THE THEOREM OF THE CUBE, SQUARE, AND APPLICATIONS

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Fix an algebraically closed field k.

Theorem 0.1 ([Mum85, §10, Theorem, p.91]). *Suppose given X, Y complete k-varieties, Z a connected variety, and $\mathcal{L} \in \operatorname{Pic}(X \times Y \times Z)$. If there exist points $x_0 \in X(k)$, $y_0 \in Y(k)$, $z_0 \in Z(k)$ with the restriction of $\mathcal{L}$ to $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ trivial, $\mathcal{L}$ is trivial.*

We require the following results:

Theorem 0.2 (Formal function theorem, [GD61, Théorème 4.1.5]). *For $X \xrightarrow{\pi} Y$ a proper morphism of noetherian schemes, $\mathcal{F}$ a coherent sheaf of $\mathscr{O}_X$-modules, and $Y' \subset Y$ a closed subscheme with ideal sheaf $\mathcal{I}$, then for each $p \geq 0$ the system of maps*

$$\mathbf{R}^p \pi_*(\mathcal{F}) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y / \mathcal{I}^n \to \mathbf{R}^p(\mathcal{F}/\mathcal{I}^n \mathcal{F})$$

induces an isomorphism of topological $\mathscr{O}_{\widehat{X}}$ modules

$$(\mathbf{R}^p \pi_*(\mathcal{F}))^\wedge \to \varprojlim \mathbf{R}^p(\mathcal{F}/\mathcal{I}^n \mathcal{F}).$$

Corollary 0.3 ([GD61, Proposition 4.2.1]). *For $X \xrightarrow{\pi} Y$ a proper morphism with Y locally noetherian and $\mathcal{F}$ coherent on X, $y \in Y$, consider the following diagram of thickenings:*

$$\begin{tikzcd}
X_n := X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r, "i_n"] \arrow[d] & X \arrow[d, "\pi"] \\
Y_n := \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r] & Y,
\end{tikzcd}$$

$\mathcal{F}_n := i_n^* \mathcal{F}$;. Then

$$(\mathbf{R}^p \pi_* \mathcal{F})_y^\wedge \cong \varprojlim_n H^p(X_n, \mathcal{F}_n)$$

as $\widehat{\mathscr{O}}_{Y,y}$-modules.

Proposition 0.4 (Künneth formula, [Stacks, Tag 0BEC]). *For X, Y locally noetherian schemes of finite type over k, there is a natural isomorphism*

$$H^n(X \times Y, \mathscr{O}_{X \times Y}) \cong \bigoplus_{i+j=n} H^i(X, \mathscr{O}_X) \otimes_k H^j(Y, \mathscr{O}_Y).$$

Lemma 0.5 ([Stacks, Tag 0FD2]). *For X a proper k-variety, $H^0(X, \mathscr{O}_X) \cong k$.*

Lemma 0.6. *For X a complete k-variety, $\mathcal{L} \in \operatorname{Pic}(X)$ is trivial if and only if both $\mathcal{L}$ and $\mathcal{L}^\vee$ have nontrivial global sections.*

Definition 0.7. For Y a topological space, and $Y \xrightarrow{f} \mathbb{Z}$ a map of sets. f is upper semicontinuous if for each $y \in Y$ there is an open $y \in U \subset Y$ such that for each $y' \in U$, $f(y') \leq f(y)$.

For $\mathcal{F}$ a sheaf of $\mathscr{O}_{X \times Y}$-modules and $y \in Y$, write $X_y := X \times \{y\}$ and $\mathcal{F}_y := \mathcal{F}|_{X_y}$.

Theorem 0.8 (Semicontinuity, [Mum85, §5, Corollary, p.50])**.** *For $X \xrightarrow{\pi} Y$ a proper morphism of locally noetherian schemes, $\mathcal{F}$ a coherent sheaf on X flat over Y, i.e. $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module for each $x \in X$. Then the function $y \mapsto \dim_{k(y)} H^p(\pi^{-1}(y), \mathcal{F}_{\pi^{-1}(y)})$ is upper semicontinuous on Y. Accordingly, the set*

$$\{y \in Y \mid \dim_{k(y)} H^p(\pi^{-1}(y), \mathcal{F}|_{\pi^{-1}(y)}) \geq n\} \subset Y$$

is closed.

