ECHT WEEK 5: THE BASS-QUILLEN CONJECTURE

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0. From Serre's problem to the Bass-Quillen conjecture

From last time: given a scheme X, the projection $X \times \mathbb{A}^n = \mathbb{A}^n_X \to X$ admits a section, and so the induced restriction map

$$\rho_{n,k} : \mathrm{Vect}_k(X) \to \mathrm{Vect}_k(\mathbb{A}^n_X)$$

is injective, where $\mathrm{Vect}_k(-)$ denotes the set of isomorphism classes of rank k vector bundles. We can ask when this is surjective.

Theorem 0.0.1 (Quillen-Suslin, 1976)**.** *For $X = \operatorname{Spec} R$, R a PID, $\rho_{n,k}$ is an isomorphism.*

This is consistent with our intuition that spectra of fields should be the scheme-theoretic analogue of the topological point.
How to generalise this? In particular, for which classes of scheme X does this hold?

Example 0.0.2. *There is a rank 2 vector bundle on $\mathbb{P}^1 \times \mathbb{A}^1$ whose restriction to $\mathbb{P}^1 \times \{0\}$ is trivial, and to $\mathbb{P}^1 \times \{1\}$ is $\mathcal{O}(1) \oplus \mathcal{O}(-1)$, and hence cannot be extended from $\mathbb{P}^1$.*

Example 0.0.3. *For k a field of characteristic 0, $R := k[x,y]/(y^2 - x^3)$ and $\mathfrak{m} := (x,y)$, any projective module over $R_\mathfrak{m}$ is necessarily trivial, but there exists a non-free rank one projective module over $R_\mathfrak{m}[T]$.*

We next observe that $\mathbb{P}^1 \times \mathbb{A}^1$ fails to be affine and R fails to be regular.

Conjecture 0.0.4 (Bass-Quillen). For R a regular commutative ring, $X := \operatorname{Spec} R$, $\rho^X_{n,k}$ is an isomorphism for all $n, k \geq 0$.

Remark 0.0.5. Note that by Quillen patching it suffices to consider R regular local: if all f.g. projective modules over $R_\mathfrak{p}[t_1, \dots, t_n]$ are free and in particular extended, for $\mathfrak{p} \in \operatorname{Spec} R$ maximal, so are f.g. projective modules over $R[t_1, \dots, t_n]$.

In this note we give an expository treatment of a result of Lindel which gives an affirmative answer to the Bass-Quillen conjecture for regular local rings essentially of finite type over fields. In preparation for this, we'll recall some algebro-geometric facts, discuss descent, and the behaviour of vector bundles in different Grothendieck topologies.

1. Recollections: regularity/smoothness, étale morphisms, Nisnevich covers

1.1. Regularity and smoothness over a field.

Given a noetherian local ring R of dimension d with maximal ideal $\mathfrak{m}$ and residue field κ, Nakayama's lemma shows that the minimal number of generators of $\mathfrak{m}$ is the dimension of the κ-vector space $\mathfrak{m}/\mathfrak{m}^2$. It follows that this is at least the Krull dimension of X, and so we have the inequality

$$\dim R \leq \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2).$$

Definition 1.1.1. A local ring $(R, \mathfrak{m}, \kappa)$ is said to be regular if $\dim R = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. A scheme X is said to be regular at $x \in X$ if $(\mathscr{O}_{X,x}, \mathfrak{m}_x, \kappa(x))$ is a regular local ring.

Since (the dual of) the Zariski tangent space to X at x is $\mathfrak{m}_x/\mathfrak{m}_x^2$, we see that regularity is precisely the requirement that the tangent space to X at x is of the expected dimension (i.e., the codimension of the Zariski closure of x in X[1]).

Example 1.1.2. *The local ring of $R := \mathbb{Q}[x,y]/(y^2-x^3)$ at the origin is of dimension 1 (since (x,y) is a maximal ideal of the one-dimensional domain R), for which the ideal cannot be generated by fewer than two elements. We thus see that $\operatorname{Spec} R$ is not regular (but has regular locus the complement of the origin) – this corresponds to the tangent space at the cusp point being of dimension 2.*

Example 1.1.3. *Since a normal variety is regular in codimension ≤ 1, any normal algebraic curve is regular.*

Definition 1.1.4. A map of schemes $X \to S$ is said to be smooth at x if it is flat, locally of finite presentation at x, and such that the fibre $X_{f(x)}$ of X at $f(x)$ is geometrically regular at x, i.e., all localisations of

$$\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,f(x)}} \overline{\kappa(f(x))}$$

are regular local rings. f is said to be smooth if it is smooth at all $x \in X$.

This rather formal definition of smoothness at $x \in X$ is equivalent to the the existence of affine neighbourhoods $x \in \operatorname{Spec} B \subset X$, $f(x) \in \operatorname{Spec} A \subset S$ with $f(\operatorname{Spec} B) \subset \operatorname{Spec} A$, such that the induced ring map is isomorphic to $A \to A[t_1,\dots,t_n]/(f_1,\dots,f_r)$, where the Jacobian $(\frac{\partial f_i}{t_j})_{ij}$ has rank r at x.
Smoothness of a morphism is a statement about the regularity of its (geometric) fibres; we promote smoothness to a property of schemes by saying an S-scheme X is smooth (over S) if the structure morphism $X \to S$ is smooth. In the case $S = \operatorname{Spec} k$ for k a field, smoothness of a locally finitely presented flat k-scheme X is equivalent to regularity of $X_{\overline{k}} := X \times_k \operatorname{Spec} \overline{k}$. Over a field, smoothness over k implies regularity, and for k perfect, regularity is equivalent to smoothness over k.

