

THE WIDOM-ROWLINSON MODEL ON THE DELAUNAY GRAPH

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ABSTRACT. We establish phase transitions for continuum Delaunay multi-type particle systems (continuum Potts or Widom-Rowlinson models) with infinite range repulsive interaction between particles of different type. Our interaction potential depends solely on the length of the Delaunay edges and is scale invariant up to a parameter replacing the role of inverse temperature. In fact we show that a phase transition occurs for all activities for sufficiently large potential parameter confirming an old conjecture that if phase transition occurs on the Delaunay graph it will be independent of the activity. This is a proof of an old conjecture of Lebowitz and Lieb extended to the Delaunay structure. Our approach involves a Delaunay random-cluster representation analogous to the Fortuin-Kasteleyn representation of the Potts model. The phase transition manifests itself in the mixed site-bond percolation of the corresponding random-cluster model. Our proofs rely mainly on geometric properties of the Delaunay tessellations in \mathbb{R}^2 and on recent studies [DDG12] of Gibbs measures for geometry-dependent interactions. The main tool is a uniform bound on the number of connected components in the Delaunay graph which provides a novel approach to Delaunay Widom Rowlinson models based on purely geometric arguments.

1. INTRODUCTION AND RESULTS

1.1. Introduction. Although the study of phase transitions is one of the main subjects of mathematical statistical mechanics, examples of models exhibiting phase transition are mainly restricted to lattice systems. In the continuous setting results are much harder to obtain, e.g., the proof of a liquid-vapor phase transition in [LMP99], or the spontaneous breaking of rotational symmetry in two dimensions for a Delaunay hard-equilaterality like interaction [MR09]. These phase transitions manifest themselves in breaking of a continuous symmetry. There is another specific model for which a phase transition is known to occur: the model of Widom and Rowlinson [WR70]. This is a multi-type particle system in $\mathbb{R}^d, d \geq 2$, with hard-core exclusion between particles of different type, and no interaction between particles of the same type. The phase transition in this model was established by Ruelle [Rue71]. Lebowitz and Lieb [LL72] extended his result by replacing the hard-core exclusion by a soft-core repulsion between unlike particles. Finally, phase transition results for a general class of continuum Potts models in $\mathbb{R}^d, d \geq 2$, have been derived in [GH96]. The repulsive interaction between particles of different type in [GH96] is of finite range and a type-independent background potential has been added. The phase transitions for large activities in all these systems reveal themselves in breaking of the symmetry in the type-distribution. The results in [GH96] are based on a random-cluster representation analogous to the Fortuin-Kasteleyn representation of lattice Potts models, see [GHM]. In [BBD04] the soft repulsion in [GH96] between unlike particles has been replaced by another kind of soft repulsion based on the structure of some graph. More

Key words and phrases. Delaunay tessellation, Widom-Rowlinson, Gibbs measures, Random cluster measures, mixed site-bond percolation, phase transition, coarse graining, multi-body interaction.

precisely, the finite range repulsion between particles of different type acts only on a nearest neighbour subgraph of the Delaunay graph in \mathbb{R}^2 .

In this paper we establish the existence of a phase transition for a class of continuum Delaunay Widom-Rowlinson (Potts) models in \mathbb{R}^2 . The repulsive interaction between unlike particles is of infinite range, and it depends on the geometry of the Delaunay tessellation, i.e. length of the edges. The potential is formally given as

$$\phi_\beta(\ell) = \log\left(\frac{\ell^4 + \beta}{\ell^4}\right), \quad \beta > 0, \ell \geq 0,$$

where $\ell \geq 0$ is the length of an Delaunay edge and $\beta > 0$ is the potential parameter serving as an inverse temperature. The main novelty of our paper is a uniform bound on the number of connected components in the Delaunay random cluster model which is purely based on geometrical properties of Delaunay tessellations in two dimensions. We believe that these new techniques can be applied to higher dimensions in the future as well. The potential ensures that for large values of the parameter $\beta > 0$, Delaunay edges with shorter lengths are more likely to be connected than those with longer edges, enabling us to bound the number of connected components for clouds of points with vanishing point-wise distances. This paper is an extensive development of the recent work [AE16] where all models had an additional background hard-core potential introducing a length scale, whereas the present work has no a priori length scale fixed. Our results extend [BBD04] and [GH96] in two ways by having an infinite range repulsion and a geometry-dependency of the type interaction. Gibbs models on Delaunay structures have been studied in [BBD99, BBD02, BBD04, Der08, DDG12, DG09, DL11], and our results rely on the existence of Gibbs measures for geometry-dependent interactions established in [DDG12]. Our approach is based on a Delaunay random-cluster representation similar to [BBD04] and [GH96], the difference being that we replace edge percolation by an adaptation of lattice hyperedge percolation [Gri94] to our continuum setting. A phase transition for our Delaunay Potts models follows if we can show that the corresponding percolation process contains an infinite cluster. A similar program was carried out by Chayes et al. in [CCK95] for the hard-core Widom-Rowlinson model. In that case, the existence of infinite clusters follows from a stochastic comparison with the Poisson Boolean model of continuum percolation, while our framework uses a coarse graining method to derive a stochastic comparison with mixed site-bond percolation on \mathbb{Z}^2 . Our results are extension of [LL72] and [CCK95] to the Delaunay structure replacing hard-core constraint by our soft-core repulsion. In particular we obtain phase transition for all activities once the interaction parameter $\beta > 0$ (inverse temperature) is sufficiently large. This confirms the conjecture that phase transitions on the Delaunay structure should once they occur be independent of the activity. This idea actually goes back to [Hag00] in which percolation for the Poisson Voronoi model has been established. We note that our random-cluster representation requires the symmetry of the type interaction. In the non-symmetric Widom-Rowlinson models, the existence of a phase transition has been established by Bricmont et al. [BKL], and recently by Suhov et al. [MSS].

1.2. Remarks on Delaunay tessellations. We conclude with some remarks on the particular features models defined on the Delaunay hypergraph structure show. The most simple Delaunay Potts model would be one with no more interaction than constant Delaunay edge interaction as then the percolation would be independent of the activity parameter. Our triangle model is a step towards understanding this system. We draw attention to some differences between

geometric models on the Delaunay hypergraph structure and that of classical models such as the Widom-Rowlinson model and its soft-core variant of Lebowitz and Lieb [LL72]. The first is that edges and triangles in the Delaunay hypergraph are each proportional in number to the number of particles in the configuration. However, in the case of the complete hypergraph the number of edges is proportional to the number of particles squared and the number of triangles is proportional to the number of particles cubed. Secondly, in the complete graph of the classical models, the neighbourhood of a given point depends only on the distance between points and so the number of neighbours increases with the activity parameter z of the underlying point process. This means that the system will become strongly connected for high values of z . This is not the case for the Delaunay hypergraphs which exhibit a self-similar property. Essentially, as the activity parameter z increases, the expected number of neighbours to a given point in the Delaunay hypergraph remains the same, see [Mø94]. Therefore, in order to keep a strong connectivity in our geometric models on Delaunay hypergraphs, we use a type interaction between particles of a hyperedge with a non-constant mark. Finally, and perhaps most importantly, is the question of additivity. Namely, suppose we have an existing particle configuration ω and we want to add a new particle x to it. In the case of classical many-body interactions, this addition will introduce new interactions that occur between x and the existing configuration ω . However, the interactions between particles of ω remain unaffected, and so classical many-body interactions are additive. On the other hand, in the Delaunay framework, the introduction of a new particle to an existing configuration not only creates new edges and triangles, but destroys some too. The Delaunay interactions are therefore not additive, and for this reason, attractive and repulsive interactions are indistinct. In the case of a hard exclusion interaction, we arrive at the possibility that a configuration ω is excluded, but for some x , $\omega \cup x$ is not. This is called the non-hereditary property [DG09], which seems to rule out using techniques such as stochastic comparisons of point processes [GK97].

1.3. Setup. We consider configurations of points in \mathbb{R}^2 with internal degrees of freedom, or marks. Let $M = \{1, \dots, q\}$, $q \in \mathbb{N}$, $q \geq 2$, be the finite set of different marks. That is, each marked point is represented by a position $x \in \mathbb{R}^2$ and a mark $\sigma(x) \in M$, and each marked configuration ω is a countable subset of $\mathbb{R}^2 \times M$ having a locally finite projection onto \mathbb{R}^2 . We denote by Ω the set of all marked configurations with locally finite projection onto \mathbb{R}^2 . We will sometimes identify ω with a vector $\omega = (\omega^{(1)}, \dots, \omega^{(q)})$ of pairwise disjoint locally finite sets $\omega^{(1)}, \dots, \omega^{(q)}$ in \mathbb{R}^2 (we write Ω for the set of all locally finite configurations in \mathbb{R}^2). Any ω is uniquely determined by the pair (ω, σ) , where $\omega = \cup_{i=1}^q \omega^{(i)}$ is the set of all occupied positions, and where the mark function $\sigma: \omega \rightarrow M$ is defined by $\sigma(x) = i$ if $x \in \omega^{(i)}$, $i \in E$. For each measurable set B in $\mathbb{R}^2 \times M$ the counting variable $N(B): \omega \rightarrow \omega(B)$ on Ω gives the number of marked particles such that the pair (position, mark) belongs to B . We equip the space Ω with the σ -algebra \mathcal{F} generated by the counting variables $N(B)$ and the space Ω of locally finite configurations with the σ -algebra \mathcal{F} generated by the counting variables $N_\Delta = \#\{\omega \cap \Delta\}$ for $\Delta \Subset \mathbb{R}^2$ where we write $\Delta \Subset \mathbb{R}^2$ for any bounded $\Delta \subset \mathbb{R}^2$. As usual, we take as reference measure on (Ω, \mathcal{F}) the marked Poisson point process Π^z with intensity measure $z \text{Leb} \otimes \mu_u$ where $z > 0$ is an arbitrary activity, Leb is the Lebesgue measure in \mathbb{R}^2 , and μ_u is the uniform probability measure on M .

We let $\Omega_f \subset \Omega$ (resp. $\Omega_f \subset \Omega$) denote the set of all finite configurations. For each $\Lambda \subset \mathbb{R}^2$ we write $\Omega_\Lambda = \{\omega \in \Omega: \omega \subset \Lambda \times E\}$ for the set of configurations in Λ , $\text{pr}_\Lambda: \omega \rightarrow \omega_\Lambda := \omega \cap \Lambda \times E$

for the projection from Ω to Ω_Λ (similarly for unmarked configurations), $\mathcal{F}'_\Lambda = \mathcal{F}|_{\Omega_\Lambda}$ for the trace σ -algebra of \mathcal{F} on Ω_Λ , and $\mathcal{F}_\Lambda = \text{pr}_\Lambda^{-1}\mathcal{F}'_\Lambda \subset \mathcal{F}$ for the σ -algebra of all events that happen in Λ only. The reference measure on $(\Omega_\Lambda, \mathcal{F}'_\Lambda)$ is $\Pi_\Lambda^z := \Pi^z \circ \text{pr}_\Lambda^{-1}$. In a similar way we define the corresponding objects for unmarked configurations, $\Pi^z, \Pi_\Lambda^z, \Omega_\Lambda, \text{pr}_\Lambda, \mathcal{F}'_\Lambda$, and \mathcal{F}_Λ . Finally, let $\Theta = (\vartheta_x)_{x \in \mathbb{R}^2}$ be the shift group, where $\vartheta_x: \Omega \rightarrow \Omega$ is the translation of the spatial component by the vector $-x \in \mathbb{R}^2$. Note that by definition, $N_\Delta(\vartheta_x \omega) = N_{\Delta+x}(\omega)$ for all $\Delta \subset \mathbb{R}^2$.