Corollary 0.9 (Grauert)**.** *For X, Y and $\mathcal{F}$ as above with Y integral, suppose that for some i, $y \mapsto \dim_{k(y)} H^i(\pi^{-1}(y), \mathcal{F}|_{\pi^{-1}(y)}$ is constant on Y. Then $(\mathbf{R}^i\pi_*\mathcal{F})^+$ is locally free on Y, and*

$$(\mathbf{R}^i\pi_*\mathcal{F})^+ \otimes k(y) \cong H^i(\pi^{-1}(y), \mathcal{F}_y)$$

naturally, where $(-)^+$ is the sheafification.

Theorem 0.10 ([Mum85, §5, Corollary 6, p.54])**.** *Suppose given X, Y varieties over k with X complete, and $\mathcal{L}, \mathcal{M} \in \operatorname{Pic}(X \times Y)$ such that for each $y \in Y$ closed, $\mathcal{L}_y \cong \mathcal{M}_y$. Then there exists $\mathcal{N} \in \operatorname{Pic}(Y)$ with $\mathcal{L} \cong \mathcal{M} \otimes \pi^*\mathcal{N}$, for $X \otimes Y \xrightarrow{\pi} Y$ the projection.*

Proof. Note that $X_y := X \times \{y\}$ is complete, and so $H^0(X_y, \mathcal{L}_y \otimes \mathcal{M}_y^{-1}) \cong H^0(X_y, \mathcal{O}_{X_y}) \cong k(y)$ for each $y \in Y$ closed. By Grauert's corollary, we have $\pi_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \otimes k(y) \cong H^0(X_y, \mathcal{L}_y \otimes \mathcal{M}_y^{-1}) \cong k(y)$, and so $\pi_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ is an invertible sheaf on Y. We claim that the $\pi^* \dashv \pi_*$ counit $\varepsilon : \pi^*\pi_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \to \mathcal{L} \otimes \mathcal{M}^{-1}$ is an isomorphism.
Consider the pullback

$$\begin{tikzcd} X_y \arrow[r, "j"] \arrow[d, "\pi'"] & X \times Y \arrow[d, "\pi"] \\ \operatorname{Spec}(k(y)) \arrow[r, "i"] & Y \end{tikzcd}$$

with π and hence π' flat. Writing $\mathcal{F} := \mathcal{L} \otimes \mathcal{M}^{-1}$, we have $j^*\pi^*\pi_*\mathcal{F} \cong \pi'^*i^*\pi_*\mathcal{F} \cong \pi'^{-1}\mathcal{O}_{k(y)} \cong \mathcal{O}_{X_y}$, and so $\varepsilon_y : j^*\pi^*\pi_*\mathcal{F} \to j^*\mathcal{F} \cong \mathcal{O}_{X_y}$ is an isomorphism.

It thus suffices to show that given a map $\mathcal{E} \xrightarrow{f} \mathcal{O}_{X \times Y}$ with f fibrewise an isomorphism, f is an isomorphism; Nakayama's lemma implies that f is surjective, and comparing ranks we see that it is injective. ☐

The proof below follows Akhil Mathew's exposition in [Mat12]

Proof of Theorem 0.1. Set $Z' \subset Z$ the set of points z with $\mathcal{L}|_{X \times Y \times \{z\}}$ trivial; this is the case if and only if $\dim_{k(z)} H^0(\mathcal{L}|_{X \times Y \times \{z\}})$ and $\dim_{k(z)} H^0(\mathcal{L}^\vee|_{X \times Y \times \{z\}}) > 0$, and by semicontinuity this is closed; note that $z_0 \in Z'$.
Fix $z' \in Z'$. We first show for any local finite-dimensional k-algebra A and infinitesimal thickening $\operatorname{Spec}(A) \to Z$ of z' that $\mathcal{L}|_{X \times Y \times \operatorname{Spec}(A)}$ is trivial. Set $d := \dim_k(A)$, and note that the case $d = 1$ follows by hypothesis. Suppose the required triviality holds for any such A of k-dimension $< d \geq 1$. There exists $a \in A$ nonzero with $\mathfrak{m}_A a = 0$, inducing a surjection of k-algebras $A \to A/a = A/ka$; then we have an exact sequence of sheaves (on z'):

