Last time: <u>Bounding the price of anarchy</u>

$$\rho = \frac{C(f_{Nash})}{C(f^*)}$$

Case of affine cost functions $c_e(x) = a_e x + b_e$

<u>Thm</u>: For affine cost fns $\rho \leq \frac{4}{3}$

Note: equality attained in Pigou & Braess examples of last wk

Today we'll prove this (+ other classes)

NB: Past lectures are on my website under Topics in Complexity Science tab.

To prove Thm, apply 2 results from last time:

f Nash iff $\forall i, P, P' \in \mathcal{P}_i, f_P > 0$

$$\sum_{e \in P} a_e f_e + b_e \leq \sum_{e \in P'} a_e f_e + b_e$$

f^* optimal iff ———— $\sum_{e \in P} 2a_e f_e^* + b_e \leq \sum_{e \in P'} 2a_e f_e^* + b_e$

So f Nash for $r \Rightarrow \frac{f}{2}$ optimal for $\frac{r}{2}$; and $C\left(\frac{f}{2}\right) \geq \frac{1}{4} C(f)$

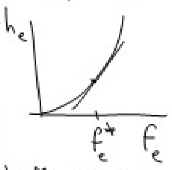
because $\sum_e \left(\frac{1}{2} a_e f_e + b_e\right) \frac{f_e}{2} \geq \frac{1}{4} \sum (af + b) f$

Now augment $\frac{f}{2}$ to a flow for r.

Recall marginal cost fns $c_e^*(x) = 2a_e x + b_e$

<u>Lemma</u> If f^* optimal for r then $\forall \delta > 0$ and f feasible for $(1+\delta) r$ then $C(f) \geq C(f^*) + \delta \sum_e c^*(f_e^*) f_e^*$

<u>Proof</u>: $x c_e(x)$ convex so $h_e(f) = c_e(f_e) f_e \geq c_e(f_e^*) f_e^* + (f_e - f_e^*) c_e^*(f_e^*)$



So $C(f) \geq C(f^*) + \sum_e (f - f^*) c^*(f^*)$

In proof of Thm 2 we showed $\sum h'(f^*) f^* \leq \sum h'(f) f$ $\forall$ feasible f for rates r. $h' = c^*$.

So for f feasible for $(1+\delta) r$ we have

$\sum c^*(f^*) f^* \leq \sum c^*(f^*) \frac{f}{1+\delta}$. Hence result $\square$

Proof of Thm ($\rho \leq \frac{4}{3}$ for affine costs):

f Nash $\Rightarrow \frac{f}{2}$ optimal for $\frac{r}{2}$ & has $c^*\left(\frac{f}{2}\right) = c(f)$

Take $\delta = 1$ in lemma. f^* is feasible for $(1+\delta)\frac{r}{2}$

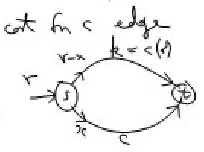
so $C(f^*) \geq C\left(\frac{f}{2}\right) + \sum c^*\left(\frac{f}{2}\right)\frac{f}{2}$

$\geq \frac{1}{4} C(f) + \frac{1}{2} \sum c(f) f = \frac{3}{4} C(f)$. $\square$

<u>General cost fns</u> For a cost fn c let

<u>anarchy value</u> $\alpha(c) = \sup_{x, r \geq 0} \frac{r c(r)}{x c(x) + (r - x) c(r)}$

= worst case (over k & r) for ρ for Pigou example with a constant k cost edge & a cost fn c edge



<u>Proposition</u>: If c C^1 semiconvex then $\alpha(c) = \sup_{r \geq 0} \frac{1}{\lambda\mu + 1 - \lambda}$ where λ solves $c^*(\lambda r) = c(r)$ & $\mu = c(\lambda r)/c(r)$

For a set $\mathcal{C}$ of cost fns let $\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \alpha(c)$

<u>Thm</u>: $\rho \leq \alpha(\mathcal{C})$

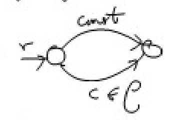
<u>Proof</u>: Note $x c(x) \geq \frac{r c(r)}{\alpha(\mathcal{C})} + (x - r) c(r)$

