Registration for TCC : please sign up

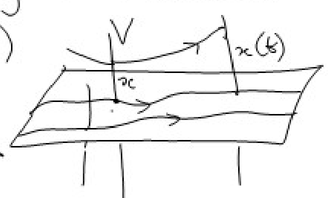
(Warwick people : a form circulating today)

First half : uniform hyperbolicity

Second half : assessment – who'd like to write what.

Uniform hyperbolicity is a property of sets of linearised dynamics about trajectories of an underlying system: $\dot{x}(t) = A(t)x(t)$

$x \in V$ normed vector space (or really vector bundle over the underlying trajectory)



A set of such linear systems is <u>uniformly hyperbolic</u> if $\exists K > 0$ s.t. $\forall$ bounded continuous functions $f : \mathbb{R} \to V$

$\dot{x} = Ax + f$ <u>has</u> a <u>unique</u> bounded solution x and $|x|_1 \leq K^{-1}|f|$

where $|f| = \sup_{t \in \mathbb{R}} |f(t)|$,

$|x|_1 = \max(|x|, \tau|\dot{x}|)$ some $\tau > 0$

e.g. $\dot{x} = 2x$ on $V = \mathbb{R}$

$$\dot{x} = 2x + f$$

$$(x e^{-2t})^{\cdot} = e^{-2t} f(t)$$

$$x e^{-2t} = x(0) + \int_0^t ds\, e^{-2s} f(s)$$

$$x(t) = x(0) e^{+2t} + \int_0^t ds\, e^{2(t-s)} f(s)$$

Unique bdd sln is $\int_t^{+\infty} ds\, e^{2(t-s)} f(s)\, ds$

not u. hyp $\dot{x} = 0.x$

$$\dot{x} = 0.x + f$$

$$x(t) = x(0) + \int_0^t ds\, f(s)$$

so eg. $f(t) = 1 \Rightarrow x(t) = x(0) + t$ has no bdd slns

Equivalently let $L : C^1(\mathbb{R}, V) \to C^0(\mathbb{R}, V)$

by $L[x](t) = \dot{x}(t) - A(t)x(t)$

Then set of lin systems is u. hyp iff $\exists K > 0$ s.t.

L is invertible & $\|L^{-1}\| \leq K^{-1}$

because then $\dot{x} = Ax + f$ has unique bdd sln $x = L^{-1}[f]$.

Robustness of uniform hyperbolicity: if $\dot{x} = Ax$ is u. hyp then so is $\dot{x} = \tilde{A}x$ for $|\tilde{A} - A|$ small enough.

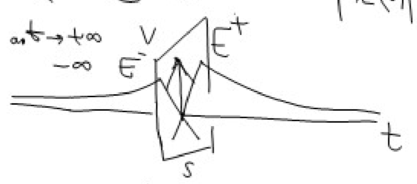
Use Lemma 2.1: If L invertible with L^{-1} bounded & ΔL is a perturbation to L with $|\Delta L| < \|L^{-1}\|^{-1}$ then $L - \Delta L$ is invertible with $\|(L-\Delta L)^{-1}\|^{-1} \geq \|L^{-1}\|^{-1} - |\Delta L|$.

Theorem 2.2: $\dot{x} = Ax$ u. hyp

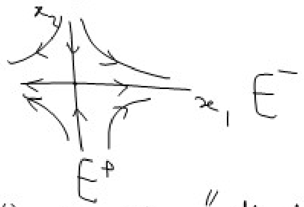
$\Rightarrow \exists$ projections $P_{\pm}(s) : V \to V$ $(P^2 = P)$, $P_+ + P_- = I$, bounded invariant & $\forall \mu < K$ $\exists C$ s.t. for $x(s) \in E_{\pm}(s) = \text{range } P_{\pm}(s)$ then

$$|x(t)| \leqslant C e^{-\mu|t-s|} |x(s)| \text{ for } \begin{cases} t \geq s \\ t \leq s \end{cases}$$

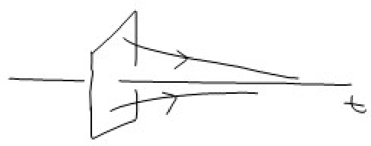
$\pm$ = exp decay as $t \to +\infty$, $-\infty$



cf. $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 \end{cases}$



Special case: $P_-(s) = 0$ $\forall s$ "attracting" case



Define matrix soln of $\frac{\partial}{\partial t} X(t,s) = A(t) X(t,s)$

starting from $X(s,s) = I$ $\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\\vdots\end{bmatrix} \cdots$