The interaction between the points depend on the geometry of their location. We describe this in terms of hypergraph structures. A hypergraph structure is a measurable set of $\Omega_f \times \Omega$ (resp. $\Omega_f \times \Omega$). We outline the definitions for the unmarked configurations first with obvious adaptations to the case of marked point configurations. The set Del of Delaunay hyperedges consist of all pairs (η, ω) with $\eta \subset \omega$ for which there exists an open ball $B(\eta, \omega)$ with $\partial B(\eta, \omega) \cap \omega = \eta$ that contains no points of ω . For $m = 1, 2, 3$, we write $\text{Del}_m = \{(\eta, \omega) \in \text{Del}: \#\eta = m\}$ for the set of Delaunay simplices with m vertices. Given a configuration ω the set of all Delaunay hyperedges $\eta \subset \omega$ with $\#\eta = m$ is denoted by $\text{Del}_m(\omega)$. It is possible that $\eta \in \text{Del}(\omega)$ consists of four or more points on a sphere with no points inside. In fact, for this not to happen, we must consider configurations in general position as in [Mø94]. More precisely, this means that no four points lie on the boundary of a circle and every half-plane contains at least one point. Fortunately, this occurs with probability one for our Poisson reference measures, and in fact, for any stationary point process. Note that the open ball $B(\eta, \omega)$ is only uniquely determined when $\#\eta = 3$ and η is affinely independent. Henceforth, for each configuration ω we have an associated Delaunay triangulation

$$\{\tau \subset \omega: \#\tau = 3, B(\tau, \omega) \cap \omega = \emptyset\} \quad (1.1)$$

of the plane, where $B(\tau, \omega)$ is the unique open ball with $\tau \subset \partial B(\tau, \omega)$. The set in (1.1) is uniquely determined and defines a triangulation of the convex hull of ω whenever ω is in general position ([Mø94]). In a similar way one can define the marked Delaunay hyperedges, i.e., \mathbf{Del} and $\mathbf{Del}_m(\omega)$ as measurable sets in $\Omega_f \times \Omega$ where the Delaunay property refers to the spatial component only.

Given a configuration $\omega \in \Omega$ (or ω) we write $\Omega_{\Lambda, \omega} = \{\zeta \in \Omega: \zeta \setminus \Lambda = \omega\}$ (resp. $\Omega_{\Lambda, \omega}$) for the set of configurations which equal ω off Λ . For any edge $\eta \in \text{Del}_2$ we denote its length by $\ell(\eta)$, i.e., $\ell(\eta) = |x - y|$ if $\eta = \{x, y\}$. We write $\Lambda \Subset \mathbb{R}^2$ in the following when Λ is non-empty and finite, i.e., $0 < |\Lambda| < \infty$. The interaction is given by the following Hamiltonian in Λ with boundary condition $\omega \in \Omega$,

$$H_{\Lambda, \omega}(\zeta) := \sum_{\substack{\eta \in \text{Del}_{2, \Lambda}(\zeta): \\ \eta \in \text{Del}_{2, \Lambda}(\zeta)}} \phi_\beta(\ell(\eta))(1 - \delta_\sigma(\eta)), \quad \zeta \in \Omega_{\Lambda, \omega}, \quad (1.2)$$

where $\text{Del}_{2, \Lambda}(\zeta) := \{\eta \in \text{Del}_2(\zeta): \exists \tau \in \text{Del}_3(\zeta), \eta \subset \tau, \partial B(\tau, \zeta) \cap \Lambda \neq \emptyset\}$. Here ϕ_β is a measurable function of the length $\ell(\eta)$ of an edge defined for any $\beta \geq 0$,

$$\phi_\beta(\ell) = \log \left(\frac{\ell^4 + \beta}{\ell^4} \right), \quad (1.3)$$

and

$$\delta_\sigma(\eta) = \begin{cases} 1 & , \text{ if } \sigma_\eta(x) = \sigma_\eta(y) \text{ for } \eta = \{x, y\}, \\ 0 & , \text{ otherwise.} \end{cases}$$

Note the following scaling relation for the potential

$$\phi_\beta(L\ell) = \phi_{\beta/L^4}(\ell), \quad \text{for any } L > 0. \quad (1.4)$$

Following [DDG12] we say a configuration $\omega \in \Omega$ (or $\omega \in \Omega$) is **admissible** for $\Lambda \in \mathbb{R}^2$ and activity z if $H_{\Lambda,\omega}$ is $\mathbf{\Pi}^z$ -almost surely well-defined and $0 < Z_\Lambda(\omega) < \infty$, where the partition function is defined as

$$Z_\Lambda(\omega) = \int_{\Omega_{\Lambda,\omega}} e^{-H_{\Lambda,\omega}(\zeta)} \mathbf{\Pi}^z(d\zeta^{(1)}) \cdots \mathbf{\Pi}^z(d\zeta^{(q)}).$$

We denote the set of admissible configurations by Ω_Λ^* . The Gibbs distribution for ϕ_β , and $z > 0$ in Λ with admissible boundary condition ω is defined as

$$\gamma_{\Lambda,\omega}(A) = \frac{1}{Z_\Lambda(\omega)} \int_{\Omega_{\Lambda,\omega}} \mathbb{1}_A(\zeta \cup \omega) e^{-H_{\Lambda,\omega}(\zeta)} \mathbf{\Pi}_\Lambda^z(d\zeta), \quad A \in \mathcal{F}. \quad (1.5)$$

It is evident from (1.5) that, for fixed $\zeta \in \Omega_\Lambda$, the conditional distribution of the marks of $\zeta = (\zeta^{(1)}, \dots, \zeta^{(q)})$ relative to $\gamma_{\Lambda,\omega}$ is that of a discrete Potts model on ζ embedded in the Delaunay triangulation with position-dependent interaction between the marks. This justifies calling our model *Delaunay Potts model* or *Delaunay Widom-Rowlinson model*.

Definition 1.1. *A probability measure μ on Ω is called a Gibbs measure for the Delaunay Potts model with activity $z > 0$ and interaction potentials ψ and ϕ if $\mu(\Omega_\Lambda^*) = 1$ and*

$$\mathbb{E}_\mu[f] = \int_{\Omega_\Lambda^*} \frac{1}{Z_\Lambda(\omega)} \int_{\Omega_{\Lambda,\omega}} f(\zeta \cup \omega) e^{-H_{\Lambda,\omega}(\zeta)} \mathbf{\Pi}_\Lambda^z(d\zeta) \mu(d\omega) \quad (1.6)$$

for every $\Lambda \in \mathbb{R}^2$ and every measurable function f .

The equations in (1.6) are the DLR equations (after Dobrushin, Lanford, and Ruelle). They express that the Gibbs distribution in (1.5) is a version of the conditional probability $\mu(A|\mathcal{F}_{\Lambda^c})(\omega)$. The measurability of all objects is established in [Eye14, DDG12].

1.4. Results and remarks. We outline the results for our triangle and edge models all of which are generic examples of a more general class of coarse grain ready potentials, see Section 2.2 for further details.

Proposition 1.2 (Existence of Gibbs measures). *For any $z > 0$ there exist at least one Gibbs measure for the Delaunay Widom-Rowlinson (Potts) model with parameter $\beta \geq 0$.*

Remark 1.3 (Gibbs measures). *The proof is using the so-called pseudo-periodic configurations (see Appendix A or [DDG12]). Large activities then ensure that these pseudo-periodic configurations have sufficient mass under the reference process. Existence of Gibbs measures for related Delaunay models have been obtained in [BBD99, Der08, DG09]. Note that for $q = 1$ our models have no marks and Gibbs measures do exist as well ([DDG12]).*

Gibbs measures for the Delaunay Potts model do exist for activity $z > 0$. A phase transition is said to occur if there exists more than one Gibbs measure for the Delaunay Potts model. The following theorem shows that this happens for all activities z and sufficiently large parameter β depending on z . Note that β is a parameter for the type interaction and not the usual inverse temperature.

Theorem 1.4 (Phase transition). *For all $z > 0$ there is $\beta_0 = \beta_0(z) > 0$ such that for all $\beta \geq \beta_0$ there exist at least q different Gibbs measures for the Delaunay Widom Rowlinson (Potts) model.*

Remark 1.5 (Free energy). *One may wonder if the phase transitions manifest themselves thermodynamically by a non-differentiability ("discontinuity") of the free energy (pressure). Using the techniques from [Geo94] and [DG09], it should be possible to obtain a variational representation of the free energy, see also [ACK11] for free energy representations for marked configurations. Then a discontinuity of the free energy can be established using our results above. For continuum Potts models this has been established in [GH96, Remark 4.3]. For a class of bounded triangle potential [DG09] shows that the Gibbs measures are minimisers of the free energy. The free energy is the level-3 large deviation rate functional for empirical processes, see [Geo94, ACK11].*

Remark 1.6 (Uniqueness of Gibbs measures). *To establish uniqueness of the Gibbs measure in our Delaunay Potts models one can use the Delaunay random-cluster measure $C_{\Lambda_n, \omega}$, to be defined in (2.2) below. In [GHM, Theorem 6.10] uniqueness is established once the probability of an open connection of the origin to infinity is vanishing for the limiting lattice version of the random-cluster measure, that is, for some set $\Delta \in \mathbb{R}^2$ containing the origin,*

$$\lim_{n \rightarrow \infty} C_{\Lambda_n, \omega}(\Delta \longleftrightarrow \Lambda_n^c) = 0,$$

for a sequence of boxes $\Lambda_n \in \mathbb{R}^2$ with $\Lambda_n \uparrow \mathbb{R}^2$ as $n \rightarrow \infty$. One way to achieve this, is to obtain a stochastic domination of the Delaunay random-cluster measure by the so-called random Delaunay edge model of hard-core particles. Using [BBD02] we know that the critical probabilities for both, the site and bond percolation on the Delaunay graph, are bounded from below. Extension to our tile (hyperedge) percolation using [Gri94] can provide a corresponding lower bound as well. Thus, if our parameter β is chosen sufficiently small, then there is no percolation in our Delaunay random-cluster measures and therefore uniqueness of the Gibbs measure.

It goes without saying that the results addressed here are merely a first step towards a closer study of phase transitions with geometry-dependent interactions. The study for Widom-Rowlinson or Potts models with geometry-dependent interaction is by far not complete, one may wish to extend the single tile (edge or triangle) interaction to mutual adjacent Voronoi cell interaction. The common feature of all these 'ferromagnetic' systems is that phase transitions are due to breaking the symmetry of the type distribution. Breaking of continuous symmetries is a much harder business, see e.g. [MR09], which shows a breaking of rotational symmetry in two dimensions, and which can be seen as a model of oriented particles in \mathbb{R}^2 with a Delaunay hard-equilaterality interaction. Models studied in [DDG12] may be natural candidates for the existence of a crystallisation transition. A recent ground state study (zero temperature) [BPT] for multi-body interactions of Wasserstein types shows optimality of the triangular lattice. It seems to be promising to analyse this model for non-zero temperature in terms of crystallisation transition. We will investigate this further.

Another direction for the class of 'ferromagnetic' models with geometry-dependent interaction is to analyse closer the percolation phenomena, and in particular to study the conformal invariance of crossing probabilities as done for the Voronoi percolation ([BS98, BR06], and [T14]).

Last but not least it is interesting to study the higher dimensional cases in \mathbb{R}^d , $d \geq 3$, as well (see [GH96]). We believe that our results still hold for these cases but there are some technical issues related to the general quadratic position such that we defer that analysis for future study.

The rest of the paper is organised as follows. In Section 2.1 we define the Delaunay random-cluster measure for edge configurations, and in Section 2.2. The main novelty is the extensive proof of the uniform bound on the number of connected components using purely geometric properties in Section 3. Finally, in Section 4 we give details of our remaining proofs.

2. THE RANDOM CLUSTER METHOD

In this Section the main method for proving existence of multiple Gibbs measures is outlined. In Section 2.1 we introduce the Delaunay random cluster model and show percolation for this model in Section 2.2. We conclude in Section 2.3 with our proof of Theorem 1.4. The key step of the proof of percolation for the Delaunay random cluster model is the novel uniform estimate of the number of connected components in Section 3. The uniform bound on the number of connected components uses solely geometric arguments and constitutes a major part of this work.

2.1. Delaunay Random Cluster measure. For $\Lambda \Subset \mathbb{R}^2$ and parameters z and ϕ_β we define a joint distribution of the Delaunay Potts model and an edge (tile) process which we call Delaunay random-cluster model. The basic idea is to introduce random edges between points in the plane. Let

$$E_{\mathbb{R}^2} = \{\eta = \{x, y\} \subset \mathbb{R}^2 : x \neq y\}$$

be the set of all possible edges of points in \mathbb{R}^2 , likewise, let E_Λ be the set of all edges in Λ and E_ζ for the set of edges in $\zeta \in \Omega_{\Lambda, \omega}$. We identify ω with $\omega^{(1)}$ and $\boldsymbol{\omega} = (\omega^{(1)}, \emptyset, \dots, \emptyset)$. This allows only monochromatic boundary conditions whereas the general version involves the so-called Edwards-Sokal coupling (see [GHM] for lattice Potts models). We restrict ourselves to the former case for ease of notation. We write

$$\mathcal{E} = \{E \subset E_{\mathbb{R}^2} : E \text{ locally finite}\}$$

for the set of all locally finite edge configurations.

The joint distribution is built from the following three components.

The *point distribution* is given by the Poisson process Π^{zq} for any admissible boundary condition $\omega \in \Omega_\Lambda^*$ and activity zq .

