(5) Large deviations and Statistical Mechanics

In this section we outline the main principles of statistical mechanics and introduce the level-3 large deviation principle for Gibbs measures. The latter one leads directly to a proof of the variational principle for Gibbs measures. This principle says that any equilibrium measure (Gibbs measure) is minimising a certain rate function or thermodynamic function. This function will be the free energy and we relate it ~~the~~ to the relative entropy of probability measures with respect to an interacting system of Gibbs distributions (ensembles).
We refer to the Dublin lecture notes and add only relevant additional information in this section.

(5.1) Mathematical statistical mechanics and Gibbs measures.

Aims: Derive macroscopic laws / descriptions of thermodynamic systems rigorously from detailed microscopic description of the underlying interacting system.
This derivation is done using entirely probabilistic means – famously the law of large numbers, the CLT, and ergodic theorems. As it will turn out the major tool is

large deviation principle methods and techniques. The main reason is that in interacting system we are concerned with families of interacting random variables which sometimes experience strong dependence structure.
As mathematicians we are seeking to derive limiting objects to obtain a macroscopic description. To avoid unnecessary technical details we will work entirely on lattice systems in the following. As outlined in the notes, the basic principles of classical equilibrium statistical mechanics are derived from interacting systems (particles) in the continuum space $\mathbb{R}^d$!
We focus on interacting systems living on $\mathbb{Z}^d$ where the single lattice sites are indices for random variables σ_x taking values in some state space E. Let E be a finite alphabet, e.g., $E = \{-1, +1\}$ for the Ising model. Henceforth we are concerned with families $(\sigma_x)_{x \in \mathbb{Z}^d}$ of random spins with $\sigma_x \in E$.

$\Omega = \{\omega : \mathbb{Z}^d \to E\} = E^{\mathbb{Z}^d}$ configuration space (infinite system)

$\Omega_\Lambda = E^\Lambda$, $\Lambda \subset \mathbb{Z}^d$ finite ; configuration space of a finite system in Λ.

The underlying idea is to describe the dependency structure on the finite level ~~and~~ and passing to the limiting system when $\Lambda \uparrow \mathbb{Z}^d$ afterwards.

To model interaction among the random spins σ_x we are seeking an appropriate change of measure away from the i.i.d. case where $\mathbb{P}(\sigma_x = e) = \frac{1}{|E|}$ $\forall e \in E$.

We denote the uniform measure on E by λ, i.e., $\lambda(e) = \frac{1}{|E|}$, $e \in E$; and call $\lambda \in \mathcal{M}_1(E)$ the reference or a-priori measure.

As motivated in Section 3.2 ~~as~~ the Gibbs conditioning principle we are seeking a function of the random spins σ_x which describes some interaction of the spins.

Formally, we let $\Phi = (\phi_A)_{\substack{A \subset \mathbb{Z}^d \\ |A| < \infty}}$ be a family of functions

$\phi_A : \Omega \to \mathbb{R}$ such that ϕ_A is $\mathcal{F}_A$-measurable (Borel-σ-algebra for $A \hat{=}$ cylinder events). We call Φ an interaction potential, and we define for $\Lambda \subset \mathbb{Z}^d$ finite

$$H_\Lambda(\omega) = \sum_{A \subset \mathbb{Z}^d, A \cap \Lambda \neq \emptyset} \phi_A(\omega), \quad \omega \in \Omega.$$

$H_\Lambda(\omega)$ is called the Hamiltonian in Λ and $e^{-\beta H_\Lambda(\omega)}$ is called the Boltzmann factor for Hamiltonian H_Λ and parameter $\beta \geq 0$ where $\beta \,\hat{=}$ inverse temperature.

example: $\phi_A = 0$ whenever $|A| > 2$ and $|A| = 1$

$$\phi_A(\omega) = \begin{cases} -J\,\omega_i \omega_j & \text{if } A = \{i,j\} \wedge |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

ferromagnetic Ising model (see below and the notes).

We have to discuss boundary conditions when deriving probability measures on Ω_Λ.

