

Existence of invariant circles for infinitely renormalisable area-preserving maps

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Abstract

Existence of an invariant circle for any orientation-preserving 2D map whose orbit under renormalisation remains forever in a certain bounded subset is proved. The construction dates back to 1984. It was stimulated by a preprint by David Rand doing the same for the dissipative case. To include the general case, notably area-preserving, required a variation on his idea.

1 Background

This paper is based on notes that I wrote in April 1984. They were inspired by a preprint by David Rand (eventually published as [R]) in which he proved the analogous result for dissipative annulus maps.

My notes were the starting point for one chapter of the PhD thesis of Nicolai Hoidn whom I supervised from April to September 1984 (eventually published in [H], under his pseudonym), but the idea ended up somewhat obscured under technical analysis there.

The idea was reinvented by Andreas Stirnemann [S], who presented it in the framework of iterated function systems. This made the argument very clear. I was not familiar with the concept back in 1984, but with hindsight one can see it was the right tool, from Fig. 4.4.2.2 of my (1982) PhD thesis (reprinted in [M]), Rand's preprint, and the construction of my notes.

I publish this exposition of my notes now, firstly to put on the record that I had obtained this general existence result back then and secondly to acknowledge David Rand's great influence.

2 Setting

Let (U, T) be a pair of orientation-preserving diffeomorphisms of domains in \mathbb{R}^2 to ranges in \mathbb{R}^2 , which commute on the subset for which both compositions UT, TU are defined. The basic example is (FR, F) where F is a lift to \mathbb{R}^2 of an orientation-preserving degree-one map of a cylinder $\mathbb{T} \times \mathbb{R}$ to itself (with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) (i.e. $F(x+1, y) = F(x, y) + (1, 0)$)

and $R(x, y) = (x - 1, y)$ is a deck transformation. The maps U, T do not have to preserve area, nor have twist, but the case with both is the main motivation.

Assumptions

- A1.** The domains of U and T are assumed to contain a vertical line segment L , without loss of generality in $x = 0$, where both compositions UT, TU are defined and such that $U(L)$ is to the left of L , $T(L)$ to the right of L . The domain of U is assumed to connect L to $T(L)$. The domain of T is assumed to connect L to $U(L)$. See Fig. 1.
- A2.** The horizontal is scaled so that some notion of the horizontal width of the union of the domains of U and T is 1.

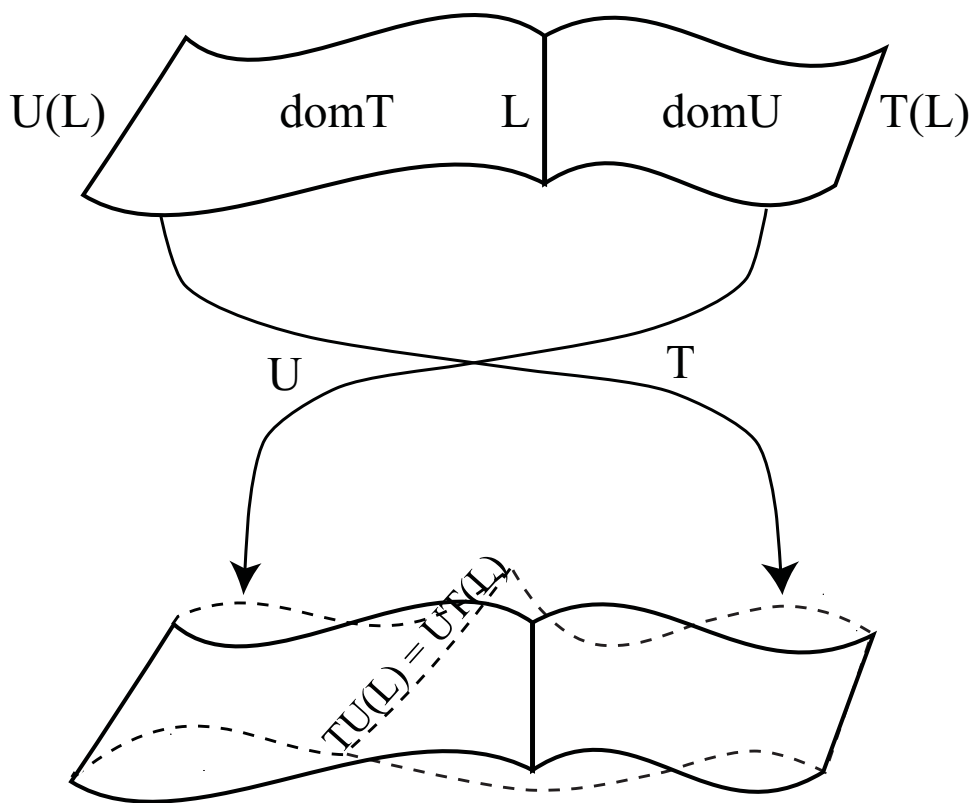


Figure 1: Domains and ranges of U, T .

