Mathematics for Fusion Power part 4

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Outline

1. Velocity (Hamiltonian) symmetries of FGCM
2. Approximate (Hamiltonian) symmetries of FGCM

First-order guiding-center motion (FGCM) - Exact treatment

For a given magnetic field B on a 3-dimensional manifold M, the Hamiltonian structure of FGCM is

$$\omega = -\beta - d(p_{||} b^\flat) \tag{1}$$

$$H = \frac{p_{||}^2}{2} + \mu |B| \tag{2}$$

on the 4-dimensional GC bundle N over M, where

- $\beta = i_B \Omega =$ magnetic flux form,
- $b = B/|B|$ and $b^\flat = i_b g$,
- $g =$ metric tensor on M
- $\Omega =$ associated volume form on M
- $\mu =$ magnetic moment

and normalised units $m = e = 1$.

FGCM - Exact treatment (continued)

In terms of the *modified* magnetic field,

$$\tilde{B} = B + p_{\|} c \tag{3}$$

where $c = \operatorname{curl} b$, the symplectic form can be written as

$$\begin{aligned} \omega &= -\beta - d(p_{\|} b^{\flat}) = -i_B \Omega - dp_{\|} \wedge b^{\flat} - p_{\|} db^{\flat} \\ &= -i_B \Omega - p_{\|} i_c \Omega - dp_{\|} \wedge b^{\flat} \\ &= -\tilde{\beta} - dp_{\|} \wedge b^{\flat} \end{aligned} \tag{4}$$

where $\tilde{\beta} = i_{\tilde{B}} \Omega$ is the modified flux.

Note that $\operatorname{div} \tilde{B} = 0$. This means that $\tilde{\beta}$ is closed on M. But it's not closed on N. To see these, write $\tilde{\beta} = \beta + p_{\|} db^{\flat}$ to derive $d\tilde{\beta} = dp_{\|} \wedge db^{\flat}$ and therefrom $d\tilde{\beta} \wedge dp_{\|} = 0$.

FGCM - Exact treatment (continued)

The GC 2-form ω is nondegenerate if and only if $\tilde{B}_\parallel \neq 0$, where $\tilde{B}_\parallel = b \cdot \tilde{B}$.

Proof. ω is nondegenerate if and only if $i_U\omega = 0 \Leftrightarrow U = 0$, where $U = (u, w)$ is a vector field on N. Now, using (4)

$$i_U\omega = -i_u i_{\tilde{B}}\Omega - wb^\flat + (u \cdot b)dp_\parallel = -(\tilde{B} \times u)^\flat - wb^\flat + (u \cdot b)dp_\parallel$$

So $i_U\omega = 0$ splits to $u \cdot b = 0$ and $\tilde{B} \times u + wb = 0$, which in turn splits to

$$\begin{cases} (\tilde{B} \times u) \times b = 0 \\ w\, b \cdot \tilde{B} = 0 \end{cases} \Rightarrow \begin{cases} (\tilde{B} \cdot b)u - (u \cdot b)\tilde{B} = 0 \\ \tilde{B}_\parallel\, w = 0 \end{cases} \Rightarrow \begin{cases} \tilde{B}_\parallel\, u = 0 \\ \tilde{B}_\parallel\, w = 0 \end{cases}$$

i.e. $U = 0$ is the only solution if-f $\tilde{B}_\parallel \neq 0$. □

Velocity symmetries of FGCM - Exact treatment

Consider a symmetry generated by a vector field

$$U = (u, w)$$

on the guiding-centre phase space N, where

- u is the 3D part on the physical space M
- w is 1D in the $p_\parallel$-direction
- u, w depend on both $(Q, p_\parallel)$
- u, w are considered independent of μ (at least for now)

Recall that the conditions for a Hamiltonian symmetry are

$$L_U \omega = 0$$
$$L_U H = 0$$

Velocity symmetries of FGCM - Exact treatment (continued)

1. From $L_U H = 0$, we have $w p_\parallel + \mu L_u |B| = 0$. For all values of μ, this splits to

$$w = 0 \tag{5}$$

$$L_u |B| = 0 \tag{6}$$

2. For $w = 0$, $L_U \omega$ reduces to $L_u \omega$ and

$$\begin{aligned} L_u \omega &= -L_u \beta - L_u d(p_\parallel b^\flat) = -i_u d\beta - d(i_u \beta) - dL_u(p_\parallel b^\flat) \\ &= -d(i_u i_B \Omega) - d(p_\parallel L_u b^\flat) = -d(i_u i_B \Omega + p_\parallel L_u b^\flat) \end{aligned}$$

since $d\beta = 0$. Thus, $L_U \omega = 0$ if and only if

$$\boxed{i_u i_B \Omega + p_\parallel L_u b^\flat = d\psi} \tag{7}$$

for some function ψ (defined at least locally) on N.