1.2. **Étale maps.** An étale map is the algebro-geometric incarnation of a local diffeomorphism, and remedies the failure of the Zariski topology to provide an algebraic analogue of the inverse function theorem. Local rings in the étale topology are strictly henselian local rings, for which the algebraic analogue of the implicit function theorem holds.

Definition 1.2.1. A map of schemes $X \xrightarrow{f} Y$ is unramified at $x \in X$ if the following hold:

(i) f is locally of finite presentation at x;
(ii) the residue field extension $\kappa(f(x))/\kappa(x)$ is finite separable;
(iii) $\mathfrak{m}_{f(x)}\mathscr{O}_{X,x} = \mathfrak{m}_x$.

Example 1.2.2. *For k a field of characteristic 0, the map $\mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $t \mapsto t^r$, i.e. induced by the inclusion $k[x^r] \hookrightarrow k[x]$ is ramified at the origin: the corresponding map of local rings is*

$$k[x^r]_{(x^r)} \to k[x]_{(x)},$$

with $(x^r)k[x]_{(x)} \subsetneq (x)k[x]_{(x)}$.

Example 1.2.3. *Any open immersion is unramified (since it is a local isomorphism).*

Definition 1.2.4. A map of schemes $X \xrightarrow{f} Y$ is étale if it is flat and unramified.

Étale morphisms enjoy the following properties:

(i) stability under composition and base change;

[1] The codimension of an irreducible closed subscheme $Z \subset X$ is usually defined to be the maximal length of a chain $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d = X$ of irreducible closed subchemes; this coincides with $\dim \mathscr{O}_{X,\eta}$, for η the generic point of Z.

(ii) étale maps are smooth and of relative dimension 0;
(iii) an étale map of local rings is necessarily faithfully flat (and in particular an inclusion);
(iv) an étale map $X \to Y$ induces an isomorphism on tangent cones; in this sense, étale maps are the algebraic analogue of local diffeomorphisms of manifolds.

Example 1.2.5. *A field extension $k \to L$ is étale precisely when it is finite separable; we can thus think of a Galois extension L/k as corresponding to a covering map $\operatorname{Spec} L \to \operatorname{Spec} k$ with group of deck transformations $\mathrm{Gal}(L/k)$.*

Example 1.2.6 (Standard étale maps)**.** *Let R be a commutative ring, and $f, g \in R[t]$ two polynomials such that f is monic, and the formal derivative $f' \in (R[t]_g/(f))^\times$. Then the map $R \to R[t]_g/(f)$ is étale, and said to be standard étale. It can be shown that any étale map of schemes is locally standard étale.*

Remark 1.2.7. Given the definition of étale, we have that a map $X \to S$ is smooth if X admits a Zariski cover $\{U_i\}_{i \in I}$ such that each U_i admits étale maps $U_i \to \mathbb{A}^{n_i}_S$ over S, i.e., smooth maps are precisely those which locally look like the composition of an étale map with projection from affine space.

1.3. **Nisnevich covers.**

Definition 1.3.1. A Zariski covering of a scheme X is a jointly surjective collection of open immersions $\{U_i \xrightarrow{\varphi_i} X\}_{i \in I}$. An étale cover of X is a jointly surjective family of maps $\{U_i \xrightarrow{\varphi_i} X\}_{i \in I}$ such that each φ_i is étale.

Given an étale cover $\{u_i \to X\}_i$, for each κ-valued point $\operatorname{Spec} \kappa \xrightarrow{x} X$ there exists some index i and $\operatorname{Spec} L \xrightarrow{u} U_i$ such that $\varphi_i(u) = x$, and the induced residue field extension L/κ is finite separable.

Definition 1.3.2. A Nisnevich cover $\{U_i \xrightarrow{\varphi_i} X\}_{i \in I}$ of X is an étale cover with the property that every κ-point of X lifts to some U_i. For $x \in X$, we call an étale X-scheme $U \to X$ with image containing x a Nisnevich neighbourhood if we have a lift

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} \kappa \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

Since open immersions are étale local isomorphisms, every Zariski cover is Nisnevich, and clearly every Nisnevich covers is étale.

Example 1.3.3. *A Nisnevich cover of* $\operatorname{Spec} k$ *is a family of finite separable field extensions* $\operatorname{Spec} L_i \to \operatorname{Spec} k$ *such that one of the L_i coincides with k; this corresponds to a Galois k-algebra $A \cong k \times L_1 \times \cdots \times L_m$.*

Example 1.3.4. *Suppose given an étale map of rings $R \xrightarrow{f} S$ such that there exists $x \in R$ with $f(x) \in S$ a nonzerodivisor, and such that f induces an isomorphism $R/x \cong S/f(x)$. Then the family $\{\operatorname{Spec} R_x \to \operatorname{Spec} R, \operatorname{Spec} S \to \operatorname{Spec} R\}$ is a Nisnevich cover: for a prime $\mathfrak{p} \in \operatorname{Spec} R$ not containing f, i.e. $\mathfrak{p} \in \operatorname{Spec} R_x$, we may lift a $\kappa(\mathfrak{p})$-point along the open immersion $\operatorname{Spec} R_x \to \operatorname{Spec} R$; for $f \in \mathfrak{p}$, the map $\operatorname{Spec} \kappa(\mathfrak{p}) \to \operatorname{Spec} R$ factors over the closed subscheme $\operatorname{Spec} R/x \hookrightarrow \operatorname{Spec} R$, and we may lift this along the isomorphism $\operatorname{Spec} R/x \cong \operatorname{Spec} S/f(x)$.*

Definition 1.3.5. A tuple $(R \xrightarrow{f} S, x)$ as above is said to be an affine distinguished Nisnevich square.