The map UT^{-1} takes $T(L)$ to $U(L)$ so the union of the domains of U and T can be considered to be an annulus cut along this line, and the pair (U, T) can be considered as a map of this annulus to an annulus with the same cut, by applying U to points between L and $T(L)$ and T to points between $U(L)$ and L . For points on L we apply both U and T , obtaining two different points, but which are identified under UT^{-1} ; their images are the same. Thus we can talk about an orbit segment of a point under (U, T) .

3 Invariant circles

Say a curve γ is an *invariant circle* for (U, T) if it joins $U(L)$ to $T(L)$, crosses L once, the image by T of its segment from $U(L)$ to L is defined and contained in γ and the image by U of its segment from L to $T(L)$ is defined and contained in γ . See Fig. 2. It is called a “circle” because it becomes a circle under the identification of $T(L)$ with $U(L)$ via UT^{-1} .

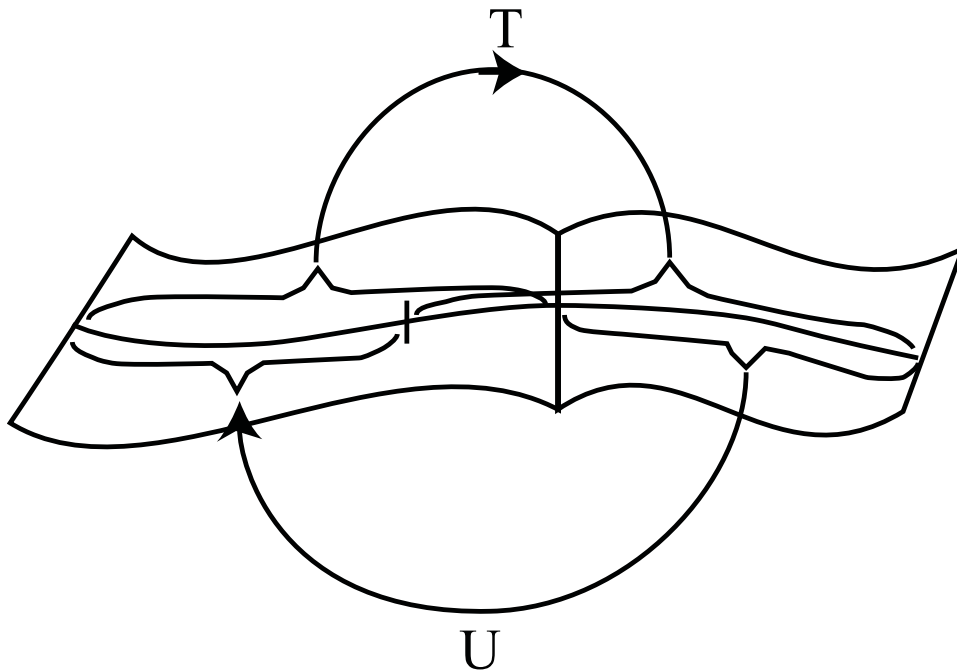


Figure 2: An invariant circle for (U, T) .

An invariant circle has a *rotation number* $\omega \in [0, 1]$, which is the fraction of iterations when U is used.

In my April 1984 notes I restricted attention to the case of *golden circles*, $\omega = \frac{\sqrt{5}-1}{2}$, but I add a section to this paper on the extension to other rotation numbers.

4 Renormalisation

If L has a subinterval L' and restrictions of the domains for T and TU can be chosen so that the conditions A1 apply to the pair (TU, T) then choose a map B of the form

$$B(x, y) = \left(\frac{x}{\alpha}, \frac{y}{\beta} - f(x) \right),$$

with $\alpha, \beta < 0$, taking L to L' with reversed orientation (thus determining β and $f(0)$), and with $-1/\alpha$ chosen to be the horizontal width of the union of the new domains. Then

the pair

$$D(U, T) := (B^{-1}TB, B^{-1}TUB)$$

satisfies the conditions A1 and A2. Note that B preserves the foliation by verticals. The function f is chosen to try to keep the domains for $D(U, T)$ “horizontal”. There are various recipes for this, e.g. [M], but the choice will not be important here.