$$0 \to \mathcal{O}_k \to \mathcal{O}_A \to \mathcal{O}_{A/a} \to 0,$$

inducing

$$0 \to \mathcal{L}|_{X \times Y \times \operatorname{Spec}(k)} \to \mathcal{L}|_{X \times Y \times \operatorname{Spec}(A)} \to \mathcal{L}|_{X \times Y \times \operatorname{Spec}(A/a)} \to 0$$

on $X \times Y \times \operatorname{Spec}(A)$. We wish to find a trivialising section $s \in \Gamma(\mathcal{L}|_{X \times Y \times \operatorname{Spec}(A)})$; by induction, there exists some such $s' \in \Gamma(\mathcal{L}|_{X \times Y \times \operatorname{Spec}(A/a)})$, since $\dim_k A/a < d$. A lift of s' exists if and only if the connecting homomorphism

$$H^0(\mathcal{L}|_{X \times Y \times \operatorname{Spec}(A/a)}) \xrightarrow{\delta} H^1(\mathcal{L}|_{X \times Y \times \operatorname{Spec}(k)})$$

takes $s' \mapsto 0$. By the Künneth formula, we have

$$\begin{aligned} H^1(\mathcal{L}\mid_{X\times Y\times\{z_0\}}) &\cong H^0(\mathcal{L}\mid_{X\times\{y_0\}\times\{z_0\}}) \otimes H^1(\mathcal{L}\mid_{\{x_0\}\times Y\times\{z_0\}}) \oplus H^1(\mathcal{L}_{X\times\{y_0\}\times\{z_0\}}) \otimes H^0(\mathcal{L}\mid_{\{x_0\}\times Y\times\{z_0\}}) \\ &\cong H^0(\mathscr{O}_{X\times\{y_0\}\times\{z_0\}}) \otimes H^1(\mathcal{L}\mid_{\{x_0\}\times Y\times\{z_0\}}) \oplus H^1(\mathcal{L}_{X\times\{y_0\}\times\{z_0\}}) \otimes H^0(\mathscr{O}_{\{x_0\}\times Y\times\{z_0\}}) \\ &\cong H^1(\mathcal{L}\mid_{\{x_0\}\times Y\times\{z_0\}}) \oplus H^1(\mathcal{L}_{X\times\{y_0\}\times\{z_0\}}), \end{aligned}$$

since $X\times\{y_0\}\times\{z_0\}$ and $\{x_0\}\times Y\times\{z_0\}$ are complete. We then note that the connecting maps

$$H^0(\mathcal{L}\mid_{X\times\{y_0\}\times\operatorname{Spec}(A/a)}) \xrightarrow{\delta'} H^1(\mathcal{L}\mid_{X\times\{y_0\}\times\{z_0\}})$$

and

$$H^0(\mathcal{L}\mid_{\{x_0\}\times Y\times\operatorname{Spec}(A/a)}) \xrightarrow{\delta''} H^1(\mathcal{L}\mid_{\{x_0\}\times Y\times\{z_0\}})$$

send $s' \mapsto 0$, since $\mathcal{L}$ is trivial on $X\times\{y_0\}\times\{z_0\}$ and $\{x_0\}\times Y\times\{z_0\}$ by hypothesis, and so $\delta(s) = (\delta'(s), \delta''(s)) = 0$.