Lemma 1.3.6. *An affine Nisnevich distinguished square*

$$\begin{tikzcd} \operatorname{Spec} S_{f(x)} \arrow[r] \arrow[d] & \operatorname{Spec} S \arrow[d] \\ \operatorname{Spec} R_x \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

is both cartesian and cocartesian.

Proof. Clearly the square is a pullback (of schemes), since $S_{f(x)} \cong S \otimes_R R_x$. To show it is a pushout, consider the map $\varphi : R \to R_x \times_{S_{f(x)}} S$, $a \mapsto (a/1, f(a))$ induced by the universal property of the pullback, where we write $a/1$ for the image of a in the localisation.
Given an element $(r/x^n, s) \in R_x \times_{S_{f(x)}} S$, i.e. such that $f(r)/f(x)^n = s/1$ in $S_{f(x)}$. If $n = 0$, we have that $f(r)/1 = s/1$, and so for some $k \geq 0$, $(f(r) - s)f(x)^k = 0$. But $f(x)$ is a nonzerodivisor in S by assumption, and so $f(r) = s$, and $(r/1, s) = \varphi(r)$. For $n \geq 1$, we observe that

$$f(r)/f(x)^n = s/1 \implies (f(r) - sf(x)^n)f(x)^m = 0 \implies f(r) = sf(x)^n,$$

so $f(r) \equiv 0 \mod f(x)$. Since $R/x \cong S/f(x)$, we have $r \equiv 0 \mod x$, and we can rewrite $r = r'x$ for some $r' \in R$. Proceeding in this manner, we obtain $(r/f^n, s) = (r'/f^{n-1}, s) = \cdots = (a/1, s) = \varphi(a)$ for some $a \in R$, so φ is a surjection.
For injectivity, we note that $\varphi(a) = 0$ if and only if $ax^n = 0$ in R for some $n \geq 0$, and $f(a) = 0$ in S. We thus have for any $x \notin \mathfrak{p} \in \operatorname{Spec} R$ that the image of a in $R_{\mathfrak{p}}$ is 0.
Now for $\mathfrak{q} \in \operatorname{Spec} S$, the induced map

$$f_{\mathfrak{q}} : R_{f^{-1}(\mathfrak{q})} \to S_{\mathfrak{q}}$$

is an étale map of local rings, and hence the inclusion of a subring. We then see that $f_{\mathfrak{q}}(a/1) = 0 \in S_{\mathfrak{q}} \implies a/1 = 0 \in R_{f^{-1}(\mathfrak{q})}$. But the map $\operatorname{Spec} S \to \operatorname{Spec} R$ in particular surjects onto the subscheme $\operatorname{Spec}(R/x)$, and so the image of a is zero in any $R_{\mathfrak{p}}$ for $x \in \mathfrak{p}$. Then a is zero in every prime localisation, and hence 0. □

Example 1.3.7. *[Mor04, Ex. 2.1.5] Let k be a perfect field, and L/k a finite extension, generated by some $x \in L$ with minimal polynomial f_x, with $k[t] \to k[t]/(f_x) \cong L$ corresponding to the L-point $x_0 : \operatorname{Spec} L \to \mathbb{A}^1_k$. Let U be the open complement of the image of x_0. Since the map $\operatorname{Spec} L \to \operatorname{Spec} k$ is étale (k is perfect), the base change to $\mathbb{A}^1_L \to \mathbb{A}^1_k$ is étale. The pullback of this map along x_0 is the finite étale L-algebra $L \otimes_{k[t]} L[t] \cong L \otimes_k L$, with $\operatorname{Spec}(L \otimes_k L) \to \mathbb{A}^1_L$ a closed immersion corresponding to $L[t] \to L \otimes_k L$, $g \mapsto g(x) \otimes 1$. The map $\operatorname{Spec}(L \otimes_k L) \to \operatorname{Spec} L$ corresponding to $L \to L \otimes_k L, x \mapsto x \otimes 1$ has as a section the multiplication map $x \otimes y \mapsto xy$. $L \otimes_k L$ thus splits as $L \times A'$, for some étale L-algebra A (we could see this also by observing that $L \otimes_k L \cong k[t]/(f) \otimes_k L \cong \prod_{\sigma \in \mathrm{Gal}(L/k)} L_\sigma$, where L_σ denotes a copy of L indexed by σ). We have the following diagram:*

$$\begin{tikzcd} \operatorname{Spec} L \coprod \operatorname{Spec} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}^1_L \arrow[d] \\ \operatorname{Spec} L \arrow[r, "x_0"] & \mathbb{A}^1_k, \end{tikzcd}$$

i.e. the fibre of $\mathbb{A}^1_L \to \mathbb{A}^1_k$ at x_0 splits as a copy of $\operatorname{Spec} L$ and $\operatorname{Spec} A'$. Setting Ω to be the open complement of $\operatorname{Spec} A'$ in $\mathbb{A}^1_L$, we thus see that the square

$$\begin{tikzcd} \Omega - \operatorname{Spec} L \arrow[r] \arrow[d] & \Omega \arrow[d] \\ U \arrow[r] & \mathbb{A}^1_k, \end{tikzcd}$$

is by construction distinguished Nisnevich.