The *type picking mechanism* for a given configuration $\zeta \in \Omega_{\Lambda, \omega}$ is the distribution $\lambda_{\zeta, \Lambda}$ of the mark vector $\sigma \in E^\zeta$. Here $(\sigma(x))_{x \in \zeta}$ are independent and uniformly distributed random variables on M with $\sigma(x) = 1$ for all $x \in \zeta_{\Lambda^c} = \omega$. The latter condition ensures that all points outside of Λ carry the given fixed mark.

The *tile drawing mechanism*. Given a point configuration $\zeta \in \Omega_{\Lambda, \omega}$, we let $\mu_{\zeta, \Lambda}$ be the distribution of the random edge configuration $\{\eta \in E_\zeta : v(\eta) = 1\} \in \mathcal{E}$ with the edge configuration $v \in \{0, 1\}^{E_\zeta}$ having probability

$$\prod_{\eta \in E_\zeta} p(\eta)^{v(\eta)} (1 - p(\eta))^{1-v(\eta)}$$

with

$$p(\eta) := \mathbb{P}(v(\eta) = 1) = \begin{cases} (1 - e^{-\phi(\eta)}) \mathbb{1}_{\text{Del}_2(\zeta)}(\eta) & \text{if } \eta \in E_{\mathbb{R}^2} \setminus E_{\Lambda^c}, \\ \mathbb{1}_{\text{Del}_2(\zeta)}(\eta) & \text{if } \eta \in E_{\Lambda^c}. \end{cases} \quad (2.1)$$

The measure $\mu_{\zeta, \Lambda}$ is a point process on $E_{\mathbb{R}^2}$. Note that $\zeta \rightarrow \lambda_{\zeta, \Lambda}$ and $\zeta \rightarrow \mu_{\zeta, \Lambda}$ are probability kernels (see [Eye14]). Let the measure

$$P_{\Lambda, \omega}^{zq}(\mathrm{d}\zeta, \mathrm{d}E) = \frac{1}{Z_{\Lambda}(\omega)} P_{\Lambda, \omega}^{zq}(\mathrm{d}\zeta) \lambda_{\zeta, \Lambda}(\mathrm{d}\zeta) \mu_{\zeta, \Lambda}(\mathrm{d}E)$$

be supported on the set of all (ζ, E) with $\zeta \in \Omega_{\Lambda, \omega}$ and $E \subset E_{\zeta}$. We shall condition on the event that the marks of the points are constant on each connected component in the graph $(\zeta, E \cap E_{\zeta})$. Two distinct vertices x and y are adjacent to one another if there exists $\eta \in E_{\zeta}$ such that $\{x, y\} = \eta$. A path in the graph $(\zeta, E \cap E_{\zeta})$ is an alternating sequence $v_1, e_1, v_2, e_2, \dots$ of distinct vertices v_i and edges e_j such that $\{v_i, v_{i+1}\} = e_i$ for all $i \geq 1$. We write

$$A = \{(\zeta, E) \in \Omega \times \mathcal{E} : \sum_{\eta \in E} (1 - \delta_{\sigma}(\eta)) = 0\}$$

for the set of marked point configurations such that all vertices of the edges carry the same mark. The set A is measurable which one can see from writing the condition in the following way

$$\sum_{\eta = \{x, y\} \in E} \sum_{i=1}^q (\mathbb{1}_{\zeta^{(i)}}(x)(1 - \mathbb{1}_{\zeta^{(i)}}(y))) = 0$$

and using the fact that $(\zeta, x) \mapsto \mathbb{1}_{\zeta^{(i)}}(x), i = 1, \dots, q$, are measurable (see [GH96, Chapter 2]). Furthermore, $\Pi_{\Lambda, \omega}^{zq}(A) > 0$, which follows easily observing $\Pi_{\Lambda, \omega}^{zq}(A) \geq \Pi_{\Lambda, \omega}^{zq}(\{\tilde{\omega}\}) = e^{-zq|\Lambda|}/Z_{\Lambda}(\omega)$ where $\tilde{\omega}$ is the configuration which equals ω outside of Λ and which is empty inside Λ . Henceforth, the random-cluster measure

$$\mathbf{P} = P_{\Lambda, \omega}^{zq}(\cdot | A)$$

is well-defined. As in [GH96] we obtain the following two measures from the random-cluster measure \mathbf{P} , namely if we disregard the edges we obtain the Delaunay Gibbs distribution $\gamma_{\Lambda, \omega}$ in (1.5) (see [Eye14]). For the second measure consider the mapping $\text{sp}: (\zeta, E) \rightarrow (\zeta, E)$ from $\Omega \times \mathcal{E}$ onto $\Omega \times \mathcal{E}$ where $\zeta \mapsto \zeta = \cup_{i=1}^q \zeta^{(i)}$. For each (ζ, T) with $E \subset E_{\zeta}$ we let $K(\zeta, E)$ denote the number of connected components in the graph (ζ, E) . The Delaunay random-cluster distribution on $\Omega \times \mathcal{E}$ is defined by

$$C_{\Lambda, \omega}(\mathrm{d}\zeta, \mathrm{d}E) = \frac{1}{Z_{\Lambda}(\omega)} q^{K(\zeta, E)} \Pi_{\Lambda, \omega}^z(\mathrm{d}\zeta) \mu_{\zeta, \Lambda}(\mathrm{d}E), \quad (2.2)$$

where $\Pi_{\Lambda, \omega}^z$ is the Poisson process with activity z replacing zq and where

$$Z_{\Lambda}(\omega) = \int_{\Omega_{\Lambda, \omega}} \int_{\mathcal{T}} q^{K(\zeta, E)} \Pi_{\Lambda, \omega}^z(\mathrm{d}\zeta) \mu_{\zeta, \Lambda}(\mathrm{d}E)$$

is the normalisation. It is straightforward to show that $\mathbf{P} \circ \text{sp}^{-1} = C_{\Lambda, \omega}$ (see [Eye14]).

For our main proofs we need to investigate the geometry of the Delaunay triangulation, and in particular what happens to it when we augment $\zeta \in \Omega_{\Lambda, \omega}$ with a new point $x_0 \notin \zeta$. Some hyperedges may be destroyed, some are created, and some remain. This process is well described in [Lis94]. We give a brief account here for the convenience of the reader. We insert

the point x_0 into one of the triangles τ in $\text{Del}_3(\zeta)$. We then create three new edges that join x_0 to each of the three vertices of τ . This creates three new triangles, and destroys one. We now need to verify that the new triangles each satisfy the Delaunay condition (1.1), that is, that their circumscribing balls contain no points of ζ . If this condition is satisfied the new triangle remains, if it is not satisfied, then there is a point $x_1 \in \zeta$ inside the circumscribing ball. We remove the edge not connected to x_0 , and replace it by an edge connecting x_0 and x_1 . This results in the creation of two new triangles. Each of these triangles must be checked as above and the process continues. Once all triangles satisfy the Delaunay condition, we arrive at the Delaunay triangulation $\text{Del}_3(\zeta \cup \{x_0\})$ and their edge part $\text{Del}_2(\zeta \cup \{x_0\})$. Let

$$\begin{aligned} E_{x_0, \zeta}^{(\text{ext})} &= \text{Del}_2(\zeta) \cap \text{Del}_2(\zeta \cup \{x_0\}), \\ E_{x_0, \zeta}^{(+)} &= \text{Del}_2(\zeta \cup \{x_0\}) \setminus \text{Del}_2(\zeta) = \text{Del}_2(\zeta \cup \{x_0\}) \setminus E_{x_0, \zeta}^{(\text{ext})}, \\ E_{x_0, \zeta}^{(-)} &= \text{Del}_2(\zeta) \setminus \text{Del}_2(\zeta \cup \{x_0\}) = \text{Del}_2(\zeta) \setminus E_{x_0, \zeta}^{(\text{ext})}, \end{aligned} \quad (2.3)$$

be the set of exterior, created, and destroyed Delaunay edges respectively, see figure 1.

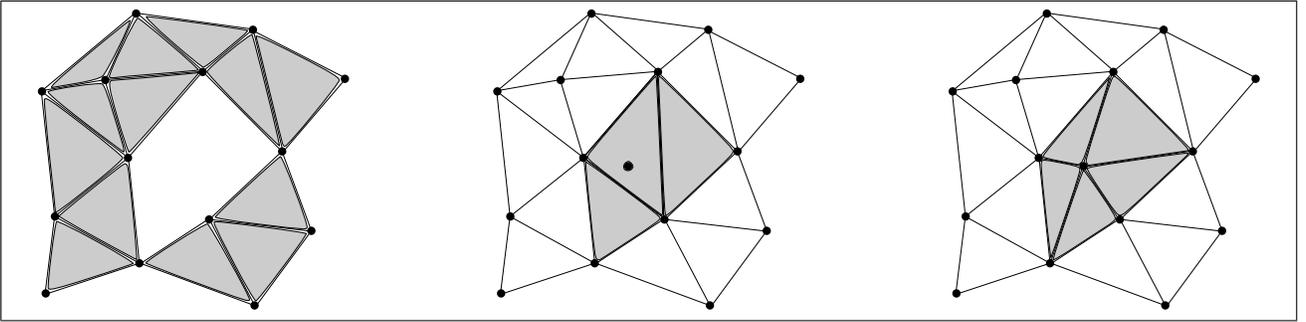


Figure 1: The Delaunay edge sets $E_{x_0, \zeta}^{(\text{ext})}$, $E_{x_0, \zeta}^{(+)}$, and $E_{x_0, \zeta}^{(-)}$ from the left to right

Note that any new triangle must contain x_0 , i.e.,

$$\{\tau \in \text{Del}_3(\zeta \cup \{x_0\}) : \tau \cap x_0 = x_0\}.$$

We let $\mu_{x_0, \zeta}^{(-)}$, $\mu_{x_0, \zeta}^{(+)}$, and $\mu_{x_0, \zeta}^{(\text{ext})}$ be the edge drawing mechanisms on $E_{x_0, \zeta}^{(\text{ext})}$, $E_{x_0, \zeta}^{(+)}$, and $E_{x_0, \zeta}^{(-)}$, respectively, which are derived from the edge drawing measure $\mu_{\zeta, \Lambda}$ above. To add to the above sets of Delaunay edges, we also define the neighbourhood of a point x_0 . For any $\zeta \in \Omega_{\Lambda, \omega}$ the neighbourhood graph of a point $x_0 \in \Lambda, x_0 \notin \zeta$, is the following random graph $G_{x_0, \zeta} = (V_{x_0, \zeta}, E_{x_0, \zeta}^{(\text{nbd})})$ where $V_{x_0, \zeta}$ is the set of points that share an edge with x_0 in $\text{Del}_2(\omega \cup \{x_0\})$ and $E_{x_0, \zeta}^{(\text{nbd})}$ is the set of edges in $\text{Del}_2(\omega \cup \{x_0\})$ that have both endpoints in $V_{x_0, \zeta}$, more precisely.

$$V_{x_0, \zeta} = \{x \in \zeta : \eta_{x, x_0} \in E_{x_0, \zeta}^{(+)}\} \quad \text{and} \quad E_{x_0, \zeta}^{(\text{nbd})} = \{\eta_{x, y} \in E_{x_0, \zeta}^{(\text{ext})} : x, y \in V_{x_0, \zeta}\}.$$

The graph $G_{x_0, \zeta}$ splits the plane into two regions, The region containing x_0 we call the neighbourhood of x_0 and $G_{x_0, \zeta}$ is called the boundary of the neighbourhood of x_0 . Having the edge drawing mechanism $\mu_{x_0, \zeta}^{(\text{ext})}$ define

$$\mu_{(\text{ext}), \zeta}^{(q)}(\text{d}E) := \frac{q^{K(\zeta, E)} \mu_{x_0, \zeta}^{(\text{ext})}(\text{d}E)}{\int q^{K(\zeta, E)} \mu_{x_0, \zeta}^{(\text{ext})}(\text{d}E)}. \quad (2.4)$$

The main task is the find an uniform upper bound (independent of ζ) for the expected number of connected components of (ζ, E) that intersect the boundary of the neighbourhood $G_{x_0, \zeta}$, where E is sampled from $\mu_{(\text{ext}), \zeta}^{(q)}$. This bound will enable us to bound certain conditional

Papangelou intensities from below and above enabling us to work with no background potential as in recent work ([BBD04, AE16]). Given $\zeta \in \Omega_{\Lambda, \omega}$ and $x_0 \in \Lambda, x_0 \notin \zeta$ and the Delaunay graph $\text{Del}_2(\zeta)$ we denote $N_{x_0}^{(\text{cc})}$ the number of connected components that intersect $G_{x_0, \zeta}$.