We now show we can extend triviality of $\mathcal{L}$ to an open containing z'. We take Z to be irreducible, without loss of generality (otherwise we restrict to each irreducible component). Write $\pi : X\times Y\times Z \to Z$ for the projection, and set $\mathcal{M} := \pi_*\mathcal{L}$, a coherent sheaf on Z, so $\mathcal{M}_{z'}$ is a finitely generated $\mathscr{O}_{Z,z'}$-module. By the corollary to the formal function theorem, we have

$$\widehat{\mathcal{M}}_{z'} \cong \varprojlim H^0(\mathcal{L}\mid_{\operatorname{Spec}\mathscr{O}_{Z,z'}/\mathfrak{m}^n_{z'}}) \cong \widehat{\mathscr{O}}_{Z,z'},$$

since

$$\begin{aligned} &H^0(X\times Y\times\operatorname{Spec}\mathscr{O}_{Z,z'}/\mathfrak{m}^n_{z'}, \mathcal{L}\mid_{X\times Y\times\operatorname{Spec}\mathscr{O}_{Z,z'}/\mathfrak{m}^n_{z'}}) \\ \cong &H^0(X\times Y, \mathscr{O}_{X\times Y}) \otimes H^0(\operatorname{Spec}\mathscr{O}_{Z,z'}/\mathfrak{m}^n_{z'}, \mathscr{O}_{\operatorname{Spec}\mathscr{O}_{Z,z'}/\mathfrak{m}^n_{z'}}) \\ \cong &H^0(\mathscr{O}_{\operatorname{Spec}\mathscr{O}_{Z,z'}/\mathfrak{m}^n_{z'}}) \end{aligned}$$

by Künneth and since $H^0(X\times Y, \mathscr{O}_{X\times Y}) \cong k$.

Since $\mathscr{O}_{Z,z'}$ is noetherian local, the completion $\mathscr{O}_{Z,z'} \to \widehat{\mathscr{O}}_{Z,z'}$ is faithfully flat, and so $\widehat{\mathcal{M}}_{z'} \cong \widehat{\mathscr{O}}_{Z,z'}$ if and only if $\mathcal{M}_{z'} \cong \mathscr{O}_{Z,z'}$. So $\mathcal{M}_{z'}$ is free of rank one, and by coherence, $\mathcal{M}$ is a line bundle in a neighbourhood V of z'. We also note that $\mathcal{M}_{z'} \to H^0(\mathcal{L}\mid_{X\times Y\times\{z'\}}) \cong k$ is surjective, and so for some neighbourhood $z' \in U \subset V$, $1 \in H^0(\mathcal{L}\mid_{X\times Y\times\{z'\}}) \cong k$ lifts to a section s of $\mathcal{L}$ over $X\times Y\times U$. Shrinking U, we may assume that s is invertible on $X\times Y\times U$, and so $\mathcal{L}_{X\times Y\times\{u\}} \cong \mathscr{O}_{X\times Y\times\{u\}}$ for each $u \in U$.

But the set of $t \in Z$ with $\mathcal{L}\mid_{X\times Y\times\{t\}}$ trivial is closed, and hence equal to Z, and we thus have that $\mathcal{L}$ is the pullback of a line bundle on Z (in fact, to $\pi^*\mathcal{M}$). Then $\mathcal{L}\mid_{\{x_0\}\times\{y_0\}\times Z} \cong \mathcal{M} \cong \mathscr{O}_Z$, and hence $\mathcal{L}$ is trivial. □

From this we immediately obtain a number of useful corollaries. For an abelian k-variety X and $S \subset \{1,2,3\}$, denote by $\pi_S : X^3 \to X$ the map defined on k-points via $(x_1, x_2, x_3) \mapsto \sum_{s\in S} x_s$, where is $s = \varnothing$ the sum is the unit of 0 of X.

Corollary 0.11. *X an abelian k-variety, $\mathcal{L} \in \operatorname{Pic}(X)$. Then the line bundle*

$$\Theta(\mathcal{L}) := \pi^*_{123}\mathcal{L} \otimes \pi^*_{12}\mathcal{L}^{-1} \otimes \pi^*_{13}\mathcal{L}^{-1} \otimes \pi^*_{23}\mathcal{L}^{-1} \otimes \pi_1 * \mathcal{L} \otimes \pi_2 * \mathcal{L} \otimes \pi_3 * \mathcal{L}$$

is trivial on X^3.