It is shown in [AHW17] that the class of affine distinguished Nisnevich squares generate the Nisnevich topology on $\mathrm{Sm}_S^{\mathrm{aff}}$ for S any qcqs base scheme. The Nisnevich topology (on Sch_S) interpolates between many of the formal properties of the Zariski and étale topologies:

(i) the Nisnevich cohomological dimension of X is bounded by the Krull dimension (this fails in the étale topology);
(ii) algebraic K-theory satisfies descent for the Nisnevich topology, for finite dimensional qcqs schemes (by results of Nisnevich, Thomason-Trobaugh, Rosenschon-Østvaer);
(iii) Nisnevich sheaf cohomology can be computed with Čech techniques.

2. Nisnevich descent

A nice reference for the discussion below is [Vis05].

2.1. Zariski patching. We saw last week that given a ring R and comaximal elements $f, g \in R$ (i.e. $fR + gR = R$), projective modules Q_0 over R_f and Q_1 over R_g, and an isomorphism of R_{fg}-modules $\Phi : (Q_0)_g = Q_0 \otimes_{R_f} R_{fg} \cong Q_1 \otimes_{R_g} R_{fg} = (Q_1)_f$, that there exists a projective R module P satisfying

$$P \otimes_R R_f \cong Q_0, \quad P \otimes_R R_g \cong Q_1;$$

we refer to such a P (which is unique up to isomorphism) as the patch of Q_0 and Q_1 along Φ. Geometrically, we have the following cartesian square of Zariski open immersions:

$$\begin{tikzcd} \operatorname{Spec} R_{fg} \arrow[r, hook] \arrow[d, hook] & \operatorname{Spec} R_f \arrow[d, hook] \\ \operatorname{Spec} R_g \arrow[r, hook] & \operatorname{Spec} R, \end{tikzcd}$$

and the data of vector bundles $\mathcal{E}_0 \to \operatorname{Spec} R_f$ and $\mathcal{E}_1 \to \operatorname{Spec} R_g$ with an isomorphism $\mathcal{E}_0 |_{\operatorname{Spec} R_{fg}} \cong \mathcal{E}_1 |_{\operatorname{Spec} R_{fg}}$. Since f and g are comaximal, $D(f) \cup D(g) = \operatorname{Spec} R$, and so we have a Zariski covering $\{\operatorname{Spec} R_f \to \operatorname{Spec} R, \operatorname{Spec} R_g \to \operatorname{Spec} R\}$ of $\operatorname{Spec} R$; the statement that we can patch projective modules along an isomorphism Φ is then an instance of Zariski-descent for vector bundles.

2.2. Notes on descent. The general setup is as follows: we have a Grothendieck topology t on some nice subcategory of schemes, and consider the small t-site over some scheme X, denoted X_t.

(i) For the Zariski topology, the site X_{Zar} has objects (U, i) for U a Zariski open subscheme of X, and $i : U \hookrightarrow X$ a specified open immersion, with arrows $(U, i) \to (V, j)$ maps $U \to V$ over X. Covers of a subscheme U are Zariski covers.
(ii) For the Nisnevich topology, X_{Nis} has objects étale schemes over X, i.e. pairs (U, π) for $\pi : U \to X$ an étale map, and arrows $(U, f) \to (V, g)$ maps $U \to V$ over X. Covers are the Nisnevich covers, i.e. jointly surjective étale covers surjective on k-points for any field k.

Definition 2.2.1. Given a t-cover of $\mathcal{U} = (U_i)_{i \in I}$ of X, write $U_{ij} := U_i \times_X U_j, U_{ijk} := U_i \times_X U_j \times_X U_k$ for $i, j, k \in I$. We define a vector bundle with descent data associated with $\mathcal{U}$ over X to be a tuple $(\{\mathcal{E}_i\}_{i \in I}, \{\Phi_{ij}\}_{i,j \in I})$, where $\mathcal{E}_i \to U_i$ is a vector bundle, and $\Phi_{ij} : \mathcal{E}_j |_{U_{ij}} \cong \mathcal{E}_i |_{U_{ij}}$, such that the following diagram commutes for any triple $i, j, k \in I$:

$$\begin{tikzcd} \mathcal{E}_k |_{U_{ijk}} \arrow[r, "\pi_{23}^* \Phi_{jk}"] \arrow[dr, "\pi_{13}^* \Phi_{ik}"'] & \mathcal{E}_j |_{U_{ijk}} \arrow[d, "\pi_{12}^* \Phi_{ij}"] \\ & \mathcal{E}_i |_{U_{ijk}}, \end{tikzcd}$$

with reference to the diagram of projections

$$
\begin{tikzcd}
& U_{ijk} \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\
U_{ij} \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & U_{ik} \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & U_{jk} \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \\
U_i & U_j & U_k
\end{tikzcd}
$$