Velocity symmetries of FGCM - Exact treatment (continued)

3. This condition is in turn equivalent to:

$$i_u i_B \Omega + p_{\|} i_u db^\flat + p_{\|} d(i_u b^\flat) = d\psi$$
$$i_u i_B \Omega + p_{\|} i_u i_c \Omega + d(p_{\|} i_u b^\flat) - (i_u b^\flat) dp_{\|} = d\psi$$

and can be written as

$$\boxed{i_u i_{\tilde{B}} \Omega - (u \cdot b) dp_{\|} = -\, dK} \tag{8}$$

where

$$\boxed{K = -\,\psi + (u \cdot b) p_{\|}} \tag{9}$$

Equation (8) splits to

$$u \times \tilde{B} = \nabla K \tag{10}$$
$$u \cdot b = \partial_{p_{\|}} K \tag{11}$$

Velocity symmetries of FGCM - Exact treatment (continued)

4. K is the invariant associated to the symmetry generator U:

$$i_U\omega = i_u\omega = i_u(-\tilde{\beta} - dp_\parallel \wedge b^\flat) = -i_u i_{\tilde{B}}\Omega + (i_u b^\flat)dp_\parallel = dK$$

5. The compatibility condition between (10)-(11) yields

$$\nabla(u \cdot b) = \partial_{p_\parallel}(u \times \tilde{B}) \tag{12}$$

while the compatibility condition of (10) is $\operatorname{curl}(u \times \tilde{B}) = 0$ and

$$\begin{aligned}\operatorname{curl}(u \times \tilde{B}) &= (\operatorname{div}\tilde{B})u - (\operatorname{div}u)\tilde{B} + (\tilde{B}\cdot\nabla)u - (u\cdot\nabla)\tilde{B} \\ &= -(\operatorname{div}u)\tilde{B} + [\tilde{B}, u],\end{aligned}$$

hence reads

$$\boxed{[u, \tilde{B}] + (\operatorname{div}u)\tilde{B} = 0} \tag{13}$$

Velocity symmetries of FGCM - Exact treatment (continued)

6a. Writing $L_u\omega = L_u(-\tilde{\beta} - dp_\| \wedge b^\flat) = -L_u\tilde{\beta} - dp_\| \wedge L_u b^\flat$, note first that yet another way of expressing $L_u\omega = 0$ is

$$L_u\tilde{\beta} + dp_\| \wedge L_u b^\flat = 0 \tag{14}$$

This implies, in particular,

$$\boxed{L_u\tilde{\beta} \wedge dp_\| = 0} \tag{15}$$

6b. Secondly, $i_{\tilde{B}} L_u\tilde{\beta} = (L_u i_{\tilde{B}} - i_{[u,\tilde{B}]})\tilde{\beta} = L_u i_{\tilde{B}}\tilde{\beta} + \operatorname{div} u\, i_{\tilde{B}}\tilde{\beta}$, using (13), and since $i_{\tilde{B}}\tilde{\beta} = i_{\tilde{B}} i_{\tilde{B}}\Omega = 0$, we have

$$\boxed{i_{\tilde{B}} L_u\tilde{\beta} = 0} \tag{16}$$

6c. Thirdly, applying $i_{\tilde{B}}$ to (14) and using (16), we also deduce

$$\boxed{i_{\tilde{B}} L_u b^\flat = 0} \tag{17}$$

Velocity symmetries of FGCM - Exact treatment (continued)

7. Using this,

$$i_{[u,\tilde{B}]}b^\flat = (L_u i_{\tilde{B}} - i_{\tilde{B}} L_u)b^\flat = L_u \tilde{B}_\parallel$$

hence the b-component of the compatibility (13) reads

$$\boxed{L_u \tilde{B}_\parallel + (\operatorname{div} u)\tilde{B}_\parallel = 0} \tag{18}$$

8. Finally, for $\bar{b} = \tilde{B}/\tilde{B}_\parallel$

$$[u, \bar{b}] = L_u(\tilde{B}_\parallel^{-1})\tilde{B} + \tilde{B}_\parallel^{-1}[u, \tilde{B}] = -\tilde{B}_\parallel^{-2} L_u(\tilde{B}_\parallel)\tilde{B} + \tilde{B}_\parallel^{-1}[u, \tilde{B}]$$

and so we deduce from (13),(18) that

$$\boxed{[u, \bar{b}] = 0} \tag{19}$$

Velocity symmetries of FGCM - Exact treatment (continued)

In summary, what we can say so far are

Theorem 1

Given a magnetic field B, a vector field $U = (u, w)$ on N generates a Hamiltonian symmetry of FGCM if-f $L_u\tilde{\beta} + dp_{\|} \wedge L_u b^\flat = 0$, $L_u|B| = 0$, $w = 0$.