<u>Defn</u> [Green function] $G(t,s) = X(t,s)P_+(s)$ for $t > s$

$-X(t,s)P_-(s)$ for $t < s$

It gives the unique bdd response to a δ fn at $t = s$

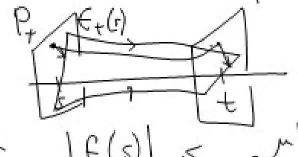
$$\dot{x} = Ax + f\,\delta(t-s) \quad f \in V$$

by $x(t) = G(t,s) f$

And gives the uniq bdd response $x = L^{-1}[f]$ to bdd cnts f by $x(t) = \int_{-\infty}^{\infty} G(t,s) f(s)\, ds$

Invariance of $P_{\pm}$ under linear flow:

$$X(t,s)P_+(s) = P_+(t)X(t,s).$$



Spse $|f(s)| \leqslant e^{\mu|s|}$ some $\mu < K$

Then $|x(0)| \leqslant \frac{1}{K-\mu}$

$\mu < K$, $T > 0$ $\quad f(s) = 0$ for $[-T, T]$

then $|x(0)| \leq \dfrac{e^{-\mu T}}{K-\mu} |f|$

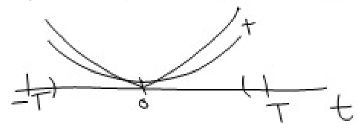
& $\quad \leq T e^{1-KT} |f|$ for $T \geq \frac{1}{K}$

<u>Thm 2.6</u> If $\alpha < K \leq \|L^{-1}\|^{-1}$, $|f(s)| \leq F$

$|f(s)| \leq \varepsilon e^{\alpha |s|}$ for $|s| \leq T$, $x = L^{-1}[f]$

then for $|t| \leq T - \frac{1}{K}$

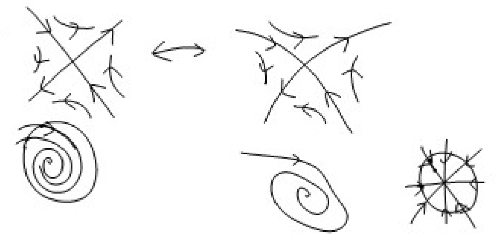
$$|x(t)| \leq \frac{\varepsilon}{K-\alpha} e^{\alpha |t|} + (T - |t|) e^{1-K(T-|t|)} F$$



Hartman – Grobman says that $\exists$ local homeomorphism near a hyperbolic eqm of a vector field conjugating the dyn to its linearisation

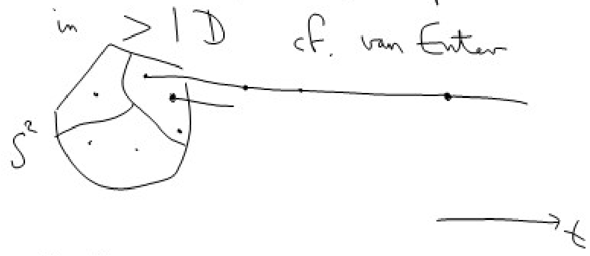
$\dot{x} = X(x, y)$
$\dot{y} = Y(x, y)$ $\quad$ linearise $\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{bmatrix} X_x & X_y \\ Y_x & Y_y \end{bmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

e.g.



Assessment is by writing (possibly in groups) a chapter of lecture notes on Aggregation of complex systems. Main topics:

Equilibrium statistical mechanics (including interpretation of min energy case as dynamic optimisation) esp. what can be done in $> 1D$ cf. van Enter



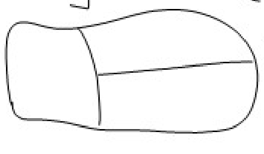
Markov processes

Needs a heuristic for which nodes to aggregate (common theme)

One possibility is "community detection" [Newman]

top-down v. bottom-up

Computer Science literature?

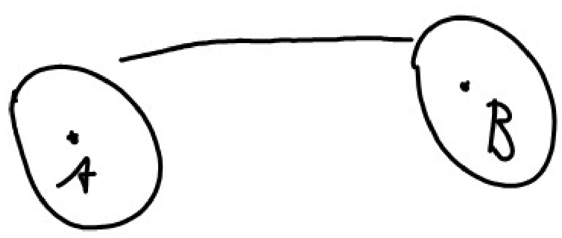


DIMAP

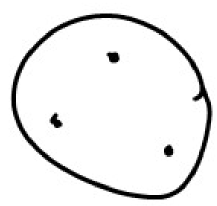
Traffic flows (selfish)

Maybe ∃ good heuristics for (almost) planar graphs

Maybe better first to do shortest route finding e.g. highway hierarchies



Interacting agents   partial orders



Synch in nets of osc.
- attracting case
- heuristic for which to aggreg.

I'll give oral exams if necessary to confirm individuals contributions to a group chapter.