This latter coherence condition is often called the cocycle condition, and ensures that on triple intersections U_{ijk} we have transitivity of our compatibility isomorphisms.
A morphism of descent data $(\{\mathcal{E}_i\}_{i\in I}, \{\Phi_{ij}\}_{i,j\in I}) \to (\{\mathcal{F}_i\}_{i\in I}, \{\Psi_{ij}\}_{i,j\in I})$ is a collection of vector bundle maps $f_i : .\mathcal{E}_i \to \mathcal{F}_i$ over U_i, commuting with Φ and Ψ, i.e. such that the following square commutes for each pair $i, j \in I$:

$$
\begin{tikzcd}
\mathcal{E}_j \mid_{U_{ij}} \arrow[r, "f_j|_{U_{ij}}"] \arrow[d, "\Phi_{ij}"] & \mathcal{F}_j \mid_{U_{ij}} \arrow[d, "\Psi_{ij}"] \\
\mathcal{E}_i \mid_{U_{ij}} \arrow[r, "f_i|_{U_{ij}}"] & \mathcal{F}_i \mid_{U_{ij}} .
\end{tikzcd}
$$

Given the above, we define the category of descent data $\mathrm{Vect}(\mathcal{U})$ associated with the cover $\mathcal{U}$. If we write $\mathrm{Vect}(X)$ for the groupoid of vector bundles over X, we have a natural functor

$$\mathrm{Vect}(X) \to \mathrm{Vect}(\mathcal{U}),$$

sending a vector bundle $\mathcal{E} \to X$ to $(\{\mathcal{E} \mid_{U_i}\}, \{\Phi_{ij}\})$. Here, $\mathcal{E} \mid_{U_i} := \mathcal{E} \times_X U_i$ is the pullback of $\mathcal{E}$ to U_i, and Φ_{ij} is the natural isomorphism

$$(\mathcal{E} \mid_{U_i}) \mid_{U_{ij}} = (\mathcal{E} \times_X U_j) \times_{U_j} U_{ij} \cong \mathcal{E} \times_X U_{ij} \cong (\mathcal{E} \times_X U_i) \times_{U_i} U_{ij} = (\mathcal{E} \mid_{U_i}) \mid_{U_{ij}} .$$

In the case $t = t_{Zar}$ and a Zariski cover $\mathcal{U} = (U_i)_{i\in I}$ of X, a vector bundle with descent data on $\mathcal{U}$ is a collection of vector bundles over each open U_i, together with compatibility data on overlaps satisfying the cocycle condition. Implicit last week was the fact that this compatibility data is sufficient to ensure that there exists a vector bundle $\mathcal{E}$ on X, such that the restriction of $\mathcal{E}$ to U_i is isomorphic to $\mathcal{E}_i$.

Definition 2.2.2. We say descent for vector bundles in the t-topology on X_t holds if for each scheme X and t-cover $\mathcal{U}$ of X, the functor

$$\mathrm{Vect}(X) \to \mathrm{Vect}(\mathcal{U})$$

is an equivalence of categories.

Remark 2.2.3. Formally, full-faithfulness of the functor $\mathrm{Vect}(X) \to \mathrm{Vect}(\mathcal{U})$ is the statement that the assignment $X \mapsto \mathrm{Vect}(X)$ is a prestack for the t-topology; this means in particular that if we have a map $f : \mathcal{E} \to \mathcal{F}$ which locally on some t-cover $\mathcal{U}$ is an isomorphism, then f is an isomorphism. Essential surjectivity of this functor is then the statement that every descent datum on X is *effective*, i.e., arises as the descent datum associated to some globally defined $\mathcal{E}$ on X. In this case, we say $\mathrm{Vect}(-)$ is a stack.

We can consider the fpqc (faithfully flat) topology on X, for which covers are roughly jointly surjective families of flat quasi-compact maps into X. We have the chain

$$t_{fpqc} < t_{ét} < t_{Nis} < t_{Zar},$$

where $<$ is taken to mean 'finer than'. It is a classical result that faithfully flat descent for vector bundles holds, and given this, descent for étale, Nisnevich, and Zariski covers follows.

In the Zariski case for $X = \operatorname{Spec} R$ affine, it suffices to consider two-fold covers[2], given by elementary distinguished opens $D(f), D(g)$ for comaximal $f, g \in R$. It turns out that for a square

$$\begin{tikzcd} \operatorname{Spec} R_{fg} \arrow[r, hook] \arrow[d, hook] & \operatorname{Spec} R_g \arrow[d, hook] \\ \operatorname{Spec} R_f \arrow[r, hook] & \operatorname{Spec} R, \end{tikzcd}$$

given $(Q \in \operatorname{Proj}(R_f), Q' \in \operatorname{Proj}(R_g), \Phi : Q \otimes_{R_f} R_{fg} \cong Q' \otimes_{R_g} R_{fg})$, we immediately have that Φ satisfies the cocycle condition for this two-fold cover. Intuitively, what's going on here is that triple intersections like $\operatorname{Spec} R_f \times_X \operatorname{Spec} R_g \times_X \operatorname{Spec} R_f$ correspond to tensor products $R_f \otimes_R R_g \otimes R_f$, and since localisation is idempotent (the multplication map $R_f \otimes_R R_f \to R_f$ is an isomorphism), our triple intersections degenerate.

2.3. **Nisnevich patching.** In the case t is a finer topology than the Zariski, we have to be more careful: a ring map $f : R \to S$ is such that the multiplication map $S \otimes_R S \to S$, $s \otimes t \mapsto st$, is an isomorphism if and only if f is an epimorphism. This is the case for localisations (epimorphisms of rings do not need to be surjections!), but not for general étale maps. In particular, a fibre product $U \times_X U$ is rarlely isomorphic (over X) to U for $U \to X$ étale, and so higher intersections like $U \times_X U$ carry nontrivial geometric data[3].