Theorem 2

If a $p_{\|}$-dependent vector field u on M generates a Hamiltonian symmetry of FGCM, then

- $[u, \tilde{B}] + (\operatorname{div} u)\tilde{B} = 0$
- $[u, \bar{b}] = 0$
- $\nabla(u \cdot b) = \partial_{p_{\|}}(u \times \tilde{B})$
- $i_{\tilde{B}} L_u b^\flat = 0$
- $i_{\tilde{B}} L_u \tilde{\beta} = 0$, $L_u\tilde{\beta} \wedge dp_{\|} = 0$

Velocity symmetries of FGCM - Exact treatment (continued)

In summary, what we can say so far are

Theorem 1

Given a magnetic field B, a vector field $U = (u, w)$ on N generates a Ham. symmetry of FGCM if-f $u \times \tilde{B} = \nabla K$, $u \cdot b = \partial_{p_\parallel} K$, $u \cdot \nabla |B| = 0$, $w = 0$, where K is the associated invariant.

Theorem 2

If a $p_\parallel$-dependent vector field u on M generates a Hamiltonian symmetry of FGCM, then

- $[u, \tilde{B}] + (\operatorname{div} u)\tilde{B} = 0$
- $[u, \bar{b}] = 0$
- $\nabla(u \cdot b) = \partial_{p_\parallel}(u \times \tilde{B})$
- $i_{\tilde{B}} L_u b^\flat = 0$
- $i_{\tilde{B}} L_u \tilde{\beta} = 0$, $L_u \tilde{\beta} \wedge dp_\parallel = 0$

FGCM & Symmetries - Approximate treatment

- FGCM is the 1st-order approximation of GCM wrt $\varepsilon = m/e \ll 1$

$$\omega = -\beta - \varepsilon\, d(p_{\|} b^{\flat})$$
$$H = \varepsilon\,(p_{\|}^2/2 + \mu|B|)$$

So, natural to consider:

- Approximate vector fields of 1st-order

$$U = U_0 + \varepsilon U_1$$

- Approximate symmetries of 1st-order

$$L_U \omega = O(\varepsilon^2)$$
$$L_U H = O(\varepsilon^2)$$

FGCM & Symmetries - Approximate treatment

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$$\omega = -\beta - \varepsilon\, d(p_{\|} b^{\flat})$$
$$H = \varepsilon\,(p_{\|}^2/2 + \mu|B|)$$

So, natural to consider:

- Approximate vector fields of 1st-order

$$U = U_0 + \varepsilon U_1$$

- Approximate symmetries of 1st-order

$$L_U \omega \approx 0$$
$$L_U H \approx 0$$

From now on, we write $A = B + O(\varepsilon^n)$ as $A \approx B$ for any two tensors of the same type. For FGCM, we take $n = 2$.

FGCM & Symmetries - Approximate treatment (continued)

Approximate version of Noether's theorem

A vector field U generates an approximate symmetry of an approximate Hamiltonian system (ω, H) if-f there exists an approximate constant of motion K such that $i_U\omega \approx dK$.

Proof. For any $K = K_0 + \varepsilon K_1 + \cdots$, a vector field U s.t. $i_U\omega \approx dK$ is well-defined for $\omega = \omega_0 + \varepsilon\omega_1 + \cdots$, since ω_0 is nondegenerate,

$$\begin{aligned} i_{U_0}\omega_0 &= dK_0 \\ i_{U_1}\omega_0 + i_{U_0}\omega_1 &= dK_1 \\ i_{U_2}\omega_0 + i_{U_1}\omega_1 + i_{U_0}\omega_2 &= dK_2 \\ &\dots \end{aligned} \tag{20}$$

