Mathematics for Fusion Power part 5

R.S.MacKay

March 12, 2024

Integrable magnetic fields

Adiabatic theory for First-order Guiding-centre motion

Omnigenity

Integrable magnetic fields

- Relax from desire to make FGCM integrable to just requiring time-averaged rate of change of a flux function be zero.
- Needs B to have a *flux function*: a function ψ such that $i_B d\psi = 0$ and $d\psi \neq 0$ a.e. Say B is *integrable*.
- Automatic for non-degenerate MHS, i.e. $i_B dB^\flat = dp$ with $dp \neq 0$ a.e. (in plasma), and AS or QS fields ($i_u i_B \Omega = d\psi$) with u, B indpt a.e. Also for ideal equilibria with flow v indpt of B ($i_v i_B \Omega = d\Phi$).
- Equivalent (modulo ψ global) to $\exists$ continuous symmetry u of β, inpdt of B a.e.: $L_u \beta = 0$ & $d\beta = 0$ imply $i_u i_B \Omega$ is closed so locally $d\psi$, some ψ. Conversely, if ψ is a flux function let $u = \xi + fB$ for $\xi = \frac{b}{|B|} \times \nabla\psi$ & fn f, $L_u \beta = d i_u i_B \Omega = d i_B (\frac{b^\flat}{|B|} \wedge d\psi) = d^2\psi = 0$.
- Note we suppose B nowhere zero in domain of interest.
- The bounded regular level sets of ψ (*flux surfaces*) are oriented by area-form $\mathcal{A} = i_n \Omega$, where $n = \nabla\psi / |\nabla\psi|^2$, and carry a nowhere-zero vector field B so by Poincaré index have Euler characteristic 0, so by classification of compact surfaces are tori.
- On a flux surface S, B preserves $\mathcal{A}$: $i_n d\psi = 1$ implies $\Omega = i_n \Omega \wedge d\psi$, so $0 = L_B \Omega = L_B i_n \Omega \wedge d\psi$. Applying i_n, $0 = i_n L_B i_n \Omega \wedge d\psi - L_B i_n \Omega$. So $L_B i_n \Omega = 0$ on pairs of tangents to $\psi = $ constant.

continued

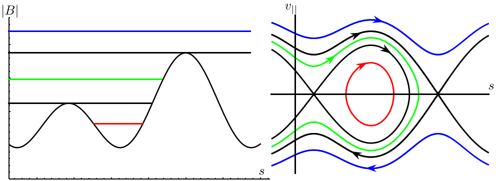
- Implies B is a *Poincaré vector field* on S, i.e. has a cross-section (a transverse circle such that every trajectory crosses it forwards and backwards in time), because only other option for a nowhere-zero field has a Reeb component (an annulus bounded by periodic orbits in opposite directions), incompatible with a preserved area-form.
- C Baesens, J Guckenheimer, S Kim, RS MacKay, Three coupled oscillators: Mode-locking, global bifurcations and toroidal chaos, Physica D 49 (1991) 387–475
- In particular, B has a winding ratio ι on S, relative to a choice of poloidal and toroidal generators of $H_1(S)$.
- $\int_\eta i_B \mathcal{A}$ along an arc η in S from a reference point makes a local coordinate that is preserved by the flow of B ($L_B \mathcal{A} = 0$ implies $i_B d i_B \mathcal{A} = 0$, so $\int_\eta i_B \mathcal{A}$ is path-indpt on S). So the return map to a cross-section is conjugate to a rigid translation.
- Consequently, B on S is conjugate to λC for some constant vector field C on $\mathbb{R}^2/\mathbb{Z}^2$ & time-change function $\lambda : S \to \mathbb{R}^+$.
- In particular, for ι rational, every fieldline is closed.
- For Diophantine winding ratio, if B is smooth enough then it is conjugate to a constant (KAM theory).

more

- Note that we don't get LA coordinates in general. To get from $L_u\beta = 0$, $d\beta = 0$ to $[u, B] = 0$ requires $\mathrm{div} u = 0$ too: $i_{[u,B]}\Omega = i_u L_B \Omega - L_B i_u \Omega = i_B d i_u \Omega = (\mathrm{div} u) i_B \Omega$.
- $\mathrm{div}\,\xi = (i_J d\psi - i_\xi d|B|^2)/|B|^2$.
- e.g. in non-degenerate MHS (meaning $dp \neq 0$ a.e.), or weaker just B integrable with p a function of ψ, then $i_J d\psi = 0$, so $\mathrm{div}\,\xi = 0$ iff $i_\xi d|B| = 0$, i.e. $\mathrm{div}\,\xi = 0$ iff ξ is a weak QS.
- Can rearrange: for integrable B, ξ weak QS implies $i_J d\psi = 0$.
- For $\eta = \xi + fB$, $\mathrm{div}\,\eta = \mathrm{div}\,\xi + i_B df$. To achieve $\mathrm{div}\,\eta = 0$ need f s.t. $i_B df = -\mathrm{div}\,\xi$. Requires $\int_\gamma \mathrm{div}\,\xi \frac{b^\flat}{|B|} = 0\ \forall$ closed field- lines γ. Sufficient for a solution $f \in C^1$ if $B \in C^3$ and $\frac{d\iota}{d\psi} \neq 0$. Newcomb WA, Magnetic differential equations, Phys Fluids 2 (1959) 362-5
- Automatic in non-degenerate MHS, because it is solvability condition for $J_\parallel$. Thus $f = J_\parallel/p'$ (plus any flux function).
- $|B|^2 \mathrm{div}\,\eta + i_\eta d|B|^2 = i_J d\psi + i_B d\tilde{f}$ with $\tilde{f} = f|B|^2$. So η weak QS implies $i_J d\psi + i_B d\tilde{f} = 0$. Conversely, under this condition, $\mathrm{div}\,\eta = 0$ iff $L_\eta |B| = 0$.
- For non-degenerate MHS & $f = \frac{J_\parallel + F(\psi)}{p'}$, η weak QS iff $i_b d((J_\parallel + F)|B|^2) = 0$.

Zeroth-order Guiding-Centre Motion

- To define time-averaged $\dot{\psi}$, compute $i_X d\psi$ due to FGCM, and average along trajectories of zeroth-order GCM.
- ZGCM is $\lim_{\varepsilon \to 0}$ of FGCM: $\dot{s} = \frac{p_\parallel}{m}$, $\dot{p}_\parallel = -\mu i_b d|B|$ along single fieldlines, with arclength parameter s.
- It has Hamiltonian formulation $H = \frac{1}{2m} p_\parallel^2 + \mu |B(s)|$, $\omega = ds \wedge dp_\parallel$. In particular, it conserves $H = E$. Suppose $\mu > 0$.



- If $h = E/\mu$ is larger than the maximum of $|B|$ along the fieldline then the motion is unidirectional.
- If the GC approaches a point where $|B| = h$ with $|B|' \neq 0$ (where $' = \frac{d}{ds}$) then the GC reverses direction there.
- If it approaches a point where $|B| = h$ with $|B|' = 0$ then the GC takes infinite time to reach it (*marginal case*).
- For a fieldline on a flux surface, most of the ZGCM is either *circulating* (unidirectional) or *periodically bouncing*.

Scaled FGCM

- For $\mu > 0$, use new time $\tau = \sqrt{\frac{\mu}{m}}\, t$, parallel velocity $u_\| = \frac{p_\|}{\sqrt{m\mu}}$, and write $\delta = \frac{\sqrt{m\mu}}{e}$ to make FGCM into

$$\frac{dQ}{d\tau} = \tilde{B}_\|^{-1}(u_\| \tilde{B} + \delta\, b \times \nabla|B|), \quad \frac{du_\|}{d\tau} = -\frac{\tilde{B}}{\tilde{B}_\|} \cdot \nabla|B|,$$

  with $\tilde{B} = B + \delta\, u_\| \operatorname{curl} b$ and $\tilde{B}_\| = \tilde{B} \cdot b = |B| + \delta\, u_\| b \cdot \operatorname{curl} b$.
- Equivalent to $\tilde{H} = \frac{H}{\mu} = \frac{1}{2}u_\|^2 + |B|$, $\tilde{\omega} = \frac{\omega}{\sqrt{m\mu}} = \frac{\beta}{\delta} + d(u_\| b^\flat)$.
- Limit of ω as $\delta \to 0$ is singular and β is degenerate, but dynamics still well-defined for $\delta = 0$ and is (scaled) ZGCM.
- This scaling absorbs all of e, m, μ into one parameter δ and makes FGCM a regular perturbation of ZGCM. Will use later.

Perpendicular drifts

- Instead of the Hamiltonian FGCM, it turns out better to use an alternative set of equations agreeing to first order: $\dot{Q} = \frac{p_\parallel}{m} b + v_d$, $\dot{p}_\parallel = -\mu(b + \frac{p_\parallel}{e|B|} c_\perp) \cdot \nabla|B|$, where $v_d = \frac{p_\parallel^2}{em|B|} c_\perp + \frac{\mu}{e|B|} b \times \nabla|B|$, $c = \mathrm{curl}\, b$ ($c_\perp$ can be written as $b \times \kappa$ with $\kappa^\flat = L_b b^\flat$). Still preserves H. [& some ω?]
- Then the rate of change of ψ is $\dot{\psi} = i_{v_d} d\psi$.
- Say B is *omnigenous* if for all bouncing orbits of ZGCM, the time-average $\langle\dot{\psi}\rangle$ of $i_{v_d} d\psi$ over one period is 0.
- Thus bouncing GCs have $O(\varepsilon^2)$ long-term rate of change of ψ (from $v_d = O(\varepsilon)$ and evaluating on FGCM instead of ZGCM).
- We'll show that $\langle\dot{\psi}\rangle$ for circulating orbits is also 0.
- And that QS implies omnigenous, so it is a generalisation.
- JW Burby, RS MacKay, S Naik, Isodrastic magnetic fields for suppressing transitions in guiding-centre motion, Nonlinearity 36 (2023) 5884–5954.

Stronger options

- Could ask for no bouncing trajectories, i.e. $L_B|B| = 0$. If DIS then $|B|$ constant on ψ constant.
- Impossible in "normal" MHS if the flux surfaces accumulate on a closed curve ("magnetic axis"), because $dp = |B|^2\kappa^\flat - |B|d_\perp|B|$ (*).
- More generally, writing $' = \frac{d}{d\psi}$, (*) requires $(p + \frac{1}{2}|B|^2)' < 0$ (assuming enclosed volume V has $V' > 0$), because take smallest sphere surrounding a flux surface T: at contact points κ is into the solid torus bounded by T.
- At contact points the Gauss curvature $K > 0$. By Gauss-Bonnet, $\exists$ region R of each flux surface with $K < 0$. Then κ inwards implies that B has to avoid a cone at each point of R. So there is an interval of winding ratios that is excluded.
- For "skinny" tori T (those near a magnetic axis), κ tends to that for the magnetic axis γ, so taking a closed curve η on T around a point of γ where $\kappa \neq 0$ we see that κ can not be inwards the whole way round η. Thus constant-strength flux surfaces is not possible for skinny tori either.