Given an affine distinguished Nisnevich square

$$(\star) \qquad \begin{tikzcd} R \arrow[r, "k"] \arrow[d, "f"] & R_x \arrow[d, "g"] \\ S \arrow[r, "h"] & S_{f(x)}, \end{tikzcd}$$

projective modules Q and Q' over R_x, S respectively, and $\Phi : Q \otimes_{R_x} S_{f(x)} \cong Q' \otimes_S S_{f(x)}$, we can ask whether Φ can be promoted to a cocycle, in which case the theory of faithfully flat descent tells us that we can patch Q and Q' to give a projective module P over R.

It is a theorem of Landsburg [Lan81, Th. 1.3] that every such Φ does extend to a cocycle. A decade later, Landsburg published a shorter proof in [Lan92] of a specific patching result for projective modules which in particular covers the affine Nisnevich case, which we now sketch. One nice aspect of this proof is that there is no explicit mention of cocycles: using a method of Milnor in [Mil71], we can patch projective modules 'by hand'.

With the notation of ($\star$), consider firstly finitely generated free modules $Q \in \operatorname{Proj}(R_x)$ and $Q' \in \operatorname{Proj}(S)$, together with an isomorphism of $S_{f(x)}$-modules.

$$\Phi : Q \otimes_{R_x} S_{f(x)} \cong Q' \otimes_S S_{f(x)}.$$

Choosing bases of $\{x_i\}_i$ and $\{y_j\}_j$ of Q and Q', Φ is represented by some matrix $A \in \mathrm{GL}_n(S_{f(x)})$. Suppose A can be expressed as $h(B) \cdot g(C)$, for some $B \in \mathrm{GL}_n(S)$ and $C \in \mathrm{GL}_n(S_x)$; then writing $g_* x_j$ and $h_* y_j$ for the image of x_j, y_j in $Q \otimes_{R_x} S_{f(x)}$ and $Q' \otimes_S S_{f(x)}$, we have

$$A \cdot g_* \underline{x} = h(B) \cdot g(C) \cdot g_* \underline{x} = h_* \underline{y}$$

$$\Longrightarrow \; g_*(C \cdot \underline{x}) = h_*(B^{-1} \cdot \underline{y}).$$

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[2] This is related to the statement that the Zariski topology on Sch_S (at least for S qcqs) is generated by a cd-structure. See [AHW17] for a nice discussion of cd structures.

[3] This kind of data for a map $A \to B$ is packaged into the Amitsur complex

$$0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \dots,$$

which features heavily in Grothendieck's proof of faithfully flat descent for modules.

Setting $\underline{x}' := C \cdot \underline{x}$ and $\underline{y}' := B^{-1} \cdot \underline{y}$, we then have that with respect to the bases $\underline{x}'$ and $\underline{y}'$, the restrictions of Q and Q' to $\mathrm{Proj}(S_{f(x)})$ can be taken to be isomorphic via the identity.
Forming the R-module P as the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "\Phi \circ g_*"] \\ Q' \arrow[r, "h_*"] & Q' \otimes_S S_{f(x)}, \end{tikzcd}$$

so P consists of pairs (p,q) for $p \in Q, q \in Q'$, such that $g_* p = h_* q$, we check that P is free with basis given by $\{(x_i, y_i)\}_i$.
The central result of [Lan92] is that given some $A \in \mathrm{GL}_n(S_{f(x)})$, the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

can be expressed as $h(B) \cdot g(C)$, as above. Given this, suppose we have a tuple (Q, Q', Φ) as above with Q and Q' still assumed free, but Φ an isomorphism represented by any $A \in \mathrm{GL}_n(S_{f(x)})$. If $\Psi : Q \otimes_{R_x} S_{f(x)} \cong Q' \otimes_S S_{f(x)}$ is the isomorphism represented by A^{-1}, the isomorphism

$$(\Phi \oplus \Psi) : (Q \oplus Q) \otimes_{R_x} S_{f(x)} \cong (Q' \oplus Q') \otimes_S S_{f(x)}$$

is represented by $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$, and so the argument above shows that we can modify the bases of $Q \oplus Q$ and $Q' \oplus Q'$ so that the corresponding restrictions are isomorphic via the identity. We then have that the pullback of $Q \oplus Q$ and $Q' \oplus Q'$ is free. Write $P(Q \oplus Q, Q' \oplus Q', \Phi \oplus \Psi)$ for the pullback; then

$$P(Q \oplus Q, Q' \oplus Q', \Phi \oplus \Psi) \cong P(Q, Q', \Phi) \oplus P(Q, Q', \Psi),$$

so the pullback of Q and Q' is a submodule of a free module and hence projective.
In the general case, Q and Q' are simply assumed to be f.g. projective; there then exist complementary summands $\widetilde{Q}$ and $\widetilde{Q'}$ such that $Q \oplus \widetilde{Q}$ and $Q' \oplus \widetilde{Q'}$ are free over R_x and S respectively; we can extend Φ to an isomorphism $(Q \oplus \widetilde{Q}) \otimes_{R_x} S_{f(x)} \cong (Q' \oplus \widetilde{Q'}) \otimes_S S_{f(x)}$ by adding (finite) free direct summands on either side as necessary, and so obtain an isomorphism of f.g. free modules $\widetilde{\Phi}$ over $S_{f(x)}$. By above, the pullback of these free modules over R is f.g. projective, and since the pullback of Q and Q' is a direct summand of this, it is f.g. projective. We have thus sketched:

Theorem 2.3.1. *[Lan92] Suppose given an affine distinguished Nisnevich square (R, S, x) as above, and f.g. projective R_x and S modules Q and Q'. If $\Phi : Q \mid_{S_{f(x)}} \cong Q' \mid_{S_{f(x)}}$ is an isomorphism of $S_{f(x)}$-modules, Φ extends to a cocycle.*

Corollary 2.3.2. *For an affine distinguished Nisnevich square (R, S, x), the following square of exact categories is (homotopy) cartesian:*

$$\begin{tikzcd} \mathrm{Proj}(R) \arrow[r] \arrow[d] & \mathrm{Proj}(R_x) \arrow[d] \\ \mathrm{Proj}(S) \arrow[r] & \mathrm{Proj}(S_{f(x)}), \end{tikzcd}$$

i.e. every tuple $(Q \in \mathrm{Proj}(R_x), Q' \in \mathrm{Proj}(S), \Phi : Q \mid_{S_{f(x)}} \cong Q' \mid_{S_{f(x)}})$ gives rise to a f.g. projective R module, and every f.g. projective R-module arises in this way:

$$(P \otimes_R R_x) \underset{(P \otimes_R S_{f(x)})}{\times} (P \otimes_R S) \cong P.$$

3. Lindel's proof of the Bass-Quillen conjecture in the geometric case

In [Lin82], Lindel shows that there is a class of rings of geometric nature for which the Bass-Quillen conjecture holds.

Definition 3.0.1. For a field k, a k-algebra A is essentially of finite type over k if it is a localisation $S^{-1}(k[X_1,\dots,X_n]/(f_1,\dots,f_r))$, of some finite type k-algebra.

Suppose we have some k-algebra A essentially of finite type and some f.g. projective module P over $R[t_1,\dots,t_n]$. By Quillen patching, to show P is extended from A, it suffices to consider the case where A is local, essentially of finite type over a field k. In this case we want to show that every projective module over $A[t_1,\dots,t_n]$ is free for each $n \geq 0$.

Lemma 3.0.2. *Suppose we have some Nisnevich square (R,S,x), and a f.g. projective S-module Q such that $Q \otimes_S S_{f(x)}$ is free; then Q is extended from some R-module P.*

Proof. Suppose $Q \otimes_S S_{f(x)}$ is free of rank n; then we can patch the pair $(Q, R_x^{\oplus n})$ along the identity matrix of $\mathrm{GL}_n(S_{f(x)})$ by Nisnevich descent, and so obtain a f.g. projective R-module P such that $P \otimes_R S \cong Q$. □

Lindel's proof goes as follows:

(i) Apply a reduction argument of Mohan-Kumar to show it suffices to assume k perfect (even prime);
(ii) Results of Quillen and Suslin show that if A is a regular noetherian local ring of Krull dimension ≤ 2, every vector bundle over $\mathbb{A}_A^n$ is extended from A. So assume $d := \dim A \geq 3$ and induct on d.
(iii) Construct a Nisnevich neighbourhood $B \subset A$, where B is of the form $k[X_1,\dots,X_d]_{\mathfrak{m}}$ for $\mathfrak{m} = (f(X_1), X_2, \dots X_n)$ maximal with f irreducible.
(iv) Show directly sthat the Bass-Quillen conjecture holds for B.
(v) Use Nisnevich descent and the induction hypothesis to show that P is extended from A.

Lemma 3.0.3. *Let $d \geq 3$. Suppose the Bass-Quillen conjecture holds for regular rings of essentially finite type over k of dimension $< d$, and let $B := k[X_1,\dots,X_d]_{(f(X_1),X_2,\dots,X_d)}$ for f irreducible. Then any f.g. projective module over $B[t_1,\dots,t_n]$ is extended from B.*

Proof. Let $B' := k[X_1,\dots,X_{d-1}]_{f(X_1),X_2,\dots,X_{d-1})}$, noting that we have a map $B'[X_d] \to B$. The induced map $B'[t_1,\dots,t_n,X_d] \to B[t_1,\dots,t_n]$ is étale and such that

$$B'[t_1,\dots,t_n,X_d]/(X_d) \cong B/(X_d),$$

so the tuple $(B'[X_d][t_1,\dots,t_n], B[t_1,\dots,t_n], X_d)$ is an affine distinguished Nisnevich square:

$$\begin{tikzcd} B'[X_d,t_1,\dots,t_n] \arrow[r] \arrow[d] & B[t_1,\dots,t_n] \arrow[d] \\ B'[X_d,t_1,\dots,t_n]_{X_d} \arrow[r] & B_{X_d}[t_1,\dots,t_n]. \end{tikzcd}$$

Note that B' and B_{X_d} are of dimension $d-1$ (the latter because X_d is in the maximal ideal of B). Given a f.g. projective $B[t_1,\dots,t_n]$-module P, we see that P_{X_d} is extended from B_{X_d}, by the induction hypothesis. Then $P_{X_d} \cong (P_{X_d}/(t_1,\dots,t_n)P_{X_d}) \otimes_{B_{X_d}} B_{X_d}[t_1,\dots,t_n]$, and since

$$P_{X_d}/(t_1,\dots,t_n)P_{X_d} \cong (P/(t_1,\dots,t_n)P) \otimes_B B_{X_d}$$

and B is local, $P/(t_1,\dots,t_n)P$ and hence P_{X_d} are free. By the lemma above,

$$P \cong Q \otimes_{B'[X_d,t_1,\dots,t_n]} B[t_1,\dots,t_n]$$

for some f.g. projective $B'[X_d,t_1,\dots,t_n]$-module Q. Now B' is local of dimension $d-1$, and so by the inductive hypothesis again, Q is extended from B', and hence free. So P is also free. □

In [Lin82], Lindel shows that for an étale extension of local rings $B \subset A$ such that A and B have that same residue field, there exists some $h \in \mathfrak{m}_B$ such that $B/h \cong A/h$, and hence the tuple (B, A, h) is an affine distinguished Nisnevich square. With this and (iii) above, the main result follows.