Theorem 2.1 (Number of connected components). *For all $\Lambda \in \mathbb{R}^2$ there exists $0 < \alpha < \infty$ such that*

$$\int N_{x_0}^{(\text{cc})}(\zeta, E) \mu_{(\text{ext}), \zeta}^{(q)}(dE) \leq \alpha$$

for all $\omega \in \Omega$ and $\zeta \in \Omega_{\Lambda, \omega}$ and for all $x_0 \in \Lambda$ with $x_0 \notin \zeta$.

We need to study the change of $K(\zeta, E), E \subset E_\zeta \subset \mathcal{E}$ when adding the point $x_0 \notin \zeta$. Adding a point x_0 to ζ without considering the change to E will always increase the number of connected components by one. On the other hand, the augmentation of a single triangle τ to E can result in the connection of a maximum of three different connected components, leaving one. Therefore,

$$\begin{aligned} K(\zeta \cup \{x_0\}, E) - K(\zeta, E) &= 1, \\ -2 \leq K(\zeta, E \cup \tau) - K(\zeta, E) &\leq 0. \end{aligned} \tag{2.5}$$

2.2. Edge-Percolation. We establish the existence of edge percolation for the Delaunay random-cluster measure $C_{\Lambda, \omega}$ when z and the parameter β are sufficiently large. Note that for any $\Delta \in \mathbb{R}^2$ we write

$$N_{\Delta \leftrightarrow \infty}(\zeta, E) = \#\{x \in \zeta \cap \Delta : x \text{ belongs to an } \infty - \text{cluster of } (\zeta, E \cap E_\zeta)\}.$$

The key step in our results is the following percolation result.

Proposition 2.2. *Suppose all the assumptions hold and that z and β are sufficiently large. Suppose that Λ is a finite union of cells $\Delta_{k,l}$ defined in (A.1) in Appendix A. Then there exists $\varepsilon > 0$ such that*

$$\int N_{\Delta \leftrightarrow \infty} dC_{\Lambda, \omega} \geq \varepsilon$$

for any cell $\Delta = \Delta_{k,l}$, any finite union Λ of cells and any admissible pseudo-periodic boundary condition $\omega \in \Omega_\Lambda^*$.

Proof of Proposition 2.2: We split the proof in several steps and Lemmata below. Our strategy to establish percolation in the Delaunay random-cluster model is to compare it to mixed site-bond percolation on \mathbb{Z}^2 (see appendix C on site-bond percolation). We extend here the strategy in [GH96] to our case in the following steps similar to [AE16]. First we employ a coarse-graining strategy to relate each site $(k, l) \in \mathbb{Z}^2$ to a cell which is a union of parallelotopes. The second step is to consider the links (bonds) of two good cells. In order to establish mixed site-bond percolation we need to define when cells are good (open) and when two neighbouring cells are linked once they are open which happens when the corresponding link (bond) is open as well. This link establishes then an open connection in our Delaunay graph structures $\text{Del}_2(\zeta)$. In order to apply mixed site-bond percolation we need to substantially adapt the coarse graining method used in [AE16].

Step 1: Coarse graining.

Let $\Lambda = \Lambda_n \subset \mathbb{R}^2$ be the paralleloptope given as the finite union of cells (A.1) with side length ℓ , i.e.,

$$\Lambda_n = \bigcup_{(k,l) \in \{-n, \dots, n\}^2} \Delta_{k,l} \quad \text{and} \quad \Delta_{k,l} = \bigcup_{i,j=0}^7 \Delta_{k,l}^{i,j},$$

where $\Delta_{k,l}^{i,j}$ are paralleloptopes with side length $\ell/8$ and each paralleloptope $\Delta_{k,l}$ has side length ℓ where the coordinate systems is the canonical one, that is $\Delta_{k,l}^{0,0}$ is the paralleloptope in the bottom right corner. The union of the 16 smaller paralleloptopes towards the centre of $\Delta_{k,l}$ is denoted

$$\Delta_{k,l}^- = \bigcup_{i,j=2}^5 \Delta_{k,l}^{i,j},$$

see Figure 2 These sites will act as the sites in the mixed site-bond percolation model on \mathbb{Z}^2 . Finally, we define the link boxes between $\Delta_{k,l}^-$ and $\Delta_{k+1,l}^-$ as

$$\Delta_{(\text{link})}^{k:k+1,l} = \left\{ \bigcup_{j=0}^3 \Delta_{k,l}^{6,j+2} \right\} \cup \left\{ \bigcup_{j=0}^3 \Delta_{k,l}^{7,j+2} \right\} \cup \left\{ \bigcup_{j=0}^3 \Delta_{k+1,l}^{0,j+2} \right\} \cup \left\{ \bigcup_{j=0}^3 \Delta_{k+1,l}^{1,j+2} \right\} \quad (2.6)$$

which act as the bonds in the mixed site-bond percolation model on \mathbb{Z}^2 , see Figure. This completes the coarse grain procedure. When we establish percolation in the mixed site-bond percolation model on \mathbb{Z}^2 , i.e., the existence of an infinite chain of open sites and open bonds, we would like to relate it to the existence of an infinite connected component of open edges in Del_2 . This infinite connected components will connect with the complement of any finite boxes, and thus this connected components corresponds to an infinite connected component of edges where all the sites carry the mark agreed for the boundary condition, that is, an infinite connected component in an appropriate Delaunay site percolation model $\tilde{C}_{\Lambda,\zeta}^{(\text{site})}$ defined in (2.17) below. To do this, we define $CB_{k:k+1,l}$ to be the straight line segment between the centres of the paralleloptopes $\Delta_{k,l}$ and $\Delta_{k+1,l}$ and let

$$H_{k:k+1,l}(\zeta) = \{x \in \zeta : \text{Vor}_\zeta(x) \cap CB_{k:k+1,l} \neq \emptyset\} \quad (2.7)$$

be the subset of points of the configuration ζ , whose Voronoi cells intersect the line segment $CB_{k:k+1,l}$, see Figures 2 and 3 .

We need to consider the distribution of the points given by the marginal distribution

$$M_{\Lambda,\omega} = C_{\Lambda,\omega}(\cdot \times \mathcal{E})$$

of the Delaunay random-cluster measure on Ω . Note that (2.2) can be written as

$$C_{\Lambda,\omega}(d\zeta, dE) = M_{\Lambda,\omega}(d\zeta) \mu_{\zeta,\Lambda}^{(q)}(dE), \quad \mu_{\zeta,\Lambda}^{(q)}(dE) = \frac{q^{K(\zeta,E)} \mu_{\zeta,\Lambda}(dE)}{\int q^{K(\zeta,E)} \mu_{\zeta,\Lambda}(dE)}.$$

We define h_Λ to be the Radon-Nikodym density of $M_{\Lambda,\omega}$ with respect to $\Pi_{\Lambda,\omega}^z$, i.e., for $\zeta \in \Omega_{\Lambda,\omega}$,

$$h_\Lambda(\zeta) := \mathcal{Z}_\Lambda(\omega)^{-1} \int q^{K(\zeta,E)} \mu_{\zeta,\Lambda}(dE).$$

In the following lemma we derive a bound for the Papangelou conditional intensity of $M_{\Lambda,\zeta}$.

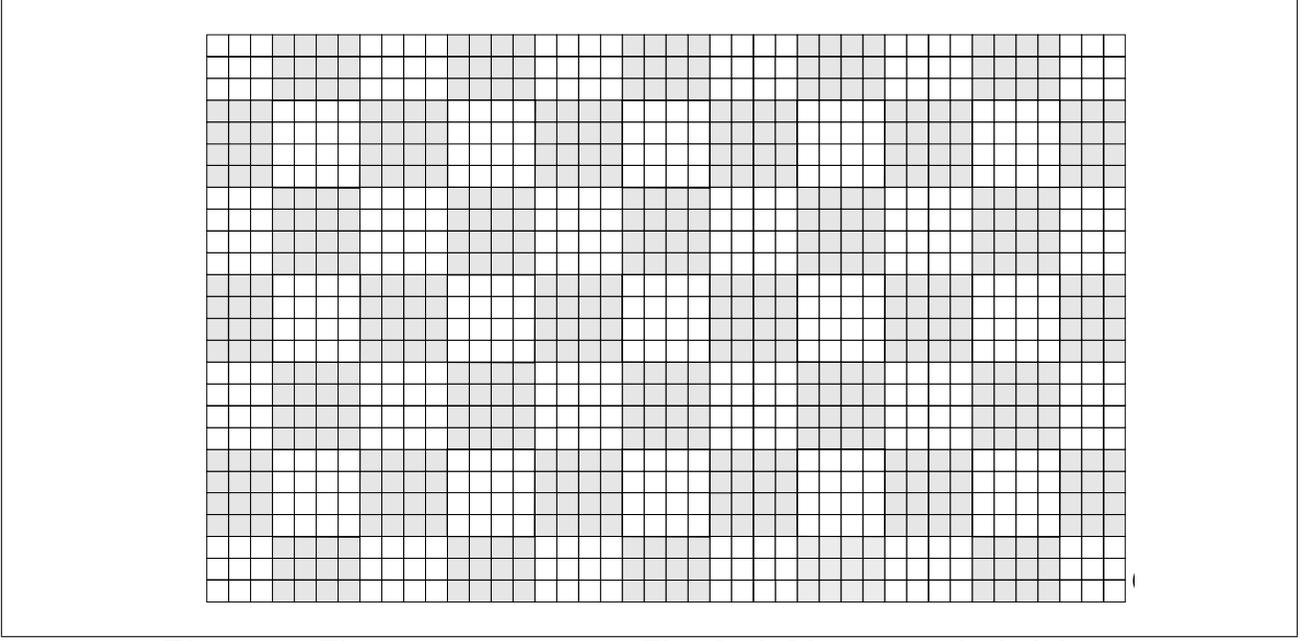


Figure 2: The l -partitioning of Λ . The shaded boxes are the link boxes

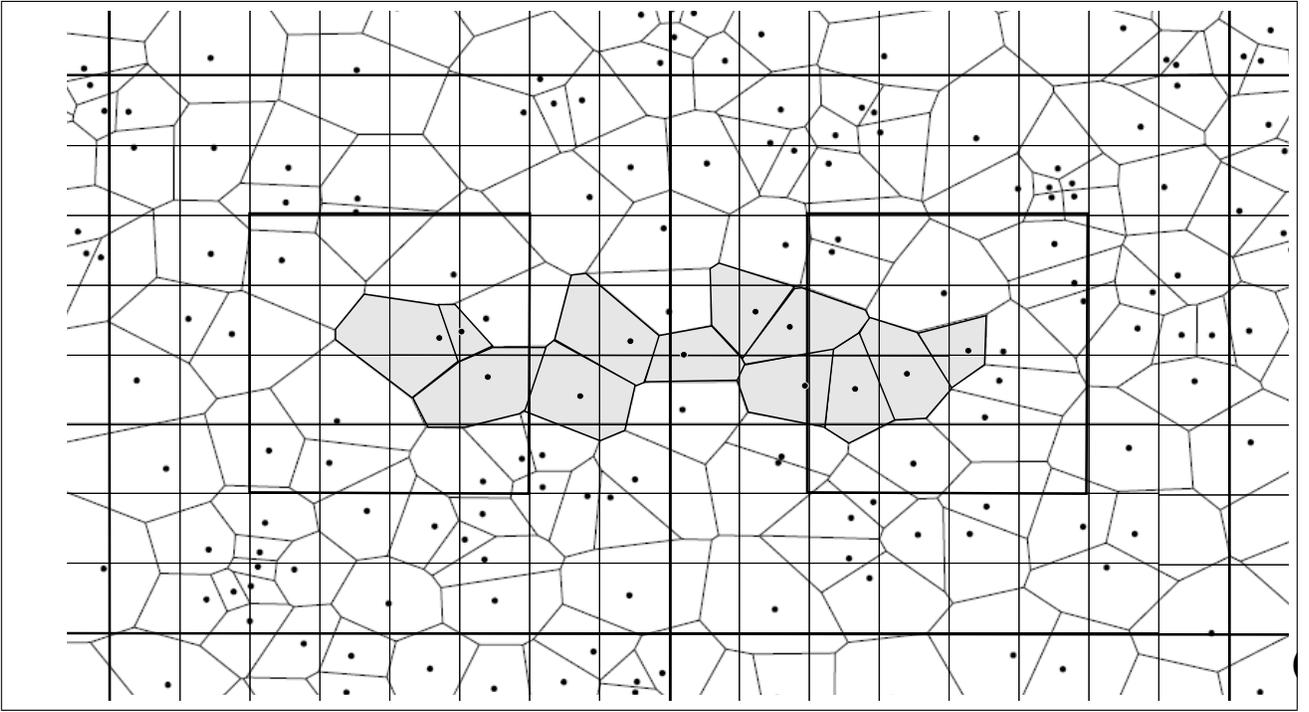


Figure 3: The shaded area is the union of the Voronoi cells with centre $\in H_{k:k+1,l}(\zeta)$

Lemma 2.3. For any $\Lambda \in \mathbb{R}^2$ and any admissible boundary $\omega \in \Omega_\Lambda^*$ and $M_{\Lambda,\zeta}$ -almost all $\zeta \in \Omega_{\Lambda,\omega}$ and a point $x_0 \in \Lambda$ with $x_0 \notin \zeta$,

$$\frac{h_\Lambda(\zeta \cup \{x_0\})}{h_\Lambda(\zeta)} \geq q^{1-\alpha}, \quad (2.8)$$

where $\alpha \in (0, \infty)$ is given in Theorem 2.1.