Note that γ is a golden circle for (U, T) iff $B^{-1}\gamma$ is a golden circle for $D(U, T)$.

The operator D is called a *renormalisation*.

5 Construction

Theorem: If $(U_n, T_n) := D^n(U_0, T_0)$ are defined for all $n \geq 0$ and lie in the set of (U, T) satisfying A1, A2 and B1–B6:

- B1.** $|\beta| \geq \beta_m > 1$;
- B2.** $|\alpha| \geq \alpha_m > 1$;
- B3.** For all $y \in L$, the slopes of the lines from y to $U(0, y)$ and $T(0, y)$ are at most s_m in absolute value;
- B4.** The derivatives of B and UB map vectors of slope at most s_m to vectors of slope at most s_m . For example, the first condition is $\frac{\alpha}{\beta}s_m + |\alpha f'(x)| \leq s_m$;
- B5.** The xx -components of the derivatives of U and T are positive;
- B6.** The derivative of T multiplies the horizontal component of tangent vectors with slope less than s_m by at least $\kappa > 1/\alpha_m$;

then (U_0, T_0) has a golden circle.

Actually, we get in addition that it is an s_m -Lipschitz graph and that the dynamics on it preserves horizontal order and is conjugate to rotation.

The proof requires the following definition (see Fig. 3).

Definition: An orbit segment C for (U, T) is a *cycle* if it forms an s_m -Lipschitz graph, considered as a function y over x , and decomposing it into subsets C_-, C_+ on the left and right of L respectively (including a point on L in both) then $U(C_+)$ is to the left of $T(C_-)$ (with possible overlap only on $UT(L) = TU(L)$) and U, T preserve horizontal order applied to C_+, C_- (ignoring the point which is last in the orbit segment).

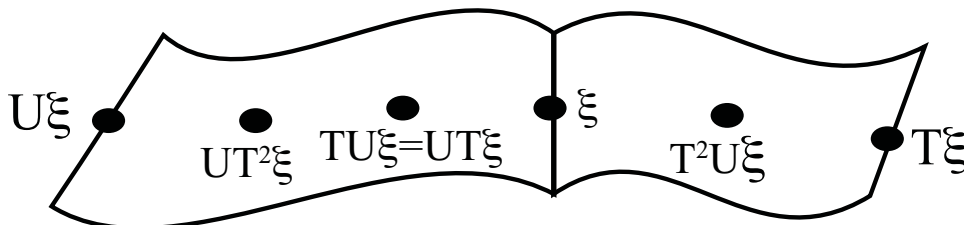


Figure 3: A cycle for (U, T) .

Proof: The sequence of line segments $B_0 \dots B_n(L)$, $n \in \mathbb{Z}_+$, is nested. By B1, it converges to a single point; call it ξ_0 .

The idea of the proof is that the closure of the orbit of ξ_0 under (U_0, T_0) is the desired golden circle. This conjecture was already clear to me in 1982, but to prove it required ideas from David Rand's preprint (building up ordered Lipschitz sets and checking no gaps), plus a replacement for his dissipativity condition in the control of Lipschitz constants.

Given $n \geq 0$, let $\xi_n = B_{n-1}^{-1} \dots B_0^{-1} \xi_0$. It is ξ_0 in the n -th coordinate system. Apply the map (U_n, T_n) once to ξ_n , obtaining an orbit segment C_n^n consisting of the three points $U_n(\xi_n), \xi_n, T_n(\xi_n)$ (by our convention about applying both U and T to points of L), ordered from left to right. It is s_m -Lipschitz by B3. Order-preservation is trivial because in the definition we chose to ignore the last point ($U_n(\xi_n)$ and $T_n(\xi_n)$ are identified).

Thus C_n^n is a cycle for (U_n, T_n) . Next we use it to make a longer cycle for (U_{n-1}, T_{n-1}) and by induction down to (U_0, T_0) .

