2.2 Varadhan's Integral Lemma

The next theorem can actually be used as a starting point for developing the large deviations paradigm. It is a very useful tool in many applications of large deviations. For example, the asymptotics of the partition function in statistical mechanics can be derived using this theorem.

<u>Theorem 5:</u> Let $(\mu_n)_{n\geq 1}$ satisfy the LDP on E with rate n and with rate function I. Let $\phi : E \to \mathbb{R}$ be a continuous function that is bounded from above. Then

$$\lim_{n\to\infty} \cancel{\sup} \frac{1}{n} \log \int_E e^{n\phi(x)} \mu_n(dx) = \sup_{x\in E} \{\phi(x) - I(x)\}$$

<u>Proof</u>: Let $J_n(S) = \int_S e^{n\phi(x)} \mu_n(dx)$, $S \subset \mathcal{B}_E$,

and put $a = \sup_{x\in E}\{\phi(x)\}$, $b = \sup_{x\in E}\{\phi(x) - I(x)\}$

$-\infty < b \leq a < \infty$.

Upper bound: We slice the space E according to the values of ϕ. Let $C = \Phi^{-1}([a,b])$, and for $N \in \mathbb{N}$ define the sets

$$C_j^N = \Phi^{-1}\left([c_{j-1}^N, c_j^N] \right) \quad , j=1,\dots,N,$$

where $c_j^N = b + {}^{j}/_{N}(a-b)$ for $j=0,1,\dots,N$.

$$C = \bigcup_{j=1}^N C_j^N$$

; and C_j^N are closed (as Φ is continuous). Hence the LDP implies that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(C_j^N) \leq - \inf_{y\in C_j^N} I(y) \quad \forall j=1,\dots,N$$

$\Phi(x) \leq c_j^N$ on C_j^N, then we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log J_n(C) \leq \max_{1\leq j\leq N} \left\{ c_j^N - \inf_{y\in C_j^N} I(y) \right\}$$

insert $c_j^N \leq \inf_{x\in C_j^N} \Phi(x) + \frac{1}{N}(a-b)$, to get

$$\limsup_{n\to\infty} \frac{1}{n} \log J_n(C) \leq \max_{1\leq j\leq N} \left\{ \inf_{x\in C_j^N} \Phi(x) - \inf_{y\in C_j^N} I(y) \right\} + \frac{1}{N}(a-b)$$

$$\leq \max_{1\leq j\leq N} \sup_{x\in C_j^N} \left\{ \Phi(x) - I(x) \right\} + {}^{1}/_{N}(a-b)$$

$$= \sup_{x\in C} \left\{ \Phi(x) - I(x) \right\} + {}^{1}/_{N}(a-b) \leq b + {}^{1}/_{N}(a-b) .$$

Letting $N \to \infty$, we get $\limsup_{n\to\infty} \frac{1}{n} \log J_n(C) \le b$.

Note $J_n(E \setminus C) \le e^{nb}$, and conclude with
$\overline{\lim_{n\to\infty}}\, \frac{1}{n} \log J_n(E) \le b$.

Lower bound: Pick $x \in E$ and $\varepsilon > 0$ arbitrary.
Then the set $O_{x,\varepsilon} = \{ y \in E : \phi(y) > \phi(x) - \varepsilon \}$ is an open neighbourhood of x.
The LDP implies

$$\underline{\lim_{n\to\infty}}\, \frac{1}{n} \log \mu_n(O_{x,\varepsilon}) \geq - \inf_{y \in O_{x,\varepsilon}} I(y) .$$

Since $\inf_{y \in O_{x,\varepsilon}} I(y) \le I(x)$, this estimate gives us

$$\liminf_{n\to\infty} \frac{1}{n} \log J_n(O_{x,\varepsilon}) \geq \Phi(x) - \varepsilon - I(x) .$$

Now use ~~$\inf J_n(E)$~~
$\inf_{x\in E} J_n(x) \geq \inf_{y \in O_{x,\varepsilon}} J_n(y)$, let $\varepsilon \downarrow 0$ and afterwards take the supremum over $x \in E$, to find $\liminf_{n\to\infty} \frac{1}{n} \log J_n(E) \geq b$

□

Remark: Instead of Φ bounded from above some versions of the Varadhan Lemma have either a tail-condition, or a moment condition. That is, Φ bounded from above is replaced by

(tail condition) $\lim\limits_{M\to\infty} \limsup\limits_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{\Phi(Z_n)} 1\!\!1\{\Phi(Z_n) \geq M\}\right] = -\infty.$

where (Z_n) is sequence of random variables taking values in E and having law/distribution given by μ_n.