PROOF OF LEMMA 2.3. Recall the different edge drawing mechanisms $\mu_{x_0, \zeta}^{(-)}$, $\mu_{x_0, \zeta}^{(+)}$, and $\mu_{x_0, \zeta}^{(\text{ext})}$ on $E_{x_0, \zeta}^{(\text{ext})}$, $E_{x_0, \zeta}^{(+)}$, and $E_{x_0, \zeta}^{(-)}$, respectively, and the definition of the probability measure $\mu_{(\text{ext}), \zeta}^{(q)}$ in (2.4). It follows that

$$\begin{aligned} \frac{h_\Lambda(\zeta \cup \{x_0\})}{h_\Lambda(\zeta)} &= \frac{\int q^{K(\zeta, E_2)} \int q^{K(\zeta \cup \{x_0\}, E_1 \cup E_2) - K(\zeta, E_2)} \mu_{x_0, \zeta}^{(+)}(dE_1) \mu_{x_0, \zeta}^{(\text{ext})}(dE_2)}{\int q^{K(\zeta, E_3)} \int q^{K(\zeta, E_3 \cup E_4) - K(\zeta, E_3)} \mu_{x_0, \zeta}^{(-)}(dE_4) \mu_{x_0, \zeta}^{(\text{ext})}(dE_3)} \\ &\geq \iint q^{K(\zeta \cup \{x_0\}, E_1 \cup E_2) - K(\zeta, E_2)} \mu_{x_0, \zeta}^{(+)}(dE_1) \mu_{(\text{ext}), \zeta}^{(q)}(dE_2) \\ &\geq q^{\iint (K(\zeta \cup \{x_0\}, E_1 \cup E_2) - K(\zeta, E_2))} \mu_{x_0, \zeta}^{(+)}(dE_1) \mu_{(\text{ext}), \zeta}^{(q)}(dE_2), \end{aligned}$$

since

$$K(\zeta, E_3 \cup E_4) - K(\zeta, E_3) \leq 0$$

due to the fact that adding any edge from $E_4 \subset E_{x_0, \zeta}^{(-)}$ will only fuse connected components of the remaining graph. The second inequality is just Jensen's inequality applied to the convex function $x \mapsto q^x$. Note that new edges from $E_{x_0, \zeta}^{(+)}$, made by the insertion of x_0 to the configuration ζ , are edges connecting x_0 to points in ζ and are open with respect to $\mu_{x_0, \zeta}^{(+)}$, and therefore

$$K(\zeta \cup \{x_0\}, E_1 \cup E_2) - K(\zeta, E_2) \geq -N_{x_0}^{(\text{cc})}(\zeta, E_2) + 1 \quad (2.9)$$

for any $E_1 \subset E_{x_0, \zeta}^{(+)}$ and any $E_2 \subset E_{x_0, \zeta}^{(\text{ext})}$, and thus by Theorem 2.1 we conclude with the statement. \square

An important component of our coarse graining method is to estimate the conditional probability that at least one point lies inside some $\Delta \subset \Lambda$. In the following we denote by $M_{\Lambda, \Delta, \omega'}$ the conditional distribution (relative to $M_{\Lambda, \omega}$) in Δ given that the configuration in $\Lambda \setminus \Delta$ is equal to ϕ' with $\omega' \cap \Lambda^c = \omega$. The details of the construction of the regular conditional probability distribution can be found in [Eye14] or Appendix C. Having a uniform lower bound for the quotient $h_\Lambda(\zeta \cup \{x_0\})/h_\Lambda(\zeta)$ allows to exhibit some control over the distribution $M_{\Lambda, \nabla, \omega'}$ for any $\nabla \subset \Lambda$. In the following we write ∇ for any cell $\Delta_{k, l}^{i, j}$, $i, j = 0, \dots, 7$. We fix

$$\varepsilon = \frac{1 - \sqrt{p_c^{(\text{site})}(\mathbb{Z}^2)}}{4}, \quad (2.10)$$

where $p_c^{(\text{site})}(\mathbb{Z}^2) \in (0, 1)$ is the critical probability for site percolation on \mathbb{Z}^2 .

Lemma 2.4. *For all $z > 0$ there exist $\ell > 0$ such that for all admissible pseudo-periodic boundary conditions $\omega \in \Omega_\Lambda^*$,*

$$M_{\Lambda, \nabla, \omega'}(\#\{\zeta \cap \nabla\} \geq 1) > 1 - \frac{\varepsilon}{64}$$

for all cells $\nabla = \Delta_{k, l}^{i, j}$, $(k, l) \in \{-n, \dots, n\}^2$, $i, j = 0, \dots, 7$, and for any admissible boundary condition $\omega' \in \Omega_\nabla^*$ with $\omega' \setminus \Lambda = \omega$.

PROOF. The statement follows immediately from

$$\frac{M_{\Lambda, \nabla, \omega'}(\#\{\zeta \cap \nabla\} = 1)}{M_{\Lambda, \nabla, \omega'}(\#\{\zeta \cap \nabla\} = 0)} = z \int_\nabla \frac{h_\Lambda(\zeta \cup \{x_0\})}{h_\Lambda(\zeta)} dx \geq z q^{-\alpha} z |\nabla|,$$

where the inequality follows from Lemma 2.3. It follows that

$$M_{\Lambda, \nabla, \omega'}(\#\{\zeta \cap \nabla\} = 0) \leq q^\alpha (z |\nabla|)^{-1},$$

and hence

$$M_{\Lambda, \nabla, \omega'}(\#\{\zeta \cap \nabla\} \geq 1) \geq 1 - q^\alpha(z|\nabla|)^{-1} = 1 - q^\alpha(z\sqrt{3}/2\ell^2)^{-1} > 1 - \frac{\varepsilon}{64}$$

for $\ell \geq \ell_0(z, q, \alpha) > 0$. \square

Remark 2.5. *Note that one can fix the length scale for the coarse graining partitions in Lemma 2.4. Then the same results holds for all $z > z_0(\ell, q, \alpha)$.* \diamond

Let $\Delta_{k,l}$ be an element of of partitioning of Λ and define $F_{k,l}^{(\text{ext})}$ to be the event that each of the smaller boxes $\Delta_{k,l}^{i,j} \subset \Delta_{k,l}$ that are not in the centre region $\Delta_{k,l}^{(-)}$, contain at least one point. We call the elements in this event 'well-behaved' configurations,

$$F_{k,l}^{(\text{ext})} = \bigcap_{\substack{i,j \in \{0, \dots, 7\}: \\ \Delta_{k,l}^{i,j} \not\subset \Delta_{k,l}^{(-)}}} \{\zeta \in \Omega_{\Lambda, \omega} : \#\{\zeta \cap \Delta_{k,l}^{i,j}\} \geq 1\}. \quad (2.11)$$

Lemma 2.6. *For any $\Lambda \in \mathbb{R}^2$ and any admissible boundary $\omega \in \Omega_\Lambda^*$ and ℓ -partitioning of Λ and configuration $\zeta \in F_{k,l}^{(\text{ext})}$ and a point $x_0 \in \Delta_{k,l}^{(-)}$ with $x_0 \notin \zeta$ for any $k, l \in \{-n, \dots, n\}$,*

$$\frac{h_\Lambda(\zeta \cup \{x_0\})}{h_\Lambda(\zeta)} \leq q^\alpha, \quad (2.12)$$

where $\alpha \in (0, \infty)$ is given in Theorem 2.1.

PROOF OF LEMMA 2.6. Then, adapting similar steps in Lemma 2.3, we see that

$$\begin{aligned} \frac{h_\Lambda(\zeta \cup \{x_0\})}{h_\Lambda(\zeta)} &= \frac{\int q^{K(\zeta, E_2)} \int q^{K(\zeta \cup \{x_0\}, E_1 \cup E_2) - K(\zeta, E_2)} \mu_{x_0, \zeta}^{(+)}(dE_1) \mu_{x_0, \zeta}^{(\text{ext})}(dE_2)}{\int q^{K(\zeta, E_3)} \int q^{K(\zeta, E_3 \cup E_4) - K(\zeta, E_3)} \mu_{x_0, \zeta}^{(-)}(dE_4) \mu_{x_0, \zeta}^{(\text{ext})}(dE_3)} \\ &\leq \frac{q \int q^{K(\zeta, E_2)} \mu_{x_0, \zeta}^{(\text{ext})}(dE_2)}{\int q^{K(\zeta, E_3)} \int q^{K(\zeta, E_3 \cup E_4) - K(\zeta, E_3)} \mu_{x_0, \zeta}^{(-)}(dE_3) \mu_{x_0, \zeta}^{(\text{ext})}(dE_4)} \\ &= q \left(\iint q^{K(\zeta, E_3 \cup E_4) - K(\zeta, E_4)} \mu_{x_0, \zeta}^{(-)}(dE_3) \mu_{(\text{ext}), \zeta}^{(q)}(dE_4) \right)^{-1}, \end{aligned}$$

where we used the inequality

$$K(\zeta \cup \{x_0\}, E_1 \cup E_2) - K(\zeta, E_2) \leq 1, \quad E_1 \subset E_{x_0, \zeta}^{(+)}, E_2 \subset E_{x_0, \zeta}^{(\text{ext})}, \quad (2.13)$$

as all new edges are connected to x_0 and thus can at most built one additional open component. We can apply Jensen's inequality to the inttegral in the denominator to obtain the upper bound

$$\frac{h_\Lambda(\zeta \cup \{x_0\})}{h_\Lambda(\zeta)} \leq q \left(q^{\iint (K(\zeta, E_3 \cup E_4) - K(\zeta, E_4))} \mu_{x_0, \zeta}^{(-)}(dE_3) \mu_{(\text{ext}), \zeta}^{(q)}(dE_4) \right)^{-1}. \quad (2.14)$$

For all cinfigurations $\zeta \in F_{k,l}^{(\text{ext})}$ and $x_0 \in \Delta_{k,l}^{(-)}$ with $x_0 \notin \zeta$ and all $x \in V_{x_0, \zeta}$ we have that $|x - x_0| < 4\sqrt{2}\ell$. This ensures that

$$V_{x_0, \zeta} \subset B_{4\sqrt{2}\ell}(x_0) \subset \Lambda,$$

where $B_{4\sqrt{2}\ell}(x_0)$ is the ball of radius $4\sqrt{2}\ell$ around x_0 . Therefore, since $x, y \in V_{x_0, \zeta}$ for all $\eta_{x,y} \in E_{x_0, \zeta}^{(-)}$, it follows that adding edges in $E_{x_0, \zeta}^{(-)}$ can only fuse together two connected components (reducing the number of connected components by one) if they each intersect $V_{x_0, \zeta}$. Hence,

$$K(\zeta, E_3 \cup E_4) - K(\zeta, E_4) \geq -N_{x_0}^{(\text{cc})}(E_4) + 1, E_3 \subset E_{x_0, \zeta}^{(-)},$$

and thus by Theorem 2.1 we conclude with the statement. \square

We can now find a lower for the conditional probability, given well-behaved configurations $\zeta \in F_{k,l}^{(\text{ext})}$, that $\Delta_{k,l}^{(-)}$ contains no more that $m = m(z) \in \mathbb{N}$ points of the well-behaved configurations ζ .

Lemma 2.7. *Given any ℓ -partitioning of Λ with admissible boundary condition $\omega \in \Omega$ and admissible boundary condition $\zeta' \in \Omega$ for $\Delta_{k,l}^{(-)}$ with $\zeta' \cap \Lambda^c = \omega$ and $\zeta' \in F_{k,l}^{(\text{ext})}$ for any $k, l \in \{-n, \dots, n\}$, the estimate*

$$M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(\#\{\zeta \cap \Delta_{k,l}^{(-)}\} \leq \lfloor m(z) \rfloor) > 1 - \varepsilon$$

holds for $m(z) = 2\varepsilon^{-1}q^{\alpha+1}|\Delta_{k,l}^{(-)}|z$.

