Today we'll show that dyn on a u. hyp. attractor is equivalent to a Gibbsian stochastic process on a symbol space (generalisation of Markov chain)

<u>Defn</u>: A Markov partition is a finite decomposition of an invariant set Λ (up to "small" overlaps) $\Lambda = \bigcup_i A_i$ s.t. letting Γ = graph with nodes A_i, edges

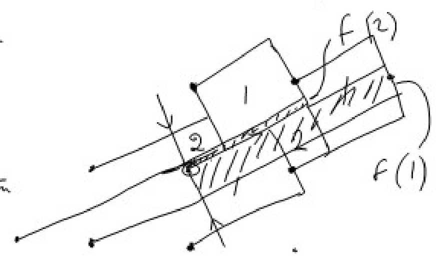
$i \to j$ if $fA_i \cap A_j \neq \emptyset$

(may need to allow multiple edges to get uniqueness below)

then every doubly ∞ path in Γ occurs as the symbolic trajectory of some orbit in Λ.

e.g. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$1 \cup 2 = \mathbb{T}^2$



Really in defn of Mptn I want each path to be realised by a <u>unique</u> orbit.

So f is topologically conjugate to the shift on doubly ∞ paths in Γ $\quad \sigma : \ldots e_{-1} . e_0 e_1 \ldots \mapsto \ldots e_{-1} e_0 . e_1 \ldots$ with product topology modulo some identifications of paths which correspond to the same orbit of f.

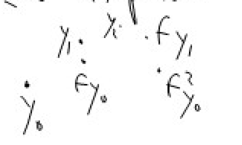
For any locally maximal bdd u. hyp set $\exists$ M ptn

Pf is via "shadowing thm": given a u hyp set Λ $\exists K, \varepsilon_0 > 0$ s.t. every ε-pseudo-orbit is $K\varepsilon$-shadowed by a true orbit of f: $F\underline{x} = \underline{x}$ $\quad d(F\underline{y}, \underline{y}) \leq \varepsilon$ in sup norm

$\underline{y}$ is an <u>ε-pseudo orbit</u> if $d(f y_n, y_{n+1}) \leq \varepsilon \ \forall n$.

<u>$\underline{x}$ δ-shadows</u> $\underline{y}$ if $d(x_n, y_n) \leq \delta \ \forall n$

i.e. $d(\underline{x}, \underline{y}) \leq \delta$ in sup-norm



Prove by constructing a good approx J to $(I - DF_{\underline{y}})^{-1}$ & then $\underline{x} \mapsto \underline{x} - JF\underline{x}$ is a contraction in a nbd of $\underline{y}$ (in some local coord sys about $\underline{y}$)

To make a M ptn choose "enough" periodic orbits of f on Λ as the "symbols" and use shadowing thm to make the orbits of Λ from true orbits shadowing segments of the chosen periodic orbits.

Can also show $x^t(\underline{\sigma})$ depends exponentially weakly on σ^s $|s-t| > N$ large

Another example: solenoid attractor $f : S^1 \times D^2 \circlearrowleft$, $S^1 = \mathbb{R}/\mathbb{Z}$

$$\begin{cases} x' = 2x & \in S^1 \\ w' = \lambda w + \mu e^{2\pi i x} & \in D^2 \subset \mathbb{R} \end{cases}$$

has M ptn $0 = \{x \in [0, \tfrac{1}{2})\}$ $1 = \{x \in [\tfrac{1}{2}, 1)\}$

Γ $\quad \circlearrowright 1 \rightleftarrows 2 \circlearrowleft$

<u>Natural measure</u> on a u.hyp attractor : If $f \in C^{1+\text{Hölder}}$

Λ u.hyp/bdd R attractor, topologically mixing, ∃N s.t. ∀ $A, B \neq \emptyset$ open $f^n A \cap B \neq \emptyset$ $\forall n \geq N$, then ν is abs cts prob on $B(\Lambda)$ "basin"

wrt Lebesgue class
$\lambda(A) = 0 \Rightarrow \nu(A) = 0$

ν has a density h in $L^1(\lambda)$.

$\Rightarrow \nu f^t_* \xrightarrow{w^*}$ some prob ρ on Λ as $t \to +\infty$

(SRB measure)

Can metrize w^* topology on $\mathcal{P}(\Lambda)$ by $d(\mu, \nu) = \sup_{g \in \text{Lip}\setminus\text{csts}} \frac{\mu(g) - \nu(g)}{\text{Lip}(g)}$

The image of ρ under symbolic coding is a Gibbs phase for "energy" $\varphi^t_-(\underline{\sigma}) = \log|\det Df|_{E^-}(x^t(\underline{\sigma}))|$

i.e. $\rho\{\sigma^{-T} \ldots \sigma^S \mid (\sigma^t)_{t<-T,\, t>+S}\} \propto e^{-\sum_{t \in \mathbb{Z}} \{\varphi^t_-(\underline{\sigma}) - \varphi^t_-(\underset{\sim}{\bar{\sigma}})\}}$ for any allowed reference seq $\bar{\sigma}$ satisfying given past & future

The Gibbs phase is unique because Λ top. mixing, dependence of φ^t_- on σ^s decays exponentially with $|s-t|$ and $t \in \mathbb{Z}$ is 1D so apply some 1D Stat. Mech. result

Next time I'll give the idea of proof & Tues 8th Dec will extend to CML

Note I'm away on 11 Dec.
How abt extra lecture on Fri 18 Dec?