Thus, $L_U\omega \approx 0$ and, if $L_X K \approx 0$, $L_U H \approx 0$ too, because

$$L_U\omega = d i_U\omega \tag{21}$$

$$L_U H = i_U dH = i_U i_X \omega = -i_X dK = -L_X K. \tag{22}$$

In the other direction, if U generates an approximate Hamiltonian symmetry, then (21) gives $i_U\omega \approx dK$ for some (suppose global) function K, and (22) gives $L_X K \approx 0$. □

FGCM & Symmetries - Approximate treatment (continued)

Complication:

For $\varepsilon = 0$ the GC 2-form, $\omega_0 = -\beta$, is degenerate of rank 2 (i.e., *presymplectic* of constant rank) for $B \neq 0$

because $i_U\beta = i_u i_B \Omega = (B \times u)^\flat$ for any vector field $U = (u, w)$ on N, and therefore setting $i_U\beta = 0$, we see that

The kernel of β (naturally pullbacked) on N consists of all the vector fields (fb, g) for arbitrary functions f, g

hence is two-dimensional.

This produces

Trivial symmetries

A trivial approximate symmetry is generated by any vector field S s.t. $i_S\omega \approx 0$. For the GC 2-form ω, $S = \varepsilon S_1$ with $S_1 \in \ker \beta$.

Approximate Symmetries of FGCM (Burby, K, MacKay)

Theorem 3

Given a magnetic field B, a v.f. $U = (u, w) = (u_0 + \varepsilon u_1, w_0 + \varepsilon w_1)$ on N generates an approximate Ham. symmetry of FGCM if-f $L_{u_0}\beta = 0$, $p_\| L_{u_0} b^\flat + i_{u_1} i_B \Omega = d\psi_1$, $L_{u_0}|B| = 0$, $w_0 = 0$ for a function ψ_1 on N.

Proof. Take $L_U H \approx 0$, $L_U \omega \approx 0$ and split up by different powers of ε, dropping any 2nd-order terms. The first condition gives

$$p_\| w_0 + \mu L_{u_0}|B| = 0$$

thus $w_0 = 0$, $L_{u_0}|B| = 0$ for all μ. For $w_0 = 0$, $L_U\omega \approx 0$ reduces to $L_u \omega \approx 0$, so from the second condition, we have

$$L_{u_0}\beta = 0$$
$$L_{u_0} d(p_\| b^\flat) + L_{u_1}\beta = 0$$

from the 0th- and 1st-order terms, respectively. Same as in the exact treatment (see eq. (7)), the latter gives $p_\| L_{u_0} b^\flat + i_{u_1} i_B \Omega = d\psi_1$ for some function ψ_1 on N. Straightforwardly, the converse is also true. □

Approximate Symmetries of FGCM (continued)

Flux surfaces. From $L_{u_0}\beta = 0$, we have $i_{u_0}\beta = d\psi_0$ for some function ψ_0 on N, because β is closed. The $p_\|$-component gives $\partial_{p_\|}\psi_0 = 0$, and since $i_{u_0}\beta = i_{u_0} i_B \Omega = (B \times u_0)^\flat$, we deduce then

$$B \times u_0 = \nabla \psi_0 \tag{23}$$

Theorem 4

If a vector field $U = (u_0 + \varepsilon u_1, \varepsilon w_1)$ on N generates an approximate Hamiltonian symmetry of FGCM, then:

- $\operatorname{div} u_0 = 0$, $[u_0, B] = 0$, $b \cdot V_0 = 0$;
- $B \cdot \nabla \psi_1 = 0$;
- $B \cdot \nabla (b \cdot u_0) = c \cdot \nabla \psi_0$;
- $p_\| u_0 \cdot \nabla (b \cdot u_0) = u_0 \cdot \nabla \psi_1 + u_1 \cdot \nabla \psi_0$;
- $p_\| [u_0, c] + [u_1, B] + (\operatorname{div} u_1) B = 0$

where $c = \operatorname{curl} b$ and $V_0 = c \times u_0 + \nabla (b \cdot u_0)$.

Approximate Symmetries of FGCM (continued)