Pick $\eta \in \Omega$ and consider $\omega_\Lambda \eta_{\mathbb{Z}^d \setminus \Lambda} \in \Omega$

which is defined such that $\omega_\Lambda \eta_{\Lambda^c}(x) = \begin{cases} \omega_x & x \in \Lambda \\ \eta_x & x \in \Lambda^c \end{cases}$.

Then we ~~denote~~ call ~~by~~ $H_\Lambda^\eta(\omega) = \sum_{A \cap \Lambda \neq \emptyset} \phi_A(\omega_\Lambda \eta_{\Lambda^c})$

the Hamiltonian in Λ with boundary $\eta \in \Omega$.

Then $\gamma_\Lambda^\phi(d\omega | \eta) := \frac{1}{Z_\Lambda(\eta)} e^{-\beta H_\Lambda^\eta(\omega)} \lambda^\Lambda(d\omega)$

where the normalisation constant $Z_\Lambda(\eta) = \int_{\Omega_\Lambda} e^{-\beta H_\Lambda^\eta(\omega)} \lambda^\Lambda(d\omega)$

is called the partition function, defines a probability measure on Ω_Λ which we call the Gibbs distribution in Λ with boundary $\eta \in \Omega$ and interaction potential Φ.

If we are to model a ferromagnetic material the most simple mathematical model is the Ising model (1924). Here the state space $E = \{-1, 1\}$ and the material favors neighbouring spins to align in the same direction

$$H_\Lambda^\eta(\omega) = -J \sum_{\substack{\langle x,y\rangle=1 \\ x\sim y \\ \{x,y\}\cap\Lambda\neq\emptyset}} \omega_x \omega_y \qquad , J>0 .$$

$J<0$ antiferromagnet (any two adjacent spins prefer to point in opposite directions)

One may add an external magnetic field $h: \mathbb{Z}^d \to \mathbb{R}$ or just a constant such that

$$H_\Lambda^\eta(\omega) = -J \sum_{\substack{x\sim y \\ \{x,y\}\cap\Lambda\neq\emptyset}} \omega_x \omega_y - h \sum_{x\in\Lambda} \omega_x .$$

Ising 1924: $\beta>0$ and $h\in\mathbb{R}$ and $J>0$ fixed; $d=1$.

Easy calculation when periodic boundary conditions are taken, i.e.,

$$H_\Lambda^{(per)}(\omega) = -J \sum_{i=1}^{|\Lambda|} \omega_i \omega_{i+1} - h \sum_{i=1}^{|\Lambda|} \omega_i \; ; \omega_{|\Lambda|+1} = \omega_1$$

for any $\omega \in \Omega$.

partition function $Z_\Lambda(per) = Z_\Lambda(per, \beta, h) = \mathrm{Trace}\left(V^{|\Lambda|}\right)$

where $V = \begin{pmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{pmatrix}$ having two eigenvalues

λ_+, λ_-. Hence,

$$Z_\Lambda(\text{per}) = \lambda_+^{|\Lambda|} + \lambda_-^{|\Lambda|}$$

the free energy in the thermodynamic limit is

$$f(\beta, J, h) = \lim_{\Lambda \uparrow \mathbb{Z}} \frac{-1}{\beta |\Lambda|} \log Z_\Lambda(\text{per}, \beta, h) = +\frac{1}{\beta} \log \lambda_+ .$$

What is the expected magnetisation?

mean magnetisation is $\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma_x =: m_\Lambda(\beta, J, h)$.

What is the limit $m_\Lambda \to m$? as $\Lambda \uparrow \mathbb{Z}$ under the Gibbs distribution?

$$m(\beta, J, h) = \partial_h f(\beta, J, h) = \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$$

$m(\beta, J, 0) = 0$ and $\lim_{h \to \pm\infty} m(\beta, J, h) = \pm 1$

$|m(\beta, J, h)| > |m(\beta, 0, h)|$.