Theorem 3.0.4. *Suppose $(A, \mathfrak{m}, \kappa)$ is a regular local ring of dimension d essentially of finite type over a perfect ground field k. Then for any $n \geq 0$, any finitely generated projective module over $A[t_1, \dots, t_n]$ is extended from A.*

Proof. Let $B := k[X_1, \dots, X_d]_{f(X_1), X_2, \dots, X_d)} \subset A$ be the étale neighbourhood above, and take $h \in B$ such that (B, A, h) is Nisnevich. Then the same holds for $(B[t_1, \dots, t_n], A[t_1, \dots, t_n], h)$. Consider then the (homotopy) cartesian square of exact categories

$$\begin{tikzcd} \mathrm{Proj}(B[t_1, \dots, t_n]) \arrow[r] \arrow[d] & \mathrm{Proj}(A[t_1, \dots, t_n] \arrow[d] \\ \mathrm{Proj}(B[t_1, \dots, t_n]_h) \arrow[r] & \mathrm{Proj}(A[t_1, \dots, t_n]_h). \end{tikzcd}$$

Since $h \in \mathfrak{m}_B \subset \mathfrak{m}_A$, A_h has dimension $< d$, and so by the inductive hypothesis, $P \otimes_{A[t_1, \dots, t_n]} A_h[t_1, \dots, t_n]$ is extended from A_h, and again free by the argument above. Then $P \cong Q \otimes_{B[t_1, \dots, t_n]} A[t_1, \dots, t_n]$ for some projective module $Q \in \mathrm{Proj}(B[t_1, \dots, t_n])$. From the lemma above, Q is free, and hence so is P. □

The main content of the proof is proving that the Nisnevich neighbourhood we've used here exists; we won't sketch the details here, but suggest the original article [Lin82], or [Man97, §7] and [Nas83, Th. 2.8] for the variant of the argument we've presented here.

4. An alternative to the étale neighbourhood lemma

Another approach to Lindel's result is to use the following theorem of Gabber (for infinite fields, with the finite field case provided by Hogadi-Kulkarni).

Theorem 4.0.1 (Gabber, Hogadi-Kolkarni)**.** *Suppose that k is a field, and that X is a smooth affine k-variety of dimension $d \geq 1$. Let $Z \subset X$ be a principal divisor defined by some $f \in \mathscr{O}_X(X)$, and $p \in Z$ a closed point. There exist*

(i) a Zariski-open neighbourhood U of the image of p in X;
(ii) a morphism $\Phi : U \to \mathbb{A}^d_k$;
(iii) an open neighbourhood $V \subset \mathbb{A}^{d-1}_k$ of the image of composite

$$\Psi : U \xrightarrow{\Phi} \mathbb{A}^d_k \xrightarrow{\pi} \mathbb{A}^{d-1}_k,$$

where π is projection away from the last coordinate),

satisfying:

(a) Φ is étale;
(b) for $Z_V := Z \cap \Psi^{-1} V$, the map $\Psi : Z_V \to V$ is finite;
(c) the map $\Phi|_{Z_V} : Z_V \to \mathbb{A}^1_V = \pi^{-1}(V)$ is a closed immersion;
(d) the restriction $\Phi|_{Z_V} : Z_V \to \Phi(Z_V)$ is an isomorphism.

In particular, there is a distinguished Nisnevich square

$$(\star\star) \qquad \begin{tikzcd} U - Z_V \arrow[r, hook] \arrow[d] & U \arrow[d, "\Phi"] \\ \mathbb{A}^1_V - \Phi(Z_v) \arrow[r, hook] & \mathbb{A}^1_V. \end{tikzcd}$$

The construction of ($\star\star$) is arguably a lot more involved than that of Lindel's étale neighbourhood, but it streamlines the proof as follows: we may assume as before that k is perfect, in which case X is smooth over $k \iff$ regular. For $X = \operatorname{Spec} A$ for A a regular local ring of dimension $d \geq 2$, with $p \in X$ the unique closed point, and $Z \subset X$ the principal divisor associated to some nonzero $f \in \mathfrak{m}$. Any open subscheme of X containing p is X itself (since if non-empty, the closed complement contains a closed point of X disjoint from U).
As in [AHW20], we can take $V = \operatorname{Spec} B$ regular affine over k, necessarily of dimension $\leq d-1$ (as an open of $\mathbb{A}_k^{d-1}$). We thus have a distinguished Nisnevich square

$$\begin{tikzcd} \operatorname{Spec} A_f \arrow[r, hook] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ W \arrow[r, hook] & \operatorname{Spec} B[t], \end{tikzcd}$$

for some open $W \subset \operatorname{Spec} B[t]$. We find ourselves in a similar situation to before, noting that A_f and B have dimension $< d$ and so can be used to induct.

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