(moment condition) for some $\gamma > 1$,

$$\limsup_{n\to\infty} 1/n \log \mathbb{E}\left[e^{n\gamma\phi(Z_n)}\right] < \infty.$$

(see Dembo and Zeitouni for detailed proof).

The following alternative version of Varadhan's Lemma allows us to generate one LDP from another via tilting.

Theorem 6: Let $(\mu_n)_{n\geq 1}$ satisfy the LDP on E with rate n and with rate function I. Let $\Phi : E \to \mathbb{R}$ be a continuous function that is bounded from above. Define

$J_n(S) = \int_S e^{n\phi(x)} \mu_n(dx)$, $S \subset E$ Borel.

Then the sequence $(\nu_n^{\Phi})_{n\geq 1}$ of probability measures defined by $\nu_n^{\Phi}(S) = \frac{J_n(S)}{J_n(E)}$, $S \subset E$ Borel,

satisfies the LDP on E with rate n and with rate function

$$I^{\Phi}(x) = \sup_{y\in E} \{\Phi(y) - I(y)\} - \{\Phi(x) - I(x)\}.$$

Proof (exercise – follows from the proceeding version of Varadhan's Lemma).

<u>Remark</u>: Let $\Lambda_n(f) = \frac{1}{n} \log \int_E e^{nf(x)} \mu_n(dx)$, $f \in \mathcal{C}_b(E)$.

If $(\mu_n)_{n\geq 1}$ is exponentially tight and $\lim_{n\to\infty} \Lambda_n(f) = \Lambda(f) \in \mathbb{R}$ exists for all $f \in \mathcal{C}_b(E)$,

then $(\mu_n)_{n\geq 1}$ satisfies the LDP with rate n and with rate function I given by

$$I(x) = \sup_{f\in C_b(E)} \{f(x) - \Lambda(f)\}.$$

This equation is the inverse of the relation

$$\Lambda(f) = \sup_{x\in E} \{f(x) - I(x)\}, \quad f \in C_b(E),$$

appearing in Varadhan's Lemma. Note that in this general context the rate function I need not be convex.

We will see in Section 2.3, in the setting of topological vector spaces, that linear functionals play an important role in establishing the LDP, particularly when convexity is involved. Note, however, that Varadhan's Lemma applies to nonlinear functions as well.

We finish this section with an inverse of Varadhan's Lemma. Let $(\mu_n)_{n\geq 1}$ be a sequence of probability measures on E.

For each Borel measurable function $f: E \to \mathbb{R}$, define

$$\Lambda(f) := \lim_{n\to\infty} \frac{1}{n} \log \int_E e^{nf(x)} \mu_n(dx)$$

<u>**Theorem 7:**</u> Suppose that $(\mu_n)_{n\geq 1}$ is exponentially tight and that the limit $\Lambda(f)$ exists for every $f \in C_b(E)$. Then $(\mu_n)_{n\geq 1}$ satisfies the LDP with the good rate function

$$I(x) = \sup_{f\in C_b(E)} \{ f(x) - \Lambda(f) \} .$$

Furthermore, for every $f \in C_b(E)$,

$$\Lambda(f) = \sup_{x\in E} \{ f(x) - I(x) \} .$$

Proof: Since $\Lambda(0)=0$, it follows that $I \geq 0$.
I is l.s.c., since it is the supremum of continuous functions.
The asserted LDP follows once the weak LDP is proved (exp. tightness). Then the ( ) follows with Varadhan's Lemma.

Upper bound: If $\Lambda(f)$ exists for each $f \in C_b(E)$, then, for every compact $\Gamma \subset E$, $\limsup_{n\to\infty} 1/n \log \mu_n(\Gamma) \leq -\inf_{x\in\Gamma} I(x)$.

(Proof see next subsection)

Lower bound: If $\Lambda(f)$ exists for each $f \in C_b(E)$, then, for every open $G \subset E$ and each $x \in G$,
$\liminf_{n\to\infty} 1/n \log \mu_n(G) \geq -I(x)$. Proof: Fix $x\in E$ and a neighbourhood G of x.

As a Polish space, E is completely regular, i.e., there exists a continuous function $f: E \to [0,1]$, such that $f(x)=1$ and $f(y)=0$ $\forall y \in G^c$. For $m>0$, define $f_m(\cdot) = m(f(\cdot)-1)$. Then
$$\int_E e^{nf_m(x)}\mu_n(dx) \leq e^{-mn}\mu_n(G^c) + \mu_n(G) \leq e^{-mn} + \mu_n(G).$$
Since $f_m \in C_b(E)$ and $f_m(x)=0$, it now follows that
$$\max\left\{\liminf_{n\to\infty}\frac{1}{n}\log\mu_n(G), -m\right\} \geq \liminf_{n\to\infty}\frac{1}{n}\log\int_E e^{nf_m(x)}\mu_n(dx) = \Lambda(f_m)$$
$$= -[f_m(x) - \Lambda(f_m)] \geq -\sup_{f\in C_b(E)}\{f(x)-\Lambda(f)\} = -I(x),$$
and the lower bound follows by letting $m \to \infty$ ■

Varadhan's Lemma is the natural extension of Laplace's method to infinite dimensional spaces.

$E = \mathbb{R}$, and assume that μ_n has a density w.r.t. Lebesgue's measure such that

$$\frac{d\mu_n}{dx} \approx e^{-nI(x)}$$

$$\int_{\mathbb{R}} e^{n\Phi(x)} \mu_n(dx) \approx \int_{\mathbb{R}} e^{n(\phi(x)-I(x))} dx$$

Assume that I and ϕ are twice differentiable, with $(\phi(x)-I(x))$ concave and possessing a unique global maximum at some $\bar{x}$. Then

$$\phi(x)-I(x) = \phi(\bar{x})-I(\bar{x}) + \frac{(x-\bar{x})^2}{2}(\phi(x)-I(x))''\Big|_{x=\xi}$$

where $\xi \in [\bar{x}, x]$.

$$\int_{\mathbb{R}} e^{n\phi(x)} \mu_n(dx) \approx e^{n(\phi(\bar{x})-I(\bar{x}))} \int_{\mathbb{R}} e^{-B(x)(x-\bar{x})^2 n/2} dx,$$

where $B \geq 0$.

example: Gamma function and Stirling's formula

$$n! = \int_0^\infty x^n e^{-x} \, dx \quad , n \in \mathbb{N}.$$

We shall study the asymptotic behaviour of real integrals of the form $\int_{\mathbb{R}} \exp(n\phi(x)) \, dx$ as $n \to \infty$ (Varadhan Lemma).

change variables to $z = x/n$; thus we shall study the integral

$$n^{n+1} \int_0^\infty \exp\{-n(z - \log z) \, dz\}$$

From asymptotic analysis it is clear that the leading order asymptotic behaviour of the integral is determined by the largest value of the integrand, in this case e^{-n} since $z - \log z$ has a minimum value of 1 at $z = 1$. It is not hard to prove that $n! \sim n^{n+1} e^{-n}$, in the sense that $n^{-1} \log\left(n!/n^{n+1} \right) \longrightarrow -1$ as $n \to \infty$. By making a quadratic approximation about $z = 1$ and suitably expanding, one derives Stirling's formula

$n! = n^n e^{-n} \sqrt{2\pi n} \, (1 + \varepsilon_n)$, ε_n is an asymptotic series in powers of $1/n$ which converges to 0 as $n \to \infty$ $\left(\varepsilon_n = \frac{1}{12n} + \cdots \right)$.

exercise: $E = [0,\infty)$ and define

$$\mu_n(dx) = n e^{-nx} \mathbb{1}\{x \geq 0\}\, dx$$

Then

$$-\inf_{x \in \Gamma^\circ} I(x) \leq \underline{\lim}_{n\to\infty} \frac{1}{n} \log \mu_n(\Gamma) \leq \overline{\lim}_{n\to\infty} \frac{1}{n} \log \mu_n(\Gamma) \leq -\inf_{x\in\overline{\Gamma}} I(x)$$

holds with $I(x) = x, \; x \in E$.

Using this in our example we can easily derive Stirling's formula.

Hint: put $\phi(x) = \log x \;, x \in [0,\infty)$

and use

$$\frac{1}{n} \int_{\mathbb{R}_+} e^{n\phi(x)} \mu_n(dx) \approx e^{-nI(1)} \frac{1}{n} \int_{\mathbb{R}_+} e^{-B(x-1)^2 n/2} dx$$

$$\approx e^{-n} \sqrt{2\pi n}$$ and using quadratic expansion.