PROOF. Note that we can write $M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(d\zeta)$ as

$$M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(d\zeta) = \frac{1}{Z_{\Lambda, \Delta_{k,l}^{(-)}}(\zeta')} h_{\Lambda}(\zeta \cup \zeta') \mathbb{P}_{\Delta_{k,l}^{(-)}}^z(d\zeta),$$

where $Z_{\Lambda, \Delta_{k,l}^{(-)}}(\zeta')$ is the normalisation. Using the well-known fact

$$\int f(\zeta) \Pi_{\Delta_{k,l}^{(-)}}^z(d\zeta) = e^{-z|\Delta_{k,l}^{(-)}|} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Delta_{k,l}^{(-)}} f(\{x_1, \dots, x_n\}) dx_1 \cdots dx_n$$

for any observable f of the underlying Poisson process and writing $N(\zeta) = \#\{\zeta \cap \Delta_{k,l}^{(-)}\}$ we obtain, setting $Z' = e^{-z|\Delta_{k,l}^{(-)}|}/Z_{\Lambda, \Delta_{k,l}^{(-)}}(\zeta')$ for brevity, the probability for any $n \in \mathbb{N}$,

$$\begin{aligned} M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(N = n + 1) &= \frac{z^{n+1}}{(n+1)!} Z' \int_{(\Delta_{k,l}^{(-)})^{n+1}} h_{\Lambda}(\{x_1, \dots, x_n\} \cup \zeta') dx_1 \cdots dx_{n+1} \\ &= \frac{z^n}{n!} \left(\frac{z}{n+1}\right) Z' \int_{(\Delta_{k,l}^{(-)})^n} \int_{\Delta_{k,l}^{(-)}} h_{\Lambda}(\{x_1, \dots, x_n\} \cup \zeta' \cup \{x\}) dx dx_1 \cdots dx_n \\ &= \frac{z^n}{n!} \left(\frac{z}{n+1}\right) Z' \int_{(\Delta_{k,l}^{(-)})^n} h_{\Lambda}(\{x_1, \dots, x_n\} \cup \zeta' \cup \{x\}) g_{\Delta_{k,l}^{(-)}}(y) dx dx_1 \cdots dx_n \end{aligned}$$

where

$$g_{\Delta_{k,l}^{(-)}}(y) = \int_{\Delta_{k,l}^{(-)}} \frac{h_{\Lambda}(y \cup \zeta' \cup \{x\})}{h_{\Lambda}(y \cup \zeta')} dx, \quad y = \{x_1, \dots, x_n\}.$$

We obtain that

$$\begin{aligned} M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(N = n + 1) &= \left(\frac{z}{n+1}\right) \int_{\Omega_{\Delta_{k,l}^{(-)}, \zeta'}} g_{\Delta_{k,l}^{(-)}}(y) \mathbb{1}\{N(\zeta) = n\} M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(d\zeta) = \left(\frac{z}{n+1}\right) \times \\ &\times M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(N = n) \int_{\Omega_{\Delta_{k,l}^{(-)}, \zeta'}} g_{\Delta_{k,l}^{(-)}}(y) M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(d\zeta | N = n). \end{aligned}$$

We can bound the function $g_{\Delta_{k,l}^{(-)}}(\zeta)$ for $\zeta \cup \zeta' \in F_{k,l}^{(\text{ext})}$ by Lemma 2.6 by $q^\alpha |\Delta_{k,l}^{(-)}|$. Therefore,

$$\begin{aligned} \frac{M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(N = n + 1)}{M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(N = n + 1)} &= \left(\frac{z}{n + 1}\right) \int_{\Delta_{k,l}^{(-)}} g_{\Delta_{k,l}^{(-)}}(\zeta) M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(d\zeta | N = n) \\ &= \frac{z}{n + 1} q^\alpha |\Delta_{k,l}^{(-)}| \int_{\Delta_{k,l}^{(-)}} M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(d\zeta | N = n) \leq \frac{q^\alpha |\Delta_{k,l}^{(-)}| z}{n + 1}. \end{aligned}$$

For all $n > m(z)$ we apply the previous step $n - \lfloor m(z) \rfloor$ times to obtain

$$\begin{aligned} M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(N = n) &\leq \frac{q^\alpha |\Delta_{k,l}^{(-)}| z}{n} M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(d\zeta | N = n - 1) \\ &\leq \frac{1}{n!} (q^\alpha |\Delta_{k,l}^{(-)}| z)^{n - \lfloor m(z) \rfloor} \lfloor m(z) \rfloor!. \end{aligned}$$

It follows that

$$\begin{aligned} M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(N > m(z)) &\leq \sum_{n = \lfloor m(z) \rfloor + 1}^{\infty} \frac{1}{n!} (q^\alpha |\Delta_{k,l}^{(-)}| z)^{n - \lfloor m(z) \rfloor} \lfloor m(z) \rfloor! \\ &\leq \sum_{n = \lfloor m(z) \rfloor + 1}^{\infty} \left(\frac{q^\alpha |\Delta_{k,l}^{(-)}| z}{\lfloor m(z) \rfloor}\right)^{n - \lfloor m(z) \rfloor}, \end{aligned}$$

and thus

$$M_{\Lambda, \Delta_{k,l}^{(-)}, \zeta'}(N > m(z)) \leq \sum_{n = \lfloor m(z) \rfloor + 1}^{\infty} \left(\frac{\varepsilon}{n}\right)^{n - \lfloor m(z) \rfloor} = \sum_{n = 1}^{\infty} \left(\frac{\varepsilon}{n}\right)^n.$$

Since $\varepsilon < \frac{1}{2}$, the right hand side of the previous inequality is less than ε . We conclude with the statement. \square

Step 2: Random-cluster measure $\tilde{C}_{\Lambda, \omega}$. We find a measure $\tilde{C}_{\Lambda, \omega}$ which is stochastically smaller than $C_{\Lambda, \omega}$. Then using coarse graining and comparison to site-bond percolation on \mathbb{Z}^2 we establish percolation for $\tilde{C}_{\Lambda, \omega}$. Percolation for $\tilde{C}_{\Lambda, \omega}$ then implies percolation for the original $C_{\Lambda, \omega}$. We base the definition of the measure $\tilde{C}_{\Lambda, \omega}$ on a coarse graining method originally introduced in [Hag00] and then used in [AE16].

For given $\zeta \in \Omega_{\Lambda, \omega}$ let

$$\text{Del}_2^*(\zeta) = \{\eta \in \text{Del}_2(\zeta) : \phi(\ell(\eta)) \geq g(\beta) =: \log(1 + \beta/64\ell^4)\}$$

and let $\tilde{\mu}_{\zeta, \Lambda}$ be the distribution of the random set $\{\eta \in E_\zeta : v(\eta) = 1\}$ with

$$\mathbb{P}(v(\eta) = 1) = \tilde{p} = \frac{1 - \exp\{-g(\beta)\}}{1 + (q - 1) \exp\{-g(\beta)\}} \mathbb{1}_{\text{Del}_2^*(\zeta)}(\eta). \quad (2.15)$$

Note that $\tilde{\mu}_{\zeta, \Lambda}$ depends on Λ only via the configuration $\zeta \in \Omega_{\Lambda, \omega}$. Note also that \tilde{p} is increasing in β , although to reduce excessive notation, we don't explicitly write this. It is easy to show that $\mu_{\zeta, \Lambda} \succcurlyeq \tilde{\mu}_{\zeta}$ by noting that

$$\frac{p(\eta)}{q(1 - p(\eta))} = \frac{1 - \exp(-\phi(\eta))}{q \exp(-\phi(\eta))} \geq \frac{1 - \exp[-g(\beta) \mathbb{1}_{\text{Del}_2^*(\zeta)}(\eta)]}{q \exp[-g(\beta) \mathbb{1}_{\text{Del}_2^*(\zeta)}(\eta)]} = \frac{\tilde{p}(\eta)}{q(1 - \tilde{p}(\eta))}, \quad \eta \in \text{Del}_2(\zeta),$$

and using [Gri94]. Hence, $C_{\Lambda, \omega} \succcurlyeq \tilde{C}_{\Lambda, \omega}$. As site percolation implies hyperedge percolation we consider site percolation given by (2.15), that is, we open vertices in $\text{Del}_1(\zeta)$ independently of

each other with probability \tilde{p} . Formally this is defined as follows. We let $\tilde{\lambda}_{\zeta,\Lambda}$ be the distribution of the random mark vector $\tilde{\sigma} \in E^\zeta$ where $(\tilde{\sigma}_x)_{x \in \zeta}$ are Bernoulli random variables satisfying

$$\begin{aligned}\mathbb{P}(\tilde{\sigma}_x = 1) &= \tilde{p} \mathbb{1}_{\text{Del}_1^*(\zeta)}(x) \\ \mathbb{P}(\tilde{\sigma}_x \neq 1) &= 1 - \tilde{p} \mathbb{1}_{\text{Del}_1^*(\zeta)}(x),\end{aligned}\tag{2.16}$$

where \tilde{p} is given in (2.15) and $\text{Del}_1^*(\zeta)$ is the set of points that build the edges of $\text{Del}_2^*(\zeta)$. Then the site percolation process is defined by the measure

$$\tilde{C}_{\Lambda,\omega}^{(\text{site})}(\text{d}\zeta) = M_{\Lambda,\omega}(\text{d}\zeta) \tilde{\lambda}_{\zeta,\Lambda}(\text{d}\zeta).\tag{2.17}$$

Step 3: Site-bond percolation.

We now establish percolation for the random-cluster measure $\tilde{C}_{\Lambda,\omega}$.

Lemma 2.8. *there is a $c > 0$ such that*

$$\tilde{C}_{\Lambda,\omega}^{(\text{site})}(\{\Delta \longleftrightarrow \Lambda^c\}) \geq c > 0$$

for any $\Delta = \Delta_{k,l} \subset \Lambda$, $(k,l) \in \{-n, \dots, n\}^2$, and any pseudo-periodic admissible boundary condition $\omega \in \Omega_\Lambda^*$.

PROOF OF LEMMA 2.8.

Step (i) Probability that small cells have at least one point: In the following we write $\Delta = \Delta_{k,l}$ and $\nabla = \Delta_{k,l}^{i,j}$ for any $k,l \in \{-n, \dots, n\}$ and for any $i,j \in \{0, 1, \dots, 7\}$. For all configuration $\zeta' \in \Omega_{\Delta^c}$ with $\zeta' \cap \Lambda^c = \omega$ we obtain with Lemma 2.4 the estimate

$$M_{\Lambda,\Delta,\zeta'}(F_{k,l}^{(\text{ext})}) \geq 1 - \sum_{i,j: \Delta_{k,l}^{i,j} \not\subset \Delta_{k,l}^{(-)}} M_{\Lambda,\Delta,\zeta'}(\#\{\zeta \cap \Delta_{k,l}^{i,j}\} = 0) < 1 - \frac{48\varepsilon}{64} = 1 - \frac{3\varepsilon}{4}.\tag{2.18}$$

Define the following two events, first the event

$$G_{k,l} = \{\zeta \in \Omega_{\Lambda,\omega}: \#\{\zeta \cap \Delta_{k,l}^{(-)}\} \leq m(z)\}$$

that there are at most $m(z)$ points in ζ in the centre $\Delta_{k,l}^{(-)}$ of $\Delta_{k,l}$, and the event that all smaller cells in the centre contains at least one point,

$$F_{k,l}^{(-)} = \bigcap_{i,j: \Delta_{k,l}^{i,j} \subset \Delta_{k,l}^{(-)}} \{\zeta \in \Omega_{\Lambda,\omega}: \#\{\zeta \cap \Delta_{k,l}^{i,j}\} \geq 1\}.$$

Both events depend on point configurations in the centre region $\Delta_{k,l}^{(-)}$, and it suffices to estimate the probability of the intersection of these two events for any boundary condition outside of $\Delta_{k,l}$ and any point configuration ζ'' in $\Delta_{k,l} \setminus \Delta_{k,l}^{(-)}$. For all boundary conditions $\zeta' \in \Omega_{\Delta_{k,l}^c}$ with $\zeta' \cap \Lambda^c = \omega$,

$$\begin{aligned}M_{\Lambda,\Delta,\zeta'}(F_{k,l}^{(-)} \cap G_{k,l} \cap F_{k,l}^{(\text{ext})}) &= \\ \int \mathbb{1}_{F_{k,l}^{(\text{ext})}}(\zeta'' \cup \zeta') \left[\int \mathbb{1}_{F_{k,l}^{(-)}}(\zeta) \mathbb{1}_{G_{k,l}}(\zeta) M_{\Lambda,\Delta,\zeta'}(\text{d}\zeta | \zeta = \zeta'' \text{ on } \Delta \setminus \Delta^{(-)}) \right] M_{\Lambda,\Delta,\zeta'}(\text{d}\zeta'').\end{aligned}$$