Theorem 5

Given a magnetic field B, a v.f. $U = (u, w) = (u_0 + \varepsilon u_1, w_0 + \varepsilon w_1)$ on N generates an approximate Ham. symmetry of FGCM up to trivial symmetries if-f $L_{u_0}\beta = 0$, $L_{u_0}|B| = 0$, $w = 0$, and

$$u_1 = b \times (p_\| V_0 - \nabla\psi_1)/|B| \tag{24}$$

$$b \cdot \nabla\psi_1 = p_\| b \cdot V_0 \tag{25}$$

$$\partial_{p_\|}\psi_1 = p_\| \, b \cdot \partial_{p_\|} u_0 \tag{26}$$

Proof. From $L_{u_0} b^\flat = i_{u_0} db^\flat + d i_{u_0} b^\flat = i_{u_0} i_c \Omega + d(b \cdot u_0)$, note that

$$L_{u_0} b^\flat = V_0^\flat + (b \cdot \partial_{p_\|} u_0) dp_\| \tag{27}$$

Thus, the condition $p_\| L_{u_0} b^\flat + i_{u_1} i_B \Omega = d\psi_1$ of Thm 3 splits to

$$B \times u_1 + p_\| V_0 = \nabla\psi_1 \tag{28}$$

and (26). Dotting (28) with b gives (25), while crossing with b we find

$$u_1 = b \times (p_\| V_0 - \nabla\psi_1)/|B| + (b \cdot u_1) b$$

Dropping the trivial symmetry $\varepsilon((b \cdot u_1) b, w_1)$ completes the proof. □

Approximate Symmetries of FGCM (continued)

Approximate invariant

The corresponding approximate constant of motion is now given by

$$K = -\psi_0 - \varepsilon(\psi_1 - p_\parallel\, b\cdot u_0) \tag{29}$$

Proposition 1

Assume the $p_\parallel$-dependent vector field $u_0 + \varepsilon u_1$ on M generates an approximate Ham. symmetry of FGCM.

1. u_0 is spatial if and only if ψ_1 is.
2. If u_0 is spatial, then $V_0 = \partial_{p_\parallel} u_1 \times B$.

Proof. From $B \times u_0 = \nabla\psi_0$ (23) we have $B \times \partial_{p_\parallel} u_0 = 0$ and together with $p_\parallel\, b \cdot \partial_{p_\parallel} u_0 = \partial_{p_\parallel}\psi_1$ (26) we deduce $\partial_{p_\parallel}\psi_1 = 0$ if-f $\partial_{p_\parallel} u_0 = 0$.

The second one follows from $B \times u_1 + p_\parallel V_0 = \nabla\psi_1$ (28), since if u_0 is spatial then so are ψ_1, V_0. □

Approximate Symmetries of FGCM (continued)

Corollary

Given a magnetic field B, a vector field $u = u_0 + \varepsilon u_1$ on M generates an approximate quasisymmetry if-f u_0 is a quasisymmetry and $L_{u_1}\beta = 0$.

Proof. From (27) we have $L_{u_0} b^\flat = V^\flat$, and from Prop 1 we have in turn $L_{u_0} b^\flat = 0$ and $\partial_{p_\parallel}\psi_1 = 0$. Therefore the symmetry condition $p_\parallel L_{u_0} b^\flat + i_{u_1} i_B \Omega = d\psi_1$ of Thm 3 reduces to $i_{u_1} i_B \Omega = d\psi_1$, which says $L_{u_1}\beta = 0$. The rest of the symmetry conditions, $L_{u_0}\beta = 0$, $L_{u_0}|B| = 0$, together with $L_{u_0} b^\flat = 0$ prove that u_0 is a quasisymmetry. □

Weak quasisymmetry (Rodríguez, Helander & Bhattacharjee)

is an approximate Hamiltonian symmetry of FGCM on M which is spatial to leading order and nontrivially linear in $p_\parallel$ to first order.

Approximate Symmetries of FGCM (continued)

Theorem 6

Let u_0 be a vector field on M with $V_0 \neq 0$. The vector field $u = u_0 + \varepsilon u_1$ generates a weak quasisymmetry up to trivial symmetries if and only if $L_{u_0}\beta = 0$, $\operatorname{div} u_0 = 0$, $L_{u_0}|B| = 0$ and $u_1 = b \times (p_\| V_0 - \nabla\psi_1)/|B|$ with ψ_1 a flux function on M.

Proof. If u generates a weak quasisymmetry then from Thms 4-5 we see that the conditions hold.
In the opposite direction, note first from (27) that $L_{u_0} b^\flat = V_0^\flat$ since u_0 is spatial. Now, $\operatorname{div} u_0 = 0$ is equivalent to $b \cdot V_0 = 0$, when $L_{u_0}\beta = 0$ and $L_{u_0}|B| = 0$. To see this, apply L_{u_0} to the relation $b^\flat \wedge \beta = |B|\Omega$ to find $L_{u_0} b^\flat \wedge \beta = |B| L_{u_0}\Omega$, where $L_{u_0}\Omega = d i_{u_0}\Omega = (\operatorname{div} u_0)\Omega$, and then i_b in turn to arrive at $(i_b L_{u_0} b^\flat)\beta = |B|(\operatorname{div} u_0) i_b \Omega$ and hence $i_b L_{u_0} b^\flat = \operatorname{div} u_0$ since $B \neq 0$. Thus, all the conditions of Thm 5 are met, with (26) trivially satisfied. Therefore u generates an approximate Hamiltonian symmetry of FGCM, and since V_0 and ψ_1 are independent of $p_\|$, it is a weak quasisymmetry. □

Approximate Symmetries of FGCM (continued)

Remarks

- For general approximate symmetries, given u_0 and ψ_1, we can construct u_1, as we can see from Thm 5.
- This becomes an advantage, in particular, for weak quasisymmetry, because in this case u_0 and ψ_1 decouple. Thus, as we see from Thm 6, the conditions for weak quasisymmetry to zeroth-order are completely uncoupled from the first-order ones. Moreover, the latter amount to simply building u_1 once u_0 is known.
- On this ground, the *existence* of weak quasisymmetry (but not weak quasisymmetry itself) is rightfully identified with a v.f. u_0 such that $L_{u_0}\beta = 0$, $\operatorname{div} u_0 = 0$, $L_{u_0}|B| = 0$, as the last condition of Thm 6 is merely a construction (assuming flux function ψ_1).
- This allows to compare the part u_0 of weak quasisymmetry with quasisymmetry u, despite their different nature. From their conditions respectively, we see then that $\operatorname{div} u_0 = 0$ relaxes $L_u b^\flat = 0$.

Approximate Symmetries of FGCM (continued)

Theorem 7 (Rodríguez, Helander & Bhattacharjee)

Let $f = B \cdot \nabla|B| \neq 0$. A weak quasisymmetry exists if and only if $\nabla\psi_0 \times \nabla|B| \cdot \nabla f = 0$ and $\psi_0 + \varepsilon\psi_1$ is a flux function.

Proof. From Thm 6, if $u = u_0 + \varepsilon u_1$ generates a weak quasisymmetry, then $L_{u_0}\beta = 0$, $\operatorname{div} u_0 = 0$, $L_{u_0}|B| = 0$, and there's a flux function ψ_1. Repeating (23), the first condition gives $B \times u_0 = \nabla\psi_0$, where ψ_0 is a flux function. Crossing with $\nabla|B|$ and using the third condition we get

$$u_0 = \nabla\psi_0 \times \nabla|B|/f \tag{30}$$

Applying then the second condition, we find $\nabla\psi_0 \times \nabla|B| \cdot \nabla f = 0$, as any $\nabla\psi_0 \times \nabla|B|$ has zero divergence.
In the other direction, given flux function ψ_0, define u_0 from (30). Then $L_{u_0}|B| = 0$. Also $\operatorname{div} u_0 = 0$, because $\nabla\psi_0 \times \nabla|B| \cdot \nabla f = 0$. Thirdly, crossing (30) with B gives $B \times u_0 = \nabla\psi_0$, since $B \cdot \nabla\psi_0 = 0$. Finally, take $u_1 = b \times (p_{\|} V_0 - \nabla\psi_1)/|B|$ given flux function ψ_1. Thus, $u = u_0 + \varepsilon u_1$ generates a weak quasisymmetry from Thm 6. □

Approximate Symmetries of FGCM (continued)

Theorem 8

For an MHS magnetic field with $dp \neq 0$ almost everywhere on M and density of irrational surfaces, an approximate symmetry of FGCM on N implies an approximate quasisymmetry.

References

RODRÍGUEZ E., HELANDER P. & BHATTACHARJEE A., Necessary and sufficient conditions for quasisymmetry, *Phys. Plasmas* **27** (2020), 062501.

BURBY J.W., KALLINIKOS N. & MACKAY R.S., Approximate symmetries of guiding-centre motion, *J. Phys. A: Math. Theor.* **54** (2021), 125202.

Food for thought

- How does MHS (or at least vacuum) combine with (exact) velocity symmetries?
- Does circle action of quasisymmetry extend to an analogue for velocity symmetries?
- Need to impose boundedness. What's the role of magnetic curvature for approximate symmetries?
- Are there special μ-dependent symmetries? Need to study GCM as a Hamiltonian reduction of charged particle motion.
- How far is weak QS from isometries compared to QS? Note that $\mathrm{tr}_g A = 0$ but $\det A \neq 0$, where $A = L_{u_0} g$ for weak QS $u = u_0 + \varepsilon u_1$