In $d=2$ one finds spontaneous magnetisation for large enough β (low enough temperature), see Peierls 1936 and Onsager 1942.

Analysing the so-called thermodynamic limit of thermodynamic functions like the free energy (pressure, entropy, magnetisation, ...) is one possibility for an macroscopic description of an interacting in thermodynamic equilibrium. Singularities in the thermodynamic function and/or their derivatives are considered to be signals for phase transitions.
A higher level of mathematical description and analysis concerns the probability measures on Ω which corresponds to the thermodynamic equilibrium. These measures are called Gibbs measures. A straightforward possible definition of Gibbs measures is to denote any weak accumulation point of Gibbs distributions a Gibbs measure and include any convex combination of these and the closure of the set of probability measures derived in this way.
However, there is more elegant way to define Gibbs measures which is analogous to stochastic processes ~~[illegible]~~ — in particular Markov processes. The time parameter is replaced by any finite subset of $\mathbb{Z}^d$, and the condition upon past events is replaced conditioning on events outside of any finite subset of $\mathbb{Z}^d$.

__Definition 28:__ A probability measure $\mu \in \mathcal{M}_1(\Omega)$ is is a Gibbs measure for interaction potential Φ and inverse temperature β if

$$\mu(A \mid \mathcal{F}_{\Lambda^c})(\eta) = \gamma_\Lambda^\Phi(A \mid \eta) \qquad \mu \text{ a.s.}$$

for all events $A \in \mathcal{F}$ and all finite $\Lambda \subset \mathbb{Z}^d$.

The set of Gibbs measures for inverse temperature β and interaction potential Φ is denoted by $\mathcal{G}_\beta(\Phi)$.

The above equation for the conditional distributions are called DLR equations (DLR $\hat{=}$ Dobrushin, Lanford, Ruelle, 1969 – 1972).

If $|\mathcal{G}_\beta(\Phi)| = 1$ there is no phase transition at β, and if $|\mathcal{G}_\beta(\Phi)| > 1$ we ~~denote~~ call this phase transition at β.

<u>example</u>: $d=2$ Ising ferromagnet, external magnetic field $h=0$; $J=1$.

For sufficiently large β there exist two shift-invariant Gibbs measures μ_+^β, μ_-^β such that

$$\mu_-^\beta(\sigma_0) = \mathbb{E}_{\mu_-^\beta}(\sigma_0) < 0 < \mathbb{E}_{\mu_+^\beta}(\sigma_0) = \mu_+^\beta(\sigma_0)$$

$\beta_c = \frac{1}{2} \log(1+\sqrt{2})$

$|\mathcal{G}_\beta(J)| = 1$ for $\beta < \beta_c$; $|\mathcal{G}_\beta(J)| > 1$ for $\beta > \beta_c$

5.2 The variational principle and level-3 large deviations

The study of the asymptotic probabilities of large fluctuations of time averages or space averages away from the mean is based on two fundamental principles: the principle of large deviations, and the maximum entropy principle (or Gibbs conditioning principle). The LDP provides the exact rate of exponential decay of the fluctuation probabilities, whereas the latter predicts the limiting conditional distribution under the condition that the fluctuations are large.

We shall be concerned with spatial averages

$$|\Lambda|^{-1} \sum_{k \in \Lambda} f \circ \theta_k$$

of bounded local function $f : \Omega \to \mathbb{R}$, where $(\theta_k \omega)(x) = \omega(k+x)$, $\omega \in \Omega$, is the shift by $k \in \mathbb{Z}^d$!

Pick $\omega \in \Omega$ and denote by $(\omega_\Lambda)^{per}$ the periodic continuation of the restriction/projection of $\omega \in \Omega$ onto Λ.