Using Lemma 2.6 it follows that

$$\begin{aligned}M_{\Lambda,\Delta_{k,l}^{(-)},\zeta'' \cup \zeta'}(F_{k,l}^{(-)} \cap G_{k,l}) &\geq 1 - M_{\Lambda,\Delta_{k,l}^{(-)},\zeta'' \cup \zeta'}((F_{k,l}^{(-)})^c) - M_{\Lambda,\Delta_{k,l}^{(-)},\zeta'' \cup \zeta'}((G_{k,l})^c) \\ &> 1 - \frac{16\varepsilon}{64} - \varepsilon = 1 - \frac{5\varepsilon}{4},\end{aligned}\tag{2.19}$$

and hence we conclude that

$$\begin{aligned} M_{\Lambda, \Delta_{k,l}, \zeta'}(F_{k,l}^{(-)} \cap G_{k,l} \cap F_{k,l}^{(\text{ext})}) &> \left(1 - \frac{5\varepsilon}{4}\right) \int \mathbb{1}_{F_{k,l}^{(\text{ext})}}(\zeta'' \cup \zeta') M_{\Lambda, \Delta, \zeta'}(d\zeta'') \\ &> \left(1 - \frac{5\varepsilon}{4}\right) \left(1 - \frac{3\varepsilon}{4}\right) > 1 - 2\varepsilon. \end{aligned} \quad (2.20)$$

Step (ii): Good cells and site percolation:

After these preparation steps we shall define when a cell $\Delta_{k,l}$ is good which will then be the site percolation at site $(k, l) \in \mathbb{Z}^2$. The next step is therefore to condition the marks of the points, that is, we pick $(k, l) \in \{-n, \dots, n\}^2$ and consider the event $C_{k,l}$ that each cell $\Delta_{k,l}^{i,j}$ has at least one point, $\Delta_{k,l}^{(-)}$ contains no more than $m(z)$ points and all points in $\Delta_{k,l}^{(-)} \cap \text{Del}_1^*(\zeta)$ are carrying mark 1,

$$C_{k,l} = \{\zeta \in \Omega_{\Lambda, \omega} : \zeta \in F_{k,l}^{(-)} \cap F_{k,l}^{(\text{ext})} \cap G_{k,l} \text{ and } \sigma_\zeta(x) = 1 \text{ for all } x \in \Delta_{k,l}^{(-)} \cap \text{Del}_1^*(\zeta)\}.$$

A cell $\Delta_{k,l}$ is declared to be 'good' or 'open' if $C_{k,l}$ occurs. Each vertex $x \in \text{Del}_1^*(\zeta)$ is open with probability \tilde{p} (see (2.16)), that is with probability

$$\tilde{p} = \frac{1 - \exp\{-g(\beta)\}}{1 + (q-1)\exp\{-g(\beta)\}} \mathbb{1}_{\text{Del}_2^*(\zeta)}(\eta).$$

It follows that

$$\begin{aligned} \tilde{C}_{\Lambda, \omega}^{(\text{site})}(C_{k,l}) &\geq \int M_{\Lambda, \Delta_{k,l}, \zeta'}(d\zeta) \mathbb{1}_{F_{k,l}^{(-)}}(\zeta) \mathbb{1}_{F_{k,l}^{(\text{ext})}}(\zeta) \mathbb{1}_{G_{k,l}}(\zeta) \tilde{p}^{\#\{\text{Del}_1^*(\zeta) \cap \Delta_{k,l}^{(-)}\}} \\ &\geq \tilde{p}^{\lfloor m(z) \rfloor} M_{\Lambda, \Delta, \zeta'}(F_{k,l}^{(-)} \cap G_{k,l} \cap F_{k,l}^{(\text{ext})}). \end{aligned}$$

Recall from Lemma 2.3 and Remark 2.5 that for all $z > 0$ there is a coarse graining length scale $\ell_0(z, q, \alpha)$ such that all our results hold for $\ell \geq \ell_0(z, q, \alpha)$. There there exists a $\beta_0 = \beta_0(z, q, \alpha, \varepsilon)$ such that for all $\beta \geq \beta_0$ we obtain the lower bound

$$\tilde{p} = \frac{1}{\frac{64q\ell^4}{\beta} + 1} \geq (1 - 2\varepsilon)^{1/\lfloor m(z) \rfloor}, \quad (2.21)$$

and hence $\tilde{p}^{\lfloor m(z) \rfloor} \geq (1 - 2\varepsilon)$. Combining the above estimates, we conclude, for all $\beta \geq \beta_0$, that

$$\tilde{C}_{\Delta_{k,l}, \zeta'}^{(\text{site})}(C_{k,l}) \geq (1 - 2\varepsilon)^2 > 1 - 4\varepsilon > (p_c^{(\text{site})}(\mathbb{Z}^2))^{1/2}. \quad (2.22)$$

Step (iii) Neighbouring good cells and link percolation:

If $\zeta \in C_{k,l}$, we say that the cell $\Delta_{k,l}$ is a 'good' cell. Two neighbouring cells $\Delta_{k,l}$ and $\Delta_{k+1,l}$ are said to be 'linked' if the box $\Delta_{(\text{link})} := \Delta_{(\text{link})}^{k:k+1, l}$ defined in (2.6) has an intersection with $\text{Del}_1^*(\zeta)$ that contains only points of mark 1. More precisely, the event that $\Delta_{k,l}$ and $\Delta_{k+1,l}$ are linked, is

$$L_{k:k+1}^l = \{\zeta \in \Omega_{\Lambda, \omega} : \sigma_\zeta(x) = 1 \text{ for all } x \in \Delta_{(\text{link})}^{k:k+1, l} \cap \text{Del}_1^*(\zeta)\}.$$

We also define

$$F_{(\text{link})} = \left(F_{k,l}^{(-)} \cap F_{k,l}^{(\text{ext})}\right) \cap \left(F_{k+1,l}^{(-)} \cap F_{k+1,l}^{(\text{ext})}\right)$$

and

$$G_{(\text{link})} = \{\zeta \in \Omega : \#\{\zeta \cap \Delta_{(\text{link})}\} \leq m(z)\},$$

and let $\zeta' \in \Omega_{\Delta_{(\text{link})}^c}$ be the boundary condition outside $\delta_{(\text{link})}$ such that $\zeta' \cap \Lambda^c = \omega$. The conditional probability that the cells $\Delta_{k,l}$ and $\Delta_{k+1,l}$ are linked, given they are both 'good' cells, is then given by

$$\begin{aligned} \tilde{C}_{\Delta_{(\text{link})}, \zeta'}^{(\text{site})}(L_{k,k+1}^l | C_{k,l} \cap C_{k+1,l}) &\geq \int \mathbb{1}_{G_{(\text{link})}}(\zeta) \tilde{p}^{\lfloor m(z) \rfloor} M_{\Lambda, \Delta_{(\text{link})}, \zeta'}(d\zeta | F_{(\text{link})}) \\ &\geq (1 - \varepsilon)(1 - 2\varepsilon) \geq (1 - 4\varepsilon) \geq (p_c^{(\text{site})}(\mathbb{Z}^2))^{1/2}, \end{aligned} \quad (2.23)$$

where the second inequality comes from an adaptation of Lemma 2.7, where $\Delta_{(\text{link})}$ takes the role of $\Delta_{k,l}^{(-)}$. Then by (2.22), (2.23) and the results of McDiarmid and Hammersley, in particular, (C.4), mixed site-bond percolation in \mathbb{Z}^2 occurs. There exists a chain of good cells/boxes joined by open links from $\Delta_{k,l} \subset \Lambda$ to Λ^c .

It remains to check that the preceding result implies $\{\Delta \leftrightarrow \Lambda^c\}$ in the Delaunay graph. For this, we recall the set $H_{k:k+1,l}(\zeta)$ from (2.7). We know by construction that all edges $\eta = \{x, y\} \in \text{Del}_2(\zeta)$ that have a non-empty intersection with $H_{k:k+1,l}(\zeta)$ satisfy $|x - y| < 2\sqrt{2}\ell$. This implies that $H_{k:k+1,l}(\zeta) \subset \text{Del}_1^*(\zeta)$. Let $x, y \in \zeta$ be such that $\text{Vor}_\zeta(x)$ and $\text{Vor}_\zeta(y)$ contain the centres of the boxes $\Delta_{k,l}$ and $\Delta_{k+1,l}$ respectively. Since $H_{k:k+1,l}(\zeta) \subset \text{Del}_1^*(\zeta)$, we can connect x and y in the graph $\text{Del}_2^*(\zeta)$ inside $\Delta_{k,L}^{(-)} \cup \Delta_{(\text{link})} \cup \Delta_{k+1,L}^{(-)}$. Hence, by (2.22) and (2.23), there exists $c > 0$ such that we obtain the following uniform lower bound

$$\tilde{C}_{\Lambda, \omega}^{(\text{site})}(\{\Delta \leftrightarrow \Lambda^c\}) > c > 0,$$

and the proof of Lemma 2.8 is finished. □

Step 4: Finish of the proof of Proposition 2.2. The Proof follows immediately from all previous steps as the percolation in the site percolation measure $\tilde{C}_{\Lambda, \omega}^{(\text{site})}$ implies percolation in the Delaunay random cluster measure $C_{\Lambda, \omega}$ due to stochastic dominance,

$$C_{\Lambda, \omega} \succcurlyeq \tilde{C}_{\Lambda, \omega}.$$

□

2.3. Phase transition. To relate the influence of the boundary condition on the mark of a single point to the connectivity probabilities in the random-cluster model we follow [GH96]. For any $\Delta \subset \Lambda$, $s \in E$, $\zeta \in \Omega_{\Lambda, \omega}$ and (ζ, E) , with $E \subset E_\zeta$ we define

$$N_{\Delta, s}(\zeta) = \#\{\zeta^{(s)} \cap \Delta\}.$$

Then

$$N_{\Delta \leftrightarrow \Lambda^c}(\zeta, E) = \#\{x \in \zeta \cap \Delta : x \text{ belongs to a cluster connected to } \Lambda^c \\ \text{in } E \cap \text{Del}_2(\zeta)\}$$

is the number of points in $\zeta \cap \Delta$ connected to any point in Λ^c in the random graph $E \cap \text{Del}_2(\zeta)$. Because of the edge-drawing mechanism, $\{\Delta \leftrightarrow \Lambda^c\} = \{N_{\Delta \leftrightarrow \Lambda^c} > 0\}$ is also the event that there exists a point in $\zeta \cap \Delta$ connected to infinity in $E \cap \text{Del}_2(\zeta)$.

The next Proposition is the key argument why percolation for the random cluster measures leads to a break of symmetry in the mark distribution, and as such it is standard and well known ([GH96]).

Proposition 2.9. *For any measurable $\Delta \subset \Lambda$,*

$$\int (qN_{\Delta,1} - N_{\Delta}) d\gamma_{\Lambda,\omega} = (q-1) \int N_{\Delta \leftrightarrow \Lambda^c} dC_{\Lambda,\omega}.$$

PROOF. This is proved in [Eye14, Lemma 2.17] following ideas in [GH96]. \square

3. NUMBER OF CONNECTED COMPONENTS

In this section we are going to prove the main technical tool for our phase transition proof, namely, Theorem 2.1. The proof of Theorem 2.1 is rather long, so we first outline the strategy. We want to bound the number of connected components that intersect the graph $G_{x_0,\zeta}$ in the Delaunay graph $E_{x_0,\zeta}^{(\text{ext})}$ under the edge drawing mechanism $\mu_{(\text{ext}),\zeta}^{(q)}$. We write ∂_{x_0} for $G_{x_0,\zeta}$ and we also define ∂_B for the contraction of ∂_{x_0} to any ball B around x_0 , see Figure 5, that is $\partial_B = (V_B, E_B)$ with $V_B = V_{x_0,\zeta} \cap B$ and $E_B = \{\eta_{x,y} : x, y \in V_B\}$. The case where x_0 lies in the opposite half plane to all $x \in V_B$ is one such example of ∂_B not forming a loop.

The pivotal point of the whole proof is to find an upper bound for the number of edges in the edge set E_B that have length greater than some fixed real number. This will allow us to construct a sequence of edge subsets of E_B , defined by edge length, to balance the unbounded number of points against the increased probability that they are connected. The shorter the edge length, the greater the possible number of edges in the subset, but also the greater the probability that these edges are open. It turns out that such an upper bound can be found in the scenario where there are no defects in the geometry. These defects which we give the logical name 'kinks' are defined below in Section 3.1. An upper bound cannot be found if the geometry of the contracted graph contains kinks, so we devise a plan to discount them.