Proof. Clearly it suffices to check that the restriction of $\Theta(\mathcal{L})$ to $\{0\}\times X\times X$, $X\times\{0\}\times X$, and $X\times X\times\{0\}$ is trivial. Write $j : \{0\}\times X\times X \hookrightarrow X^3$ for the inclusion morphism, and note that $\pi_S \circ j = \pi_{S\setminus\{1\}}$. Writing $c_0 : X^3 \to X$ for the constant morphism to $0 \in X$, we have

$$c_0^*\mathcal{L} : U \mapsto \operatorname{colim}_{c_0(U)\subset V}\Gamma(V, \mathcal{L}) \cong \mathcal{L}_0 \cong \mathscr{O}_{X,0},$$

the stalk of the structure sheaf at zero, and hence $c_0^*\mathcal{L} \cong \mathscr{O}_{X\times X}$. Then

$$\begin{aligned} j^*\Theta(\mathcal{L}) =& \pi_{23}^*\mathcal{L} \otimes \pi_2^*\mathcal{L}^{-1} \otimes \pi_3^*\mathcal{L}^{-1} \otimes \pi_{23}^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{L} \otimes \pi_3^*\mathcal{L} \\ &\cong \pi_{23}^*\mathcal{L} \otimes \pi_{23}^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{L} \otimes \pi_3^*\mathcal{L}^{-1} \otimes \otimes \pi_3^*\mathcal{L} \\ &\cong \mathscr{O}_{X\times X}, \end{aligned}$$

and similarly for $X \times \{0\} \times X$ and $X \times X \times \{0\}$. □

Corollary 0.12. *For Y a k-variety and X an abelian k-variety, given maps $f, g, h : Y \to X$ and $\mathcal{L} \in \operatorname{Pic}(X)$, we have*

$$(f+g+h)^*\mathcal{L} \cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof.

$$\begin{tikzcd} Y \arrow[r, "{(f,g,h)}"] \arrow[dr, "{(f,g,h)_S}"'] & X^3 \arrow[d, "\pi_S"] \\ & X, \end{tikzcd}$$

where for instance $(f,g,h)_{12} = f+g$. We see that

$$\begin{aligned} (f+g+h)^*\mathcal{L} &\cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1} \cong (f,g,h)^*\Theta(\mathcal{L}) \\ &\cong (f,g,h)^*\mathscr{O}_{X^3} \\ &\cong \mathscr{O}_Y, \end{aligned}$$

and we are done. □

Recall for an abelian variety X and $x \in X(k)$, we have the translation morphisms $t_x : X \to X$.

Corollary 0.13 (Theorem of the square, [Mum85, §, Corollary 4, p.59])**.** *For X an abelian k-variety and $\mathcal{L} \in \operatorname{Pic}(X)$,*

$$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}.$$

Proof. Note firstly that $t_x = \mathrm{id}_X + c_x$, for c_x the constant map at x. Setting $f := \mathrm{id}_X$, $g := c_x$, $h := c_y$ in the above corollary,

$$t_{x+y}^*\mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L} \otimes (c_x+c_y)^*\mathcal{L} \otimes c_x^*\mathcal{L}^{-1} \otimes c_y^*\mathcal{L}^{-1} \otimes \mathcal{L}^{-1} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L} \otimes \mathcal{L}^{-1},$$

since $c_z^*\mathcal{L} \cong \mathscr{O}_X$ for any $z \in X$. □

Definition 0.14. For A an abelian k-variety, we define the degree 0 part of the Picard group to consist of translation invariant line bundles:

$$\operatorname{Pic}^0(A) := \{\mathcal{L} \in \operatorname{Pic}(A) \mid t_x^*\mathcal{L} \cong \mathcal{L}\}.$$

Note that t_x^* commutes with $\otimes$ (as a left adjoint), and so $\operatorname{Pic}^0(A) \subset \operatorname{Pic}(A)$ is a subgroup.

The following is now immediate.

Corollary 0.15. *For A an abelian k-variety and $\mathcal{L} \in \operatorname{Pic}(A)$, there is a homomorphism of abelian groups*

$$\begin{aligned} A &\xrightarrow{\varphi_{\mathcal{L}}} \operatorname{Pic}^0(A) \\ a &\mapsto t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}. \end{aligned}$$

References

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