We define the periodic empirical field

$$R_\Lambda : \Omega \longrightarrow \mathcal{M}_1^\theta(\Omega)$$

$$\omega \longmapsto R_\Lambda(\omega) = R_\Lambda^\omega = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} \delta_{\theta_k(\omega_\Lambda)^{per}}.$$

For any local function $f: \Omega \to \mathbb{R}$,

$$R_\Lambda f : \Omega \to \mathbb{R}, \quad \omega \mapsto R_\Lambda^\omega f = \int f \, dR_\Lambda^\omega$$

ergodic theorem: For each ergodic $\mu \in \mathcal{M}_1^\Theta(\Omega)$ and local function $f: \Omega \to \mathbb{R}$, we have

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} R_\Lambda f = \int f \, d\mu \quad \text{in } \mathcal{L}^1(\mu),$$

and the convergence holds almost surely whenever Λ runs through an increasing sequence of cubes.

We shall be concerned with large deviations from this ergodic behaviour.

<u>Definition 29</u>: Let (μ_Λ) be a sequence of probability measures $\mu_\Lambda \in \mathcal{M}_1(E^\Lambda)$ indexed by cubes $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| \to \infty$ and $\Lambda \uparrow \mathbb{Z}^d$.

(μ_Λ) is said to satisfy a level-3 large deviation principle with rate function $I: \mathcal{M}_1^\Theta(\Omega) \to [0,\infty]$ if

$$\limsup_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-1} \log \mu_\Lambda(R_\Lambda \in C) \leq -\inf_{\nu \in \overline{C}} I(\nu)$$

$$\liminf_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-1} \log \mu_\Lambda(R_\Lambda \in C) \geq -\inf_{\nu \in C^\circ} I(\nu)$$

for any measurable $C \subset \mathcal{M}_1^\Theta(\Omega)$.

Following ideas from Csiszár (1984) we write

$$|\Lambda|^{-1} \log \lambda^\Lambda(R_\Lambda \in C) = -|\Lambda|^{-1} H(\mu_{\Lambda,C} \mid \lambda^\Lambda),$$

where we are using $\mu_{\Lambda,C} = \lambda^{\Lambda}(\cdot \mid R_\Lambda \in C)$

We denote the spatial average of the corresponding periodic measure by $\tilde{\mu}_{\Lambda,C}$, and we note that

$$S(\tilde{\mu}_{\Lambda,C}) = |\Lambda|^{-1} H(\mu_{\Lambda,C} \mid \lambda^{\Lambda})$$ where S is

the negative relative entropy of $\tilde{\mu}_{\Lambda,C}$.

Furthermore,

$$\liminf_{\Lambda \uparrow \mathbb{Z}^d} S(\tilde{\mu}_{\Lambda,C}) \geq \inf_{\nu \in \overline{co}\, C} H(\nu \mid \lambda^{\mathbb{Z}^d}),$$

and thus $\underline{I}(\nu) = H(\nu \mid \lambda^{\mathbb{Z}^d})$, that is,

$$I(\nu) = \lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-1} H(\underbrace{\nu_\Lambda}_{\text{marginal on } E^\Lambda} \mid \lambda^{\Lambda})$$ is the

limiting or specific relative entropy with respect to the a-priori measure. We obtain in addition

$$\operatorname*{acc}_{\Lambda \uparrow \mathbb{Z}^d} \tilde{\mu}_{\Lambda,C} \subset \left\{ \nu \in \overline{co}\, C : I(\nu) = \inf_{\tilde{\nu} \in \overline{C}} I(\tilde{\nu}) \right\}$$

and

$$\operatorname*{acc}_{\Lambda \uparrow \mathbb{Z}^d} \lambda^{\Lambda}(\cdot \mid R_\Lambda \in C) \subset \left\{ \nu \in \overline{co}\, C : I(\nu) = \inf_{\tilde{\nu} \in \overline{C}} I(\tilde{\nu}) \right\}.$$

This is a version of the maximum entropy principle.
We have a level-3 LDP for $\mu_\Lambda = \lambda^\Lambda$ and rate function $I(\nu) = \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} H(\nu_\Lambda | \lambda^\Lambda)$.

We shall be concerned with level-3 large deviations whenever the sequence (μ_Λ) is given by the Gibbs distributions. The level-3 LDP for the reference measure in conjunction with versions of ~~Varad~~ Varadhan's lemma give the corresponding LDP for Gibbs distributions. In order to introduce to new thermodynamic functions which lead to the representation of the rate function, we are seeking criteria for Gibbs measures.
For that we assume that Ω is finite and we write formally

$$\mu(H) = \sum_{\omega\in\Omega} \mu(\omega) H(\omega) = \mathbb{E}_\mu(H), \quad \mu\in\mathcal{M}_1(\Omega),$$

for the mean energy. Recall the entropy is given by $\mathcal{H}(\mu) = -\sum_{\omega\in\Omega} \mu(\omega)\log\mu(\omega)$, and define

$F(\mu) := \mu(H) - \mathcal{H}(\mu)$ the free energy of $\mu\in\mathcal{M}_1(\Omega)$.
For any probability measure $\mu\in\mathcal{M}_1(\Omega)$:

$$F(\mu) \geq -\log Z \text{ and } F(\mu) = -\log Z \text{ iff } \mu = \gamma,$$

where γ is the Gibbs measure $\gamma(\omega) = \frac{1}{Z} e^{-\beta H(\omega)}$.

This can be easily seen using Jensen's inequality for the function $x \log x$ (exercise).

<u>Definition 30:</u>

(a) $H(\nu|\mu) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} H(\nu_\Lambda | \mu_\Lambda)$ specific relative entropy of $\nu \in \mathcal{M}_1(\Omega)$ with respect to $\mu \in \mathcal{M}_1(\Omega)$.

(b) $S_\Lambda(\mu) = - H(\mu_\Lambda | \lambda^\Lambda)$ is called the entropy of $\mu \in \mathcal{M}_1(\Omega)$ in Λ.

$h(\mu) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} S_\Lambda(\mu)$ is the specific entropy of $\mu \in \mathcal{M}_1(\Omega)$.

<u>Remark:</u> The limit $h(\mu)$ can be proven using the sub-additivity property of S_Λ.

We let $\Phi = (\Phi_A)_{A \subset \mathbb{Z}^d}$ be a family of translation invariant interaction potential, and we shall be concerned with averages of the form

$$f_\Phi = \sum_{A \ni 0} |A|^{-1} \Phi_A$$

Theorem 3.1: Let $\mu \in \mathcal{M}_1^{\Theta}(\Omega)$, $\eta \in \Omega$.

(a) The specific energy $\mathbb{E}_\mu(f_\Phi) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mu(H_\Lambda^\eta)$

exist for μ a.e. $\eta \in \Omega$.

We call $f(\mu) = \mu(f_\Phi) - h(\mu)$ the specific free energy of μ for potential Φ

(b) $P(\Phi) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_\Lambda(\eta)$ exists

for all $\eta \in \Omega$.

(c) Let $\mu \in \mathcal{M}_1^{\Theta}(\Omega)$ and $\gamma \in \mathcal{G}_\beta^{\Theta}(\Phi)$.
Then the limit of the specific relative entropy, i.e.,

$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} H_\Lambda(\mu | \gamma)$, exists and is given by

$$h(\mu | \Phi) := P(\Phi) + \mu(f_\Phi) - h(\mu) = P(\Phi) + f(\mu)$$

(limit only depends on Φ and β).
$P(\Phi)$ is called the pressure or specific Gibbs free energy

Proof: See notes or book by Georgii. $\blacksquare$

The latter theorem leads immediately to the variational principle for Gibbs measures (translation-invariant).

<u>**Theorem 32:**</u> Variational Principle (Φ transl. inv.)

(a) $\mu \in \mathcal{M}_1^\Theta(\Omega)$: $h(\mu|\Phi) \geq 0$.
If $\mu \in \mathcal{G}_\beta^\Theta(\Phi)$ then $h(\mu|\Phi) = 0$.

(b) Let $\mu \in \mathcal{M}_1^\Theta(\Omega)$, $\nu \in \mathcal{G}_\beta^\Theta(\Phi)$ be such that $\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} H_\Lambda(\mu_\Lambda | \gamma_\Lambda) = 0$,

then $\mu \in \mathcal{G}_\beta^\Theta(\Phi)$.
Furthermore $h(\mu|\Phi) = 0 \iff \mu \in \mathcal{G}_\beta^\Theta(\Phi)$.

$h(\cdot|\Phi) : \mathcal{M}_1^\Theta(\Omega) \to [0,\infty]$ is an affine and lower semicontinuous function which attains its minimum 0 on the set $\mathcal{G}_\beta^\Theta(\Phi)$, or equivalently, $\mathcal{G}_\beta^\Theta(\Phi)$ is the set on which $f : \mathcal{M}_1^\Theta(\Omega) \to \mathbb{R}$ attains its minimum $-P(\Phi)$.

<u>Remark:</u> The parameter β can be absorbed into the interaction potential.

The preceding results have been obtaind from the following LDP. In the second part we study conditions of the following form. In statistical mechanics one is often concerned on the asymptotic upon conditioning on certain mean or averaged ~~rate~~ energy of the interacting system.

If we employ periodic boundary condition we easily see that the average energy is a function of the periodic empirical field,

$$\langle R^{\omega}_{\Lambda}, \Phi \rangle = |\Lambda|^{-1} H_{\Lambda}^{(per)}(\omega) \quad , \omega \in \Omega .$$

For any potential Ψ we define $\mathcal{T}_{\Lambda}^{\Psi}(\omega) = \langle R^{\omega}_{\Lambda}, \Psi \rangle$, $\omega \in \Omega$.

<u>Theorem 33</u>: LDP for Gibbs measures

(a) Let $\mu \in \mathcal{G}^{\Theta}_{\beta}(\Phi)$ be given. Then $(\mu \circ R_{\Lambda}^{-1})_{\Lambda \subset \mathbb{Z}^d}$ satisfies the LDP (with rate $|\Lambda|^d$) with rate function $I^{\Phi}(\nu) = h(\nu | \mu) = h(\nu | \Phi)$
$= P(\Phi) + f(\nu) = P(\Phi) + \nu(f\Phi) - h(\nu)$

(b) $F \subset \mathcal{M}_1(\Omega)$ closed and $G \subset \mathcal{M}_1(\Omega)$ open

$$\limsup_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \sup_{\eta \in \Omega} \gamma_{\Lambda}^{\Phi, \eta}(R_{\Lambda} \in F) \leqslant - \inf_{\nu \in F} I^{\Phi}(\nu)$$

$$\liminf_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \inf_{\eta \in \Omega} \gamma_{\Lambda}^{\Phi, \eta}(R_{\Lambda} \in G) \geqslant - \inf_{\nu \in G} I^{\Phi}(\nu) .$$

(c) Let V be the vector space of all interaction potentials. Then τ_Λ^Ψ defines in the limit a linear functional $\tau \in V^*$.

$K \subset V^*$ be measurable. Then

$$\limsup_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \sup_{\eta \in \Omega} \gamma_\Lambda^{\Phi,\eta}(\tau_\Lambda \in K) \leq -\inf_{\tau \in \bar{K}} J_V^\Phi(\tau)$$

$$\liminf_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \inf_{\eta \in \Omega} \gamma_\Lambda^{\Phi,\eta}(\tau_\Lambda \in K) \geq -\inf_{\tau \in K^\circ} J_V^\Phi(\tau),$$

with $J_V^\Phi(\tau) = P(\Phi) + \inf_{\Psi \in V} \left\{ \tau(\Psi) + P(\Psi + \Phi) \right\}$

<u>Remark</u>: The proof and the statement can be used to ~~proof~~ prove the equivalence of Gibbs ensembles on the level of probability measures.

Proof. The proof follows using the previous theorem and Varadhan's lemma in conjunction with the LDP for $\mu_\Lambda = \lambda^\Lambda$ (see above ~~theorem~~ 2.1. for the LDP following Csiszár's ideas 1984).