Note that $E_B \subset E_{x_0,\zeta}^{(\text{ext})}$ does not hold in general. This creates a problem when dealing with our edge drawing mechanism on $E_{x_0,\zeta}^{(\text{ext})}$. To overcome this, we introduce an edge drawing mechanism on the contracted graph and build a structure that will allow us to compare events between the two different probability spaces. All our techniques rely heavily on geometric properties of the Delaunay tessellation, and therefore we are able to present a purely geometric proof of our phase transition result.

3.1. Notation and geometric facts. We introduce a polar coordinate system in \mathbb{R}^2 with x_0 being the pole, and we denote L the polar axis in horizontal direction. For any $z \in \mathbb{R}^2$, denote \hat{z} be the angular coordinate of z taken counter clockwise from the axis L . Given two points $x, y \in \mathbb{R}^2$, \overleftrightarrow{xy} denotes the unique straight line that intersects x and y in the plane, \overleftarrow{xy} denotes the half line that stops at y and \overline{xy} denotes the line segment between x and y only. Given two straight lines $\ell_1, \ell_2 \subset \mathbb{R}^2$ that intersect a point $z \in \mathbb{R}^2$, $\angle(\ell_1, \ell_2)$ denotes the angle between them. More precisely, it is the angle in order to rotate ℓ_1 onto ℓ_2 with z as the centre of rotation. Notice that it is certainly not true that $\angle(\ell_1, \ell_2) = \angle(\ell_2, \ell_1)$, however, it holds that $\angle(\ell_1, \ell_2) + \angle(\ell_2, \ell_1) = \pi$. When we consider a triangle in the plane with vertices x, y, z , for example, we refer to the interior angle at y as \widehat{xyz} . In this case, as we specify the interior angle, $\widehat{xyz} = \widehat{zyx}$.

Given a set of points $V = \{x_i \in \mathbb{R}^2: 1 \leq i \leq n\}$ with $\hat{x}_1 < \dots < \hat{x}_n$, the graph

$$\Gamma = \left(V, \bigcup_{i=1}^{n-1} \eta_{x_i, x_{i+1}} \right)$$

is called a **spoked chain** if $\eta_{x_0, x_i} \in \text{Del}_2(V \cup \{x_0\})$ for all $1 \leq i \leq n$. The polygon $P(\Gamma, x_0)$ created by adding the point x_0 and edges η_{x_0, x_1} and η_{x_n, x_n} to Γ is called the induced polygon of Γ , see Figure 4.

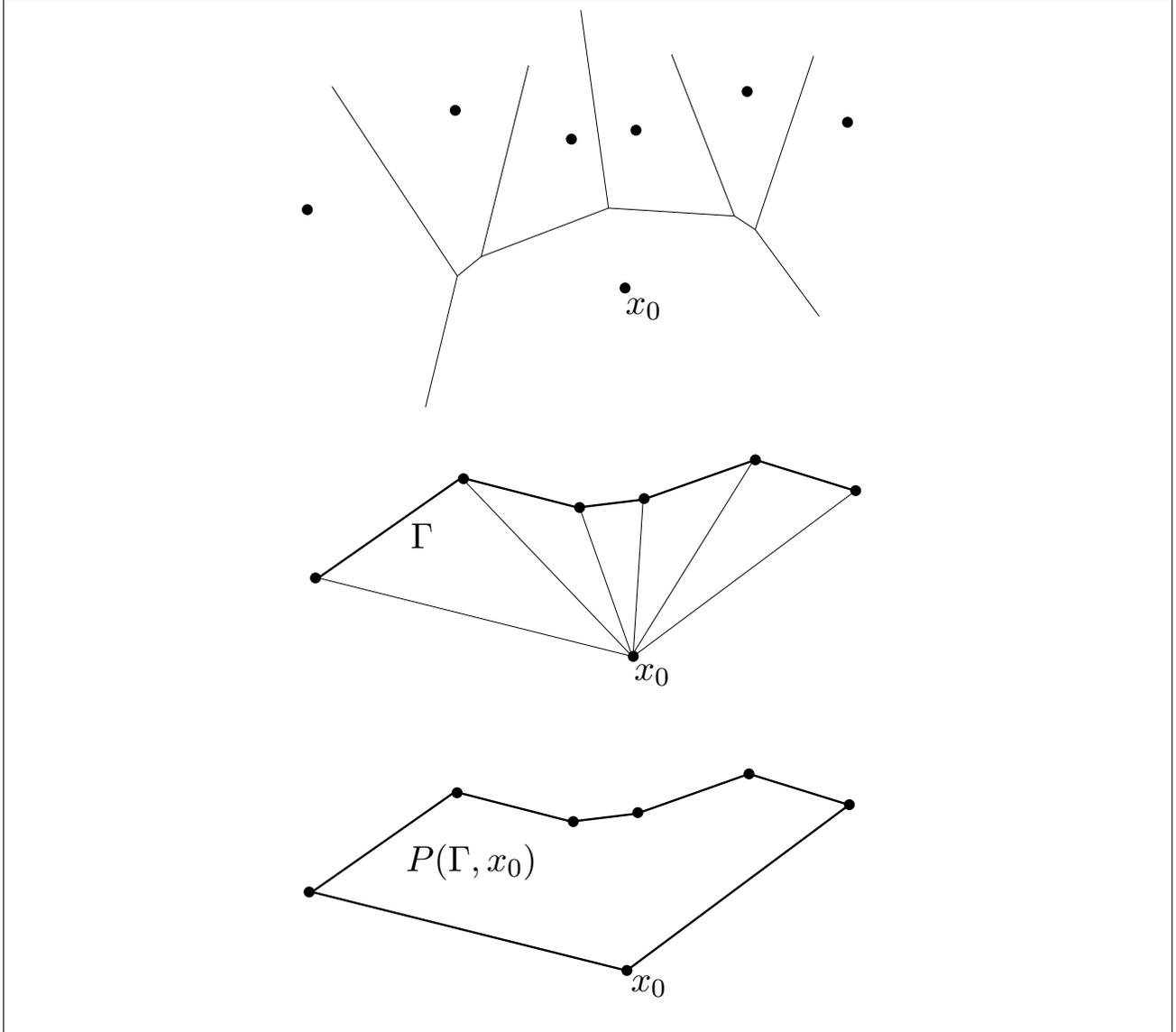


Figure 4: From top to bottom we have: (1.) A collection of points that neighbour x_0 in the Delaunay/Voronoi tessellation. (2.) A spoked chain Γ . (3.) The induced polygon $P(\Gamma, x_0)$.

In order to quantify the number of connected components that intersect V_B , we analyse the shape of the contracted boundary, ∂_B . For that we split B into four quadrants, $Q_i \subset \mathbb{R}^2, i = 1, 2, 3, 4$, where

$$Q_i = \left\{ z \in B: \frac{\pi}{2}(i-1) \leq \hat{z} < \frac{\pi}{2}i \right\}$$

This allows to bound the number of connected components in one quadrant, and then multiply that bound by four to obtain the final bound. The reason for doing this are twofold: it not only provides us a framework to define kinks, but also ensures that any two points that we consider will differ in angle by no more than $\pi/2$. This allows to find a lower bound for the probability that the two points belong to the same connected component.

Definition 3.1 (Kinks). Let $\Gamma = (V, E)$ be a spoked chain. Suppose that $x_i, x_j, x_k \in V$ such that $\hat{x}_i < \hat{x}_j < \hat{x}_k$. We say that x_i, x_j and x_k form a **kink** in Γ if the following holds.

- (1) $\widehat{x_i x_j x_k} < \pi/2$,
- (2) $\widehat{x_{i'} x_{j'} x_{k'}} \geq \pi/2$ for all $x_{i'}, x_{j'}, x_{k'} \in V$ with $\hat{x}_i < \hat{x}_{i'} < \hat{x}_{j'} < \hat{x}_{k'} < \hat{x}_k$,
- (3) $\widehat{x_{i'} x_{j'} x_{k'}} \geq \pi/2$ for all $x_{i'}, x_{j'}, x_{k'} \in V$ with $\hat{x}_j < \hat{x}_{i'} < \hat{x}_{j'} < \hat{x}_{k'} \leq \hat{x}_k$.

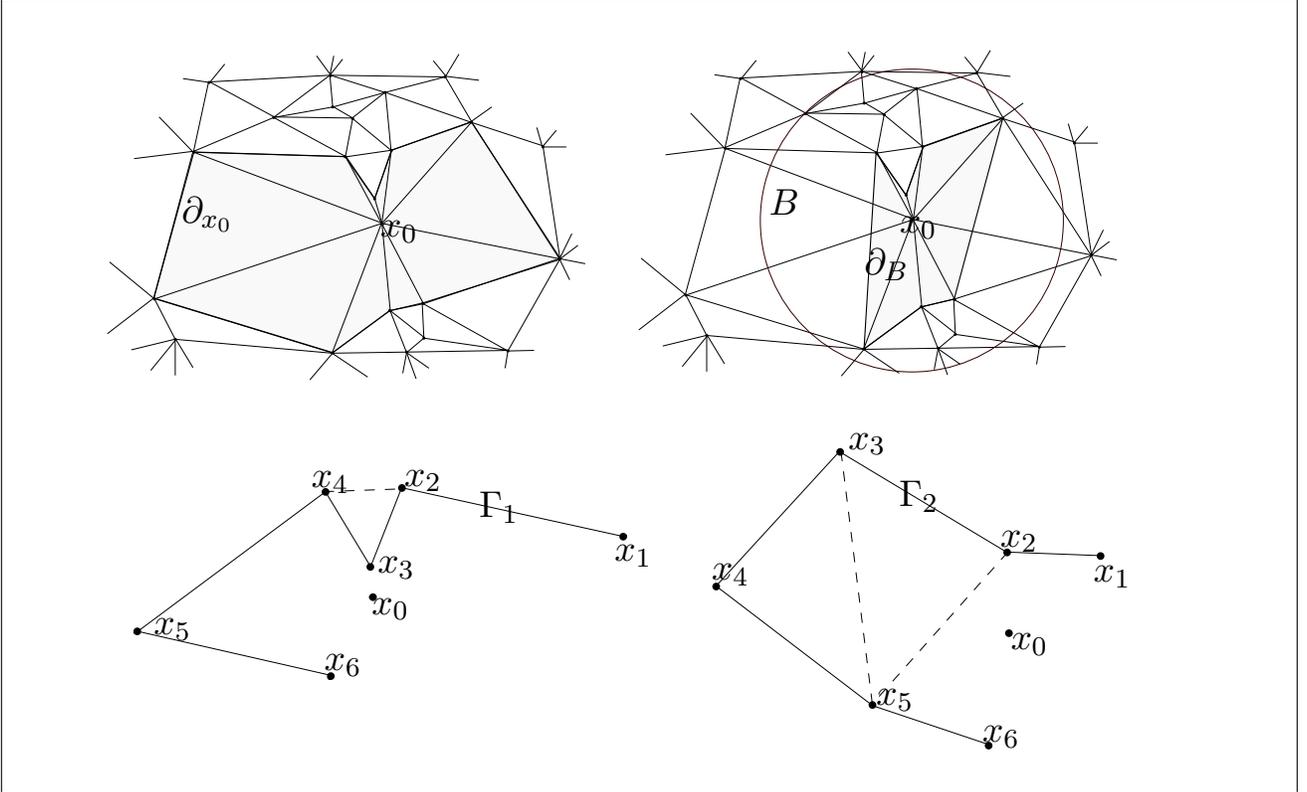


Figure 5: From left to right, top to bottom we have: (1.) The shaded area is the neighbourhood of x_0 , whilst the bold edges form ∂_{x_0} . (2.) The contraction of ∂_{x_0} to ∂_B . (3.) The points x_2, x_3 and x_4 form an intruding kink in Γ_1 . (4.) The points x_2, x_3 and x_5 form a protruding kink in Γ_2

Suppose that x_i, x_j and x_k form a kink in the spoked chain $\Gamma = (V, E)$. The kink is called **intruding** if the line segment $\overline{x_i x_k}$ lies outside of the induced Polygon $P(\Gamma, x_0)$ and **protruding** if it lies inside $P(\Gamma, x_0)$.

Lemma 3.2. Let $\Gamma = (V, E)$ be a spoked chain with $V = \bigcup_{i=1}^n \{x_i\}$ and $\hat{x}_1 < \dots < \hat{x}_n$. A kink in Γ is either intruding or protruding.

PROOF. Suppose the statement is false. Then there exist $1 \leq i < j < k \leq n$ such that x_i, x_j and x_k form a kink in Γ and $\overline{x_i x_k}$ lies neither inside nor outside of $P(\Gamma, x_0)$. Let $U \subset \mathbb{R}^2$ be the connected component of $\mathbb{R}^2 \setