Sheet 6:

Question (1c) Let T be the time which elapses before a simple random walk is absorbed at either of the absorbing barriers at 0 and N, having started at k where $0 \leq k \leq N$.

Show that $\mathbb{P}(T < \infty) = 1$ and $\mathbb{E}(T^k) < \infty$ for all $k \geq 1$.

Solution: First note that we consider a Bernoulli random walk with probability p.

We can relate that simple random walk to an infinite sequence of tosses of a coin, any one of which turns up heads with probability p (corresponds to move to the right).

With probability one there will appear a run of N heads sooner or later.

Hence absorption is almost surely certain.

Let $\tilde{T}$ be the number of tosses before the first run of N heads.

$$\mathbb{P}(\tilde{T} > Nr) \leq (1-p^N)^r$$

Nr tosses divided into r blocks of N tosses

$(1-p^N)$ probability of not having N heads

$$\mathbb{P}(\tilde{T} = s) \leq \mathbb{P}(\tilde{T} \leq s) \leq \left(1 - p^{s/r}\right)^{\lfloor B/N \rfloor}$$

$$\left(1 - p^{N}\right)^{\lfloor s/N \rfloor}$$

$$\mathbb{E}(T^k) = \sum_{n \geq 0} n^k \, \mathbb{P}(T = n)$$

$$\leq \sum_{n \geq 0} n^k \left(1 - p^{N}\right)^{\lfloor n/N \rfloor} < \infty .$$