If C_k^n is a cycle for (U_k, T_k) then we make one for (U_{k-1}, T_{k-1}) by taking $B_{k-1}(C_k^n)$ to be the rescaled cycle with reversed order of points, and concatenating $U_{k-1}B_{k-1}(C_k^n), B_{k-1}(C_k^n)$. The condition B4 guarantees that the points come in the right order and form an s_m -Lipschitz set. Call it C_{k-1}^n . Iterating k from n to 1 produces a cycle C_0^n for (U_0, T_0) .

Now take n to infinity. The C_0^n are a nested sequence ($C_0^n \subset C_0^{n+1}$) so the closure C_0^∞ of their union is an s_m -Lipschitz set, invariant under (U_0, T_0) and of golden rotation number.

The only obstacle to C_0^∞ being an invariant circle is that it might have gaps. If it has a gap then it lies between a consecutive pair of $U_0(\xi_0), T_0U_0(\xi_0), \xi_0, T(\xi_0)$, because these form a subcycle of C_0^∞ . Call its horizontal length ℓ_0 . If the chosen gap lies between $U_0(\xi_0), T_0U_0(\xi_0)$ then take its image by T_0 to obtain a gap between either $T_0U_0(\xi_0), \xi_0$ or $\xi_0, T(\xi_0)$. By condition B6, its horizontal length is at least κ times that of the first one. Thus in any case, we have a gap of horizontal length at least $\kappa\ell_0$. Now apply B_0^{-1} to it to obtain a gap in C_1^∞ between $\xi_1, T_1(\xi_1)$ or $U_1(\xi_1), \xi_1$ respectively. By B2, this gap has horizontal length ℓ_1 at least $\kappa\alpha_m\ell_0$. By induction, C_n^∞ has a gap of horizontal length at least $(\kappa\alpha_m)^n\ell_0$. Since $\kappa\alpha_m > 1$ this eventually exceeds 1, giving a contradiction, because the horizontal length between $U_n(\xi_n)$ and $T_n(\xi_n)$ is of order 1 so there is no room between them for such a long gap. \square

Remark: The sequence of pairs of maps $U_{k-1}B_{k-1}, B_{k-1}$ forms a (non-autonomous) iterated function system, of which the invariant circle is the unique pull-back attractor. Thus any seed suffices to construct it.

Remark: The hypotheses hold for F in an open $C^{3+\varepsilon}$ -neighbourhood of integrable area-preserving twist maps, thus giving a KAM theorem. More importantly, they appear to hold at the critical fixed point of D (whose existence has finally been proved [AK]), so if they can be verified rigorously then all maps on its stable manifold under renormalisation also have a golden circle. This was the main reason for me to develop the idea.

6 Extension to other rotation numbers

For rotation number $\omega_0 \in (0, 1)$, define its continued fraction sequence a_0, a_1, \dots by $a_n = [1/\omega_n]$, $\omega_{n+1} = \frac{1}{\omega_n} - a_n$. Define renormalisation operators

$$D_a(U, T) = (B^{-1}TB, B^{-1}T^aUB)$$

for $a \geq 1$ on those pairs (U, T) for which conditions A1 apply to (T^aU, T) (thus the previous renormalisation $D = D_1$). Then γ is an invariant circle of rotation number ω_0 for (U, T) iff $B^{-1}\gamma$ is one of rotation number ω_1 for $D_{a_0}(U, T)$.

If condition B4 of the Theorem is extended to apply also to $T^{a-1}UB$, $a \geq 1$, then the construction generalises to prove that if $(U_{n+1}, T_{n+1}) = D_{a_n}(U_n, T_n)$ are defined for all $n \geq 0$ and lie in the set satisfying A1, A2 and B1–B6 then (U_0, T_0) has an invariant circle of rotation number ω_0 .

If a_n grows too fast the hypotheses might fail to hold at the conjectured critical points of renormalisation or even on a C^∞ -neighbourhood of integrable area-preserving twist maps. Certainly there are Liouville numbers for which the latter occurs. But some growth is permitted, because the B become much more contracting and flattening as $a \rightarrow \infty$, compensating for the possible rotation effects of the subsequent composition $T^{a-1}U$. In particular, Hoidn showed that in high enough smoothness classes, the approach gives a KAM theorem for all Diophantine rotation numbers [H], and by considering the effect of the iterated function system on not just Lipschitz constants but also higher derivatives, he deduced a high degree of smooth conjugacy to rotation for the invariant circles.

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