② T-phases for uniformly hyperbolic attractors of finite-dim deterministic dyn sys.

A warm-up for S-T phases for spatially extended deterministic dyn sys.

Key idea: introduce a symbolic description of the orbits

Setting: $x^{t+1} = f(x^t)$, $x^t \in M$ mfld with norm $|\cdot|$ on tgt vectors, f C^1 diffeo

orbits $\underline{x} = (x^t)_{t\in\mathbb{Z}}$

$\longleftrightarrow$ fixed points of $F : M^{\mathbb{Z}} \circlearrowleft$

given by $F(\underline{x})^t = f(x^{t-1})$ $\quad F(\underline{x}) = \underline{x}$

Defn: An orbit $\underline{x}$ is uniformly hyperbolic if it is a non-degenerate fixed pt of F, i.e. $I - DF_{\underline{x}}$ has bounded inverse wrt supremum norm on $T(M^{\mathbb{Z}})$

$$\|\underline{\xi}\|_\infty = \sup_{t\in\mathbb{Z}} |\xi^t| \quad \text{for } \underline{\xi} \in T(M^{\mathbb{Z}}) = (TM)^{\mathbb{Z}}$$

$$I - DF_{\underline{x}} = \begin{bmatrix} \ddots & & & \\ & I & & 0 \\ & -f'(x^{-1}) & I & \\ 0 & & -f'(x^0) & I \\ & & & \ddots \end{bmatrix} \begin{matrix} t \\ \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \end{matrix}$$

A set Λ of orbits is u.hyp if $\|(I - DF_{\underline{x}})^{-1}\|$ bdd on $\underline{x} \in \Lambda$

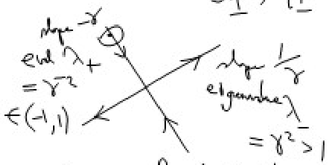
Example: $f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

Whole $\mathbb{T}^2$ is u.hyp, because

Solve $(I - DF_{\underline{x}})\underline{\xi} = \underline{\eta}$ for $\underline{\xi} \in TM^{\mathbb{Z}}_\infty$ given $\underline{\eta} \in TM^{\mathbb{Z}}_\infty$

by splitting ξ^t, η^t into components $\xi^t_\pm, \eta^t_\pm$

along eigenvectors of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$



$+$ cpt: $\xi^t_+ - \lambda_+ \xi^{t-1}_+ = \eta^t_+$

$\xi^t_+ = \eta^t_+ + \lambda_+ \xi^{t-1}_+$

$+$ means forward contracting

$-$ — backward —

traditionally $+ = s$

$- = u$

has unique bdd soln $\xi^t_+ = \sum_{n\geq 0} \lambda_+^n \eta_+^{t-n}$

$-$ cpt: $\xi^t_- - \lambda_- \xi^{t-1}_- = \eta^t_-$

$\xi^{t-1}_- = \frac{1}{\lambda_-}(\xi^t_- - \eta^t_-)$ has unique bdd soln

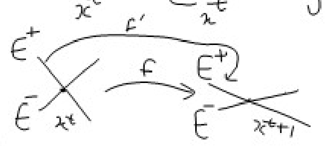
$$\xi^t_- = -\sum_{n\geq 1} \lambda_-^{-n} \eta_-^{t+n}$$

So $\|\underline{\xi}_+\| \leq \frac{\|\underline{\eta}_+\|}{1-\lambda_+}$ & $\|\underline{\xi}_-\| \leq \frac{\|\underline{\eta}_-\|}{\lambda_- - 1}$

& $\|(I - DF_{\underline{x}})^{-1}\| \leq \frac{1}{1-\lambda_+}$ (say using Euclidean norm on $T\mathbb{T}^2$) $\square$

Usual def^n of u.hyp is $\exists$ invariant splitting of

$$TM_{x^t} = E^+_{x^t} \oplus E^-_{x^t} \quad \text{along orbit } \underline{x}$$



$C \in \mathbb{R}$, $\lambda \in (0,1)$

s.t. $\xi^t \in E^{\pm}_{x^t} \Rightarrow \left|\xi^{t+s}\right| \leq C\lambda^{|s|}\left|\xi^t\right|$

for all $s \begin{cases} >0 & \text{for } E^+ \\ <0 & \text{for } E^- \end{cases}$

But this is more or less equivalent to mine

In particular, it is implied by mine if $|f'(x^t)|_{t\in\mathbb{Z}}$ is bounded (automatic if M compact)

<u>Proof</u>: $(I - DF_{\underline{x}})\underline{\xi} = \underline{\eta}$ has unique bdd soln $\underline{\xi}$ for any bdd $\underline{\eta}$. In particular take $\eta^t = 0$ for all $t \neq 0$. Let $\tilde{\xi}^t = z^{|t|}\xi^t$, some $z>1$ & try to prove $\tilde{\underline{\xi}}$ bdd.

$$\xi^t - f'\xi^{t-1} = \eta^t$$

$$\Rightarrow \tilde{\xi}^t - zf'\tilde{\xi}^{t-1} = 0 \quad t>0$$

$$\tilde{\xi}^t - \frac{f'}{z}\tilde{\xi}^{t-1} = \eta^t \quad \text{for } t \leq 0 \begin{pmatrix} =0 \text{ for} \\ t<0 \end{pmatrix}$$

So $(I - DF - E)\tilde{\xi} = \underline{\eta}$ with

$$E = \begin{array}{cc} & t \\ \begin{bmatrix} \ddots & & & \\ & 0 & & \\ & (\frac{1}{z}-1)f' & 0 & \\ & & (z-1)f' & 0 \\ & & & \ddots \end{bmatrix} & \begin{array}{c} \\ -1 \\ 0 \\ +1 \\ \\ \end{array} \end{array}$$

$$\|E\| \leq |z-1|\,l \quad \text{with } l = \sup_t |f'(x^t)|$$

and so $(I - DF - E)$ has bdd inverse if

$$\|E\| < K := \|(I-DF)^{-1}\|^{-1}$$

and $\|(I-DF-E)^{-1}\| \leq \dfrac{1}{K - \|E\|}$.

So $$\|\tilde{\xi}\| \leq \frac{|\eta^0|}{K - |z-1|\,l}$$

i.e. $$|\xi^t| \leq \frac{z^{-|t|}\,|\eta^0|}{K - |z-1|\,l}$$ decays exp both ways in t

Same for any initial time s

Then split $\eta^0 = \underset{\eta^0_+}{\xi^0}\ \underbrace{-\,f'(x^{-1})\xi^{-1}}_{\eta^0_-}$

& check that this defines a complementary pair

of projections $P^{\pm}$ $\quad P^+ \eta^\circ = \eta^\circ_+$

$P^2 = P$ $\quad P^- \eta^\circ = \eta^\circ_-$

$P^+ + P^- = I$ & invariance of splitting

& note exponential decay estimates. $\square$

Similarly, usual defn + some extra if M non-compact

$\Rightarrow$ my defn.

A nice feature of u hyp orbits $\underset{\sim}{x}$ of F is robustness

$\forall \tilde{f}$ C^1-close to f $\exists$ unique orbit $\tilde{\underset{\sim}{x}}$ of $\tilde{f}$

uniformly near to $\underset{\sim}{x}$ (Pf: apply implicit fn thm

to $F(\underset{\sim}{x}) = \underset{\sim}{x}$)

Furthermore $\tilde{x}^0(x^0)$ is Hölder continuous

and one-to-one, so given u hyp set $\Lambda \subset M$

($= \bigcup$ pts of a u hyp set of orbits)

$\forall \tilde{f}$ C^1-close to f $\exists h : \Lambda \to M$ near-id

homeomorphism onto $h(\Lambda)$ s.t. $\tilde{f} h = h f$ on Λ

Similar for t-dependent perturbations.