Isodynamic

- Or ask for *isodynamic*: $i_{v_d} d\psi = 0$, rather than $\langle i_{v_d} d\psi \rangle = 0$.
- Can write as $-\frac{1}{e} i_\xi (\frac{p_\parallel^2}{m} i_b db^\flat + \mu d|B|)$ (see Thm 1 to follow).
- But in MHS, $dp = i_B dB^\flat = i_B d(|B| b^\flat) = |B|(i_b db^\flat - d_\perp |B|)$, so for normal MHS $0 = i_\xi dp = |B|(i_\xi i_b db^\flat - d_\perp |B|)$. Thus $i_{v_d} d\psi = -\frac{1}{e}(\frac{p_\parallel^2}{m|B|} + \mu) i_\xi d|B|$ is 0 iff $i_\xi d|B| = 0$ iff $i_\xi \kappa^\flat = 0$.
- Says B-lines form a geodesic foliation for g on each flux surface. Gauss-Bonnet $\int_S k\, dS = 2\pi\chi = 0$ for $\mathbb{T}^2$ implies Gauss-curvature $k \geq 0$ somewhere. Imposes restrictions on geodesic foliations (Converse KAM), e.g. $\nexists$ for "big bump" tori.
- $i_\xi d|B| = 0$ implies $\operatorname{div} \xi = 0$ in MHS, so DIS for ξ is typical. If add DIS for ξ then get $|B|$ constant on flux surfaces and we are back to the no bouncing case.
- Helander P, Theory of plasma confinement in non-axisymmetric magnetic fields, Rep Prog Phys 77 (2014) 087001
- Palumbo D, Some considerations on closed configurations of magnetohydrostatic equilibrium, Nuovo Cim B 53 (1967) 507

Longitudinal adiabatic invariant

- If v_d is slow on scale of the period $T = \int dt = 2\int \frac{m}{p_\parallel}\, ds$ of bouncing along a segment γ of fieldline, there is a second adiabatic invariant $L = \int p_\parallel\, ds = \int_\gamma p_\parallel b^\flat$ for FGCM.
- Note that from $p_\parallel = \sqrt{2m(E - \mu|B|)}$, can write $T = 2\frac{dL}{dE}$.
- Can obtain L by defining phase of bouncing oscillation (e.g. time from lefthand end divided by T) and showing approximate symmetry of FGCM wrt phase-shift.
- Or use conservation of Poincaré invariant for loop-dynamics: $L = \int_{\phi_t D} \omega$ for disk D moving with the Hamiltonian flow ϕ. $\omega = -e\beta - d(p_\parallel b^\flat) = -d(eA^\flat + p_\parallel b^\flat)$. So for disk bounded by slowly moving "periodic" orbit γ_t, $L = \int_{\gamma_t} eA^\flat + p_\parallel b^\flat$ is conserved. For bouncing orbit of ZGCM, the contributions of opposite directions cancel for $A^\flat$ and are equal for $p_\parallel b^\flat$, so can redefine invariant $L = \int_\gamma p_\parallel b^\flat$ in just one direction.