Let f^* optimal, f Nash then

$C(f^*) = \sum_e c(f^*) f^* \geq \frac{1}{\alpha} \sum c(f) f + \sum (f^* - f) c(f)$

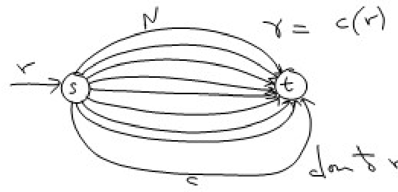
$\geq \frac{C(f)}{\alpha}$ using Cor 4. $\square$

By defn of α, ρ arbitrarily close to $\alpha(\mathcal{C})$ occurs for



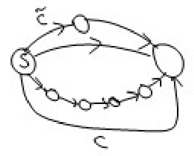
. So $\rho \leq \alpha(\mathcal{C})$ is the best inequality possible if $\mathcal{C}$ contains all constant costs.

<u>Defn</u>: set $\mathcal{C}$ is <u>diverse</u> if $\forall \gamma > 0 \ \exists c \in \mathcal{C}$ with $c(0) = \gamma$. Then $\forall \varepsilon > 0$ $\rho > \alpha(\mathcal{C}) - \varepsilon$ occurs for



close to realising $\alpha(\mathcal{C})$

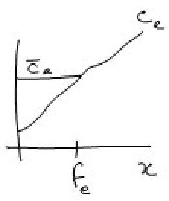
Can weaken requirement on $\mathcal{C}$ still further & still keep $\rho \leq \alpha(\mathcal{C})$ best possible bound: say $\mathcal{C}$ <u>inhomogeneous</u> if $\tilde{c}(0) \neq 0$ for some $\tilde{c} \in \mathcal{C}$. Then $\forall \varepsilon > 0$ $\rho \geq \alpha(\mathcal{C}) - \varepsilon$ occurs for a "union of paths"



A different type of bound: by what factor should rates be increased by an optimal controller to realise same cost as Nash?

<u>Thm</u> f Nash for r, f^* optimal for $2r$
$$\Rightarrow \quad C(f) \leq C(f^*)$$

<u>Proof</u>: $C(f) = \sum_i c_i(f) r_i$



Let $\bar{c}_e(x) = \begin{cases} c_e(f_e) & \text{if } x \leq f_e \\ c_e(x) & \text{if } x \geq f_e \end{cases}$

Then $\bar{C}(f) = C(f)$ & $\forall$ feasible f^* for $2r$
$$\sum \bar{c}_e(f_e^*) f_e^* - C(f^*) = \sum f_e^* \left(\bar{c}(f_e^*) - c_e(f_e^*) \right)$$
$$\leq \sum c_e(f_e) f_e = C(f)$$
and
$$\sum_p \bar{c}_p(f^*) f_p^* \geq \sum_i \sum_{p \in \mathcal{J}_i} c_i(f) f_p^* =$$
$$\sum 2 c_i(f) r_i = 2 C(f) \quad \text{(because } f^* \text{ feasible for } 2r\text{)}$$
So
$$C(f^*) \geq \sum_p \bar{c}_p(f^*) f_p^* - C(f)$$
$$\geq 2C(f) - C(f) = C(f). \quad \square$$

<u>Cor</u> Let $\tilde{f}$ be Nash for $\tilde{c}$ and f^* optimal for c where $\tilde{c}_e(x) = \frac{1}{2} c_e\left(\frac{x}{2}\right)$. Then $\tilde{C}(\tilde{f}) \leq C(f^*)$

<u>Proof</u>: Let f be Nash for $\left(\frac{r}{2}, c\right)$, f^* feasible for (r, c). By thm $\sum c_e(f_e) f_e \leq \sum c_e(f_e^*) f_e^*$
Let $\tilde{f} = 2f$ which is feasible for $r, \tilde{c}$. It is Nash for $r, \tilde{c}$ and $\sum \tilde{c}(\tilde{f}) \tilde{f} = \sum c(f) f$. $\square$