Revision lecture 7th June, example

$(X_t)_{t\geq 0}$ with $I=\{1,2,3,4,5\}$

$$Q=\begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) $X_0=1$. Find prob. that $X_t=2$ for some $t\geq 0$

(ii) Assuming $X_0 \neq 5$, find the prob. that $(X_t)_{t\geq 0}$ eventually visits every state

(i) $h_i := h_i^{\{2\}} = \mathbb{P}_i(\text{hit } 2)$

$h_2 = 1$, $h_3 = 1/3 + 1/3\, h_4$, $h_1 = 1/3 + 1/3\, h_4$, $h_4 = 2/3\, h_1$

resulting in $h_1 = \frac{3}{7}$ and $h_4 = \frac{2}{7}$

$$\mathbb{P}_i(\text{hit } j) = \begin{cases} 3/7 & i \text{ and } j \text{ adjacents } (1,2)\ (\\ 2/7 & i=1,\ j=\cancel{4}\ 3 \end{cases}$$

(ii) $= \mathbb{P}$ (visit every state) : λ initial distribution

$= \sum_{i \in I} \lambda(i)\, \mathbb{P}_i$ (visit every state) $= \mathbb{P}_1$ (visit every state)

Clearly, $\lambda(5) = 0$.
$\mathbb{P}_i(\quad) = \mathbb{P}_1(\quad)$ (Symmetry)

$\mathbb{P}_1$(hit 2, 3, and 4) $= 1 - \underbrace{\mathbb{P}_1(\{\text{avoid } 2\} \cup \{\text{avoid } 3\} \cup \{\text{avoid } 4\})}$

$\mathbb{P}_1\left(\bigcup_{i=2}^{4} \{\text{avoid } i\}\right) = \mathbb{P}_1(\{\text{avoid } 2\}) + \mathbb{P}_1(\{\text{avoid } 3\}) + \mathbb{P}_1(\{\text{avoid } 4\})$

$- \mathbb{P}_1(\{\text{avoid } 2 \text{ and } 3\}) - \mathbb{P}_1(\{\text{avoid } 2 \text{ and } 4\})$

$- \mathbb{P}_1(\{\text{avoid } 3 \text{ and } 4\}) + \mathbb{P}_1(\{\text{avoid } 2, 3, \text{ and } 4\})$

$\mathbb{P}_1(\text{avoid } j) = 4/7 \quad j = 2, 4$
$= 5/7 \quad j = 3$

$\mathbb{P}_1(\text{hit } 2 \text{ or } 4) = 2/3$ and $\mathbb{P}_1(\text{avoid } 2 \text{ and } 4) = 1/3$

$h_i^{\{3,4\}}$
$h_1^{\{3,4\}} = \frac{1}{3} + \frac{1}{3} h_2^{\{3,4\}}$ and $h_1^{\{3,4\}} = h_2^{\{3,4\}}$
$\Rightarrow h_1^{\{3,4\}} = 1/2$

$\mathbb{P}_1(\{\text{avoid } 2 \text{ and } 3\}) = \frac{1}{2} = \mathbb{P}_1(\{\text{avoid } 3 \text{ and } 4\})$

$\mathbb{P}_1(\{\text{avoid } 2, 3, 4\}) = 1/3$

$$\frac{4}{7} + \frac{5}{7} + \frac{4}{7} - \frac{1}{3} - \frac{1}{2} - \frac{1}{2} + 1/3 = 6/7$$

$$\mathbb{P}_1(\text{visit every plate}) = 1/7 \qquad \square$$