continued

- **Theorem 1**: (B, ψ) omnigenous iff L locally constant, given E, ψ.
- **Proof**: $i_{v_d} d\psi = \frac{1}{e|B|}(\frac{p_\|^2}{m} i_c + \mu i_f) d\psi = -\frac{1}{e} i_\xi (\frac{p_\|^2}{m} i_b db^\flat + \mu d|B|)$ (*) with $f = b \times \nabla|B|$, $\xi = \frac{b}{|B|} \times \nabla\psi$, since (i) $i_\xi \Omega = \frac{b^\flat}{|B|} \wedge d\psi$, so $i_\xi i_b db^\flat = i_b i_c i_\xi \Omega = -\frac{1}{|B|} i_c d\psi$, and (ii) $i_f \Omega = b^\flat \wedge d|B|$ so $i_b i_\xi i_f \Omega = -i_\xi d|B|$, but can also be written as $i_f i_b i_\xi \Omega = \frac{1}{|B|} i_f d\psi$. Thus $\frac{T}{2}\langle\dot{\psi}\rangle = \int_\gamma i_{v_d} d\psi \, dt = -\frac{1}{e}\int_\gamma i_\xi (\frac{p_\|^2}{m} i_b db^\flat + \mu d|B|) \frac{m}{p_\|} ds$. Now $i_{[B,\xi]}\Omega = L_B i_\xi \Omega - i_\xi L_B \Omega = L_B(\frac{b^\flat}{|B|} \wedge d\psi) = (L_B \frac{b^\flat}{|B|}) \wedge d\psi$. Apply i_B to get $(L_B 1) d\psi = 0$, so $[\xi, B] = fB$ for some function f. Thus ξ-flow takes B-lines to B-lines; also preserves ψ. Let $\eta = \xi + gB$ for a function g with $g = -\frac{i_\xi d|B|}{i_B d|B|}$ at ends of γ (so $i_\eta d|B| = 0$ there) and ϕ_λ its flow. For $L = \int_{\phi_\lambda \circ \gamma} p_\| b^\flat$, $\frac{dL}{d\lambda} = \int_\gamma L_\eta (p_\| b^\flat)$. To fix E, use $p_\| = \sqrt{2m(E - \mu|B|)}$. $L_\eta(p_\| b^\flat) = i_\eta d(p_\| b^\flat) + d(p_\| i_\eta b^\flat)$. Second term integrates to 0 since $p_\| = 0$ at the ends. So $\frac{dL}{d\lambda} = \int i_b i_\eta d(p_\| b^\flat)\, ds = \int i_b i_\xi (dp_\| \wedge b^\flat + p_\| db^\flat)\, ds = \int (-i_\xi dp_\| + p_\| i_b i_\xi db^\flat)\, ds$. But $dp_\| = -\frac{m\mu}{p_\|} d|B|$ so $\frac{dL}{d\lambda} = \frac{eT}{2}\langle\dot{\psi}\rangle$. □

Length function

- For segments γ of fieldline between points of equal $|B|$ with smaller $|B|$ between, let $h = |B|$ at ends and $\ell =$ length of γ.
- The space of segments of fieldline on a flux surface is a complex of 2D surfaces bounded by set Σ where $i_b d|B| = 0$.
- **Theorem 2**: (B, ψ) is omnigenous iff for each flux surface, ℓ is constant along h constant.
- **Proof**: Convenient to write $E = h\mu$ and $L = \sqrt{m\mu}\, j$. Then L constant for E constant iff $j = \int \sqrt{2(h - |B|)}\, ds$ constant for h constant. Decompose integral according to value h' of $|B|$ and let $\ell(h') = \int_{x \in \gamma : |B(x)| \leq h'} ds$, so $d\ell(h') = \sum_{x:|B(x)|=h'} ds$. Then $j(h) = \int_{-\infty}^{h} \sqrt{2(h-h')}\, d\ell(h')$ (so j is the *Abel transform* of ℓ). So if ℓ constant along h' constant for all $h' \leq h$ then j constant along h constant. Conversely, if j is constant along h' constant for all $h' \leq h$, then Abel inversion gives $\ell(h) = \frac{2}{\pi} \int_{-\infty}^{h} \frac{dj(h')}{\sqrt{2(h-h')}}$ is constant for h constant. □

Abel inversion

- **Lemma**: If $j(h) = \int_{-\infty}^{h} \sqrt{2(h-h')}\, d\ell(h')$ then $\ell(h) = \frac{2}{\pi} \int_{-\infty}^{h} \frac{dj(h')}{\sqrt{2(h-h')}}$.
- **Proof**: $dj(h') = \int_{-\infty}^{h'} \frac{d\ell(h'')}{\sqrt{2(h'-h'')}}\, dh'$.

  So $\int_{-\infty}^{h} \frac{dj(h')}{\sqrt{2(h-h')}} = \int_{-\infty}^{h} \left(\frac{1}{\sqrt{2(h-h')}} \int_{-\infty}^{h'} \frac{d\ell(h'')}{\sqrt{2(h'-h'')}} \right) dh'$.

  Interchange order of integration to obtain

  $\int_{-\infty}^{h} \left(\int_{h''}^{h} \frac{dh'}{2\sqrt{(h-h')(h'-h'')}} \right) d\ell(h'') = \frac{\pi}{2}\ell(h)$. □
- In particular, for an omnigenous field, every fieldline has to have the same sequence of local minima and maxima of $|B|$, so get non-generic case of curves of local maxima and minima.

Rational circulating trajectories

- Fieldlines γ on a rational torus S are closed. Particles with $E/\mu = h$ above the maximum of $|B|$ on γ keep going in their original direction. Have adiabatic invariant $L = \int_\gamma eA^\flat + p_\| b^\flat$. $A^\flat$ contributes just a function of ψ, because $\int_\gamma A^\flat = \int_D i_B \Omega$ for a disk D spanning γ, which is the same for all γ.
- By the same proof as for Theorem 2 in the bouncing case, if (B, ψ) is omnigenous then L is the same for all circulating particles of the same energy on the same flux surface, because it is determined by the length function.
- And by the same proof as for Theorem 1, this implies that $\langle \dot{\psi} \rangle = 0$ for them.
- Thus, omnigenity for bouncing particles implies omnigenity for rational circulating ones.

Relation to $J_\parallel$ in MHS

- The Newcomb solvability condition for $J_\parallel$ in normal MHS is $\int_\gamma i_\xi d|B| \frac{ds}{|B|} = 0$ for all closed fieldlines γ.
- Compare omnigenity for circulating ZGCM: $0 = -e\langle i_{v_d} d\psi \rangle = \int_\gamma (p_\parallel + \frac{m\mu|B|}{p_\parallel}) i_\xi d|B| \frac{ds}{|B|}$. Now $p_\parallel = \pm\sqrt{2m(E - \mu|B|)}$, so factor is $\pm\sqrt{2mE}(1 + O((\frac{\mu}{E})^2))$, thus omnigenity for the limit of rapidly circulating GCs ($E/\mu \to \infty$) is automatic in normal MHS.

Irrational circulating trajectories

- Finally, $\langle\dot{\psi}\rangle = 0$ for circulating particles on irrational flux surfaces S.
- **Proof**:
  1. ZGCM preserves $\frac{|B|}{Zp_\parallel}\mathcal{A}$ (speed factor $\frac{v_\parallel}{|B|}$ compared to B-flow, normalisation $Z = \int_S \frac{|B|}{p_\parallel}\mathcal{A}$) and for $h > |B|_{\max}$ is uniquely ergodic on irrational surfaces. So time-average of any continuous function along any trajectory equals its space-average. In particular, $\langle\dot{\psi}\rangle = \int_S \frac{|B|}{Zp_\parallel} i_{v_d} d\psi\, \mathcal{A}$.
  2. Now $\mathcal{A} \wedge d\psi = \Omega$, so $i_{v_d}\mathcal{A} \wedge d\psi + \mathcal{A}\, i_{v_d} d\psi = i_{v_d}\Omega$. $d\psi = 0$ on tangents to S, so $\langle\dot{\psi}\rangle = \int_S \frac{|B|}{Zp_\parallel} i_{v_d}\Omega$.
  3. Compare $\frac{|B|}{p_\parallel} i_{v_d}\Omega = \frac{p_\parallel}{em} i_{c_\perp}\Omega + \frac{\mu}{ep_\parallel} b^\flat \wedge d|B|$ to $d(p_\parallel b^\flat) = dp_\parallel \wedge b^\flat + p_\parallel db^\flat = -\frac{m\mu}{p_\parallel} d|B| \wedge b^\flat + p_\parallel i_c\Omega$. On pairs of tangents to S, $i_c\Omega = i_{c_\perp}\Omega$, because i_b gives the same. So $\langle\dot{\psi}\rangle = \frac{1}{emZ}\int_S d(p_\parallel b^\flat) = 0$. □[USE in Thm 1 TOO]
- Question about speed of convergence?

More consequences of omnigenity

- For omnigenous B, on each flux surface S the regular contours of $|B|$ (i.e. with $d(|B|_{|S}) \neq 0$) are transverse to B.
- **Proof**: By (*), $i_{v_d} d\psi = -\frac{1}{e} i_\xi (\frac{p_\parallel^2}{m} i_b db^\flat + \mu d|B|)$. If $i_b d|B| = 0$ at a point x on a regular contour of $|B|$ then $i_\xi d|B| \neq 0$ there. So for $\mu > 0$ and $E = \mu|B|(x)$, $i_{v_d} d\psi \neq 0$ there. If x is a local minimum of $|B|$ along B then for slightly larger E get short bouncers and $\langle\dot\psi\rangle \neq 0$ for them, contradicting omnigenity. If $|B|$ has a downhill direction from x along B then using that the fieldlines are recurrent and $i_\xi d|B| \neq 0$, every ZGCM with an end near to x has a second end. If the second end is a normal point ($i_b d|B| \neq 0$) then the period is dominated by the time near x, so $\langle\dot\psi\rangle \neq 0$ again. Argue that not possible for all segments to have abnormal second end [??]. □
- So they are all non-contractible closed curves with the same winding ratio.

continued

- For $B \in C^2$ most contours are regular, by Sard's theorem (set of critical values of a C^k map f from n to m dimensions with $k \geq 1, n - m + 1$ has measure zero): the set of values of $|B|$ for which there is a non-regular contour on a given flux surface has measure zero. Ignore the case of $|B|$ constant on the flux surface. [Is it impossible anyway?] $|B|$ is continuous, so its image is an interval. So contours are regular for all but a set of measure zero in this interval.
- In particular, omnigenity has a rational type, meaning the ratio of poloidal to toroidal turns for curves of constant $|B|$. So distinguish $OT, OP, OH(N, M)$ (toroidal, poloidal, helical).
- e.g. W7-X is approximately OP.
- Not aware of constraints on M (unlike for QS).
- Note that the sets where $|B|$ is max or min are not regular contours, but may still be smooth curves.

Relations between omnigenity & QS

- QS u implies a flux function ψ ($i_u i_B \Omega = d\psi$) and $L_u b^\flat = 0$. Let ϕ_λ be the flow of u. For γ fieldline segment with ends at $|B| = h$, $\frac{d}{d\lambda}\int_{\phi_\lambda \circ \gamma} ds = \int_\gamma L_u b^\flat = 0$. So (B, ψ) omnigenous.
- Same argument shows that weak QS implies omnigenous: weak QS implies $i_b L_u b^\flat = 0$, which suffices.
- Analyticity & omnigenity (with MHS) implies QS [Cary & Shasharina, Phys Plasma 4 (1997) 3323].
- It is claimed that can make many non-QS fields that are omnigenous (& MHS), e.g. Dudt et al, Magnetic fields with general omnigenity, arxiv:2305.08026, but constructions depend on realising $|B|$-profiles on flux surfaces by a 3D divergence-free field in Euclidean space, and I'm not convinced this is feasible. Do the numerical procedures converge? If so, it would provide construction of smooth non-AS MHS examples, suspected not to exist by Grad H, Toroidal containment of a plasma, Phys Fluids 10 (1967) 137.

Omnigenity & MHS

- Say an MHS field is *normal* if it has a flux function ψ and p is constant on ψ constant, e.g. non-degenerate ($dp \neq 0$ a.e.).
- **Lemma**: For a normal MHS field, $a = \frac{b}{|B|}, \xi = a \times \nabla\psi$ are commuting fields on each flux surface.
- **Proof of Lemma**: $i_a d\psi = i_\xi d\psi = 0$ (†), so a, ξ are tangent to $\psi =$ constant. Apply $i_{[a,\xi]} = L_a i_\xi - i_\xi L_a$ to basis $d\psi, B^\flat, \xi^\flat$, and use $L_a = |B|^{-2} L_B + d(|B|^{-2}) \wedge i_B$. $i_{[a,\xi]} d\psi = 0$ by (†). Using $L_B B^\flat = d(p + |B|^2)$ and $i_\xi dp = 0$, $i_{[a,\xi]} B^\flat = -i_\xi L_a B^\flat = -i_\xi(|B|^{-2} d|B|^2 + |B|^2 d|B|^{-2}) = 0$. Can show $i_{[a,\xi]} \xi^\flat = |B|^{-2} i_{[B,\xi]} \xi^\flat$. Using $i_\xi \Omega = a^\flat \wedge d\psi$, show $i_B i_{[B,\xi]} \Omega = 0$, so $[B, \xi]$ is parallel to B, hence $i_{[a,\xi]} \xi^\flat = 0$ too. So $[a, \xi] = 0$. [SIMPLIFY?] □
- Call LA coordinates for $[a, \xi] = 0$ *Boozer angles*. Extend to *Boozer coordinates* by adding ψ.

continued

- **Theorem**: A normal MHS field B is omnigenous iff the differences of Boozer angles along B between contours of constant $|B|$ are locally constant on each flux surface.
- **Proof of Theorem**: In Boozer angles $\theta = (\theta^1, \theta^2)$, $a = \rho(\psi)$, some $\rho : \mathbb{R} \to \mathbb{R}^2$. Thus, $d\theta^i = \rho^i |B|\, ds$ and for a segment with $|B| = h$ at the ends,
  $\Delta\theta^i(h) = \int d\theta^i = \rho^i \int |B|\, ds = \rho^i \int_{-\infty}^{h} h' d\ell(h')$.
  If B is omnigenous then $\ell(h')$ does not depend on the fieldline on the given flux surface, so neither does $\Delta\theta^i(h)$. Conversely, if for each value h of $|B|$ at the ends of a segment, $\Delta\theta^i(h)$ is indpt of the fieldline on the given flux surface, then $\ell(h')$ must be indpt of the fieldline too. □
- A consequence on irrational surfaces: $|B|_{\text{max}}$ contour is straight in Boozer angles, else the Boozer angle between successive intersections is not constant.

Illustration

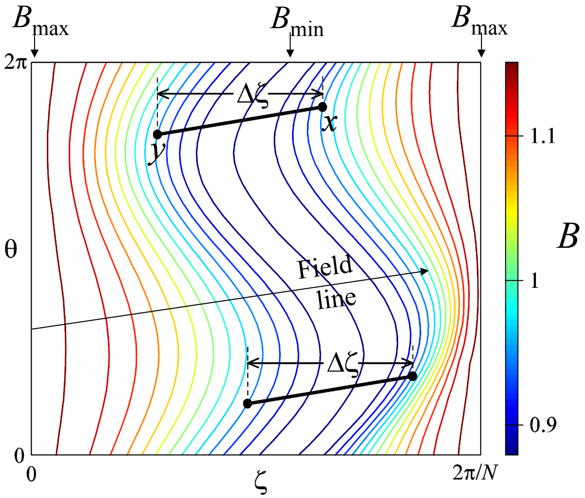


Figure: $|B|$ contours for an OP field in Boozer coordinates on a flux surface, from Landreman

Notes

- A normal MHS field is QS iff $\exists\, N, M$ s.t. $|B|$ is constant along straight lines of slope $\frac{N}{M}$ in Boozer angles on each flux surface.
- The QS is a constant field of slope N/M in Boozer coordinates and produces same ψ up to an affine transformation.
- QS u with associated ψ implies $[u, a] = 0$, $[u, \xi] = 0$. So MHS adds that $[a, \xi] = 0$, which implies simultaneous LA coordinates for all 3 on each flux surface.
- Similar theorem for non-degenerate MHS and Hamada angles.
- Current in omnigenous fields?