting results were used in turn in the proof of the remarkable Babillot-Ledrappier result on horocycle ergodicity for periodic surfaces. I subsequently gave a new proof of, which subsequently lead to my work with Ledrappier on counting problems for more general Clifford groups and p-adic groups.

Figure 3: Hadamard, de la Vallee Poussin, Landau and Ramanujan

3. Another more subtle result on the distribution of lengths is a pair correlation result comparing lengths of closed geodesics (ordered by their word length) due to Richard Sharp and myself, which is analogous to problems in Quantum Chaos.

2 Ergodicity and mixing

2.1 Decay of Correlations for hyperbolic flows

In the study of dynamical systems, one of the central themes is their long term statistical behaviour. Of particular interest is the speed at which the system approaches its equilibrium state, which is described by the "decay of correlations" (or rate of mixing). More precisely, given a smooth flow $\phi_t : M \to M$ on a compact manifold, a suitable ϕ-invariant probability measure μ and real valued square smooth F, G one can associate the correlation function

$$\rho(t) = \int F \circ \phi_t . G d\mu - \int F d\mu \int G d\mu.$$

The behaviour of this function is particularly important for "chaotic systems", one of the principle mathematical models for which are the Anosov flows $\phi_t : M \to M$.

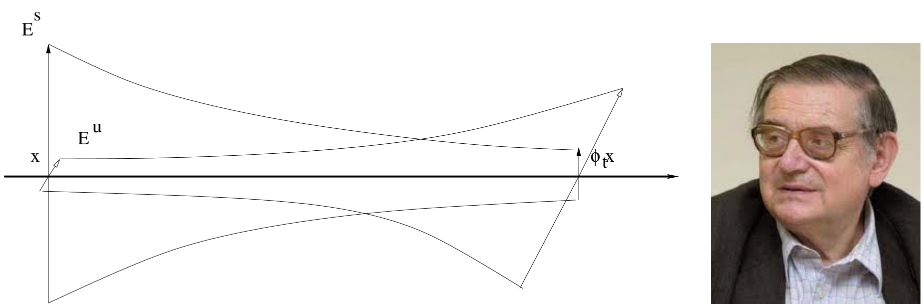


Figure 4: (a) The hyperbolicity transverse to the orbit of an Anosov flow; (b) Dmitri Victorovich Anosov.

Definition 2.1. *We say that $\phi_t : N \to N$ is a Anosov if there is a $D\phi_t$-invariant splitting $TM = E^0 \oplus E^s \oplus E^u$ such that*

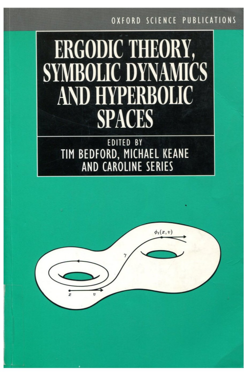
1. *E^0 is one dimensional and tangent to the flow direction;*

2. *$\exists C, \lambda > 0$ such that $\|D\phi_t|E^s\| \leq Ce^{-\lambda t}$ and $\|D\phi_{-t}|E^u\| \leq Ce^{-\lambda t}$ for $t > 0$.*

The behavior of $\rho(t)$ (where μ is a Gibbs measure for a Hölder potential and F, G are Hölder functions) is then determined by the following result:

Theorem 2.2. *The Fourier transform $\widehat{\rho}(s) = \int e^{its}\rho(t)dt$ is meromorphic for $|Im(s)| < \epsilon$. (Pollicott)*

The asymptotic behaviour of $\rho(t)$ is controlled by the poles in this extension (often now called *Pollicott-Ruelle resonances*) and this is essentially the only technique known to study the decay of correlations for flows, besides representation theory in the very special case of geodesic flows for surfaces of constant negative curvature, say.



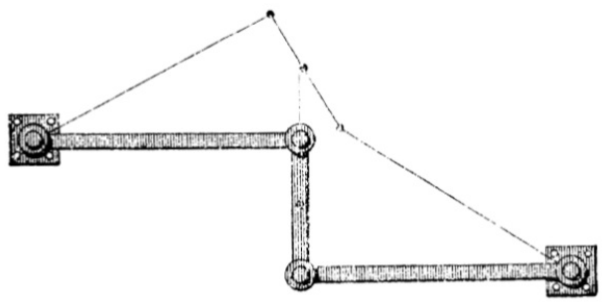


Fig. 2.

Figure 5: (a) The geodesic flow on a compact surface illustrated on a book cover; (b) A mechanical linkage whose dynamics can be described by the geodesic flow.

Example 2.3. *There are two natural classes of examples with exponential decay of correlations (i.e., $\exists C, \lambda > 0$ such that $|\rho(t)| \leq Ce^{-\lambda|t|}$):*

1. *Let V be a compact surface of variable negative curvature, then the geodesic flow $\phi_t : SV \to SV$ on the unit tangent bundle SV has exponential decay of correlations . (Dolgopyat)*

2. *For small $\delta > 0$, the suspension semiflow for a function $r(x) = 1 + \delta\sin(2\pi x)$ over doubling map $z \mapsto z^2$ on the unit circle mixes exponentially. (Ruelle conjecture; Pollicott)*

By contrast, I showed that there exist many examples of more general hyperbolic flows which mix arbitrarily slowly.

Figure 6: Ledrappier, Dolgopyat, Pesin and Burns

2.2 Frame flows

A more general class of systems than hyperbolic systems are the partially hyperbolic systems. The canonical example of this is the frame flow associated to a compact d-dimensional manifold M with negative sectional curvatures, by which orthonormal frames of tangent vectors are parallel transported along geodesics (which is a $G = SO(d-1)$-extension of the associated hyperbolic geodesic flow). The following result completed a programme of Brin and Gromov.

Theorem 2.4. *For each $d > 1$ there exists $\epsilon_d > 0$ such that if the sectional curvatures are ϵ_d-pinched then the frame flow is (stability) ergodic. (Burns-Pollicott).*

Some of my work on mixing for frame flows was an ingredient in the recent work of Markovic-Kahn.

2.3 Other systems

1. Burns, Dolgopyat, Pesin and I considered stable ergodicity for partially hyperbolic diffeomorphisms for which the neutral direction contracts on average.

2. Parry and I also developed a programme to study the stability of ergodicity (and mixing) for skew products over hyperbolic diffeomophisms. This lead to a fairly complete classification of ergodic skew products, in part through the developments in the theory of measurable Livsic theorems.

3. Johansson, Oberg and I have also studied ergodicity problems in other settings and using more probabilistic techniques I have criteria for uniqueness of g-measures, in the sense of Keane.

4. Magalhaes and I have also consider the dynamics of mechanical linkages via the associated geodesic flows.

3 Fractals and Hausdorff Dimension

3.1 Transversality and typical parameters

A basic method in studying "typical" fractal limit sets associated to a parameterised family of contractions is a technique developed with Simon called *transversality.* This was originally used to solve the following conjecture:

Theorem 3.1 (Keane-Smorodinsky-Solomyak Conjecture). *The Hausdorff dimension of*

$$\Lambda_\lambda := \left\{ \sum_{n=1}^{\infty} i_n \lambda^n \,:\, i_n \in \{0,1,3\} \right\}$$

is $-\frac{\log 3}{\log \lambda}$ *for almost all* $\frac{1}{4} \leq \lambda \leq \frac{1}{3}$. *(Pollicott-Simon)*

Moreover, this new transversality method subsequently proved to be the main tool in studying many different problems, including Solomyak's well known solution of the more famous Erdös Conjecture.

Theorem 3.2 (Erdös Conjecture). *For almost all* $\frac{1}{2} < \lambda < 1$ *the distribution*

$$\sum_{n=1}^{\infty} \pm \lambda^n$$

is absolutely continuous. (*Solmyak*)

3.2 Fat Sierpinski Gaskets

Another basic example of a "fractal set" is the Sierpinski gasket defined for a given finite family $S \subset \{0, \cdots, d-1\} \times \{0, \cdots, d-1\}$ by

$$\Sigma_\lambda := \left\{ \left(\sum_{n=1}^{\infty} i_n \lambda^n, \sum_{n=1}^{\infty} j_n \lambda^n \right) \,:\, (i_n, j_n) \in S \right\}.$$

When $\lambda = \frac{1}{d}$ this is a standard Sierpinski gasket, and the dimension is easily computed.

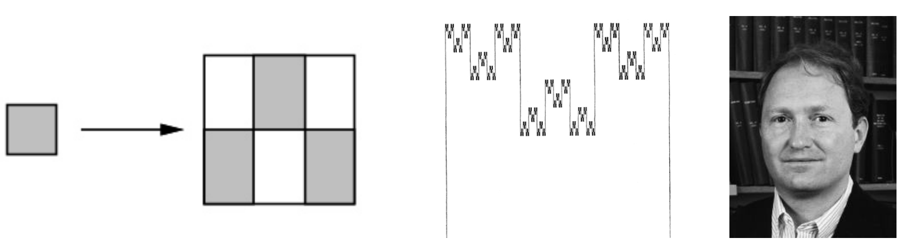


Figure 7: (a) The Beford-McMullen construction; (b) The Hironaka curve; (c) McMullen

However, for $\frac{1}{d} < \lambda < 1$ this is called a "fat Sierpinski gasket" and one of the few results known is the following:

Theorem 3.3. *For fat Sierpinski gaskets there are explicit ranges of λ where the dimension has a known value for almost all λ. There are other explicit ranges of λ where the set can be shown to have positive lebesgue measure for almost all λ. (Jordan-Pollicott)*

Moreover, developing this approach lead to resolving an old conjecture (by Peres-Solomyak) showing that there exists a self-affine set of nonzero measure and empty interior.

3.3 Computing dimension

In a different direction, I derived a new algorithm for computing for the Hausdorff dimension of hyperbolic Julia sets (and limit sets of Kleinian groups).

Theorem 3.4. *There exists a very efficient algorithm for computing the Hausdorff dimension of hyperbolic Julia sets using periodic orbit data. (Jenkinson-Pollicott)*

In particular, this method improved on an algorithm of McMullen and provides, for example, the best known numerical estimates for E_2 (the numbers in the unit interval whose continued fraction expansion contains only 1 and 2):

$$0 \cdot 5312805062772051416244686 \cdots$$

Figure 8: Solymak, Furstenberg, Simon and Jenkinson

4 Explicit values

4.1 Determinants and the Selberg zeta function revisited

A continuing theme of my work is finding explicit (and effectively computable) formulae for characteristic values. For example, for a compact Riemann surface (M, g) with negative Euler characteristic one can consider the famous Determinant of the Laplacian $\det(g)$

Theorem 4.1. *There is a formula for* $\det(g)$ *in terms of a rapidly convergent series each of whose coefficients is explicitly given in terms of the lengths of finitely many closed geodesics. (Pollicott-Rocha,)* In particular, this allows one to efficiently compute the numerical value of $\det(g)$ for given surfaces.

This method also gives an explicit expression for the Selberg zeta function for a surface of constant negative curvature. For a compact surface with curvature $\kappa = -1$, I showed that the Selberg zeta function $Z(s)$ can be written in terms of a convergent series each of whose coefficients depends on s and are explicitly given in terms of the lengths of finitely many closed geodesics. Anantharaman-Zelditch reformulated problems in Quantum Unique Ergodicity, in terms of residues of related complex functions for which we can give have explicit formulae.

4.2 Lyapunov exponents and entropy rates

In a similar spirit, I have developed a new approach to Lyapunov exponents associated to positive matrices (as defined by Furstenberg and Kesten) and showed the following:

Theorem 4.2. *There is an expliciit (and effectively computable) formulae for Lyapunov exponents in terms of a convergent series the nth term depends on of products of finitely many matrices. (Pollicott)*

Example 4.3. *If we consider the matrices*

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

then $\lambda_{n=9}$ this gives an approximation

$$\lambda = 1.1433110351029492458432518536555882994025 \cdots$$

to the Lyapunov exponent λ.

In particular, this gives an effective algorithm for computing the entropy rate for binary symmetric channels with transition matrix

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

and Bernoulli crossover probability $(1-\epsilon,)$.

Example 4.4. *Consider the case where $\epsilon = 0.1$ and $p = 0.3$ then using $2^1 2$ matrices we can estimate the entropy rate*

$$H = 0.65921241538006418846845365448691354 9$$

which is empirically accurate to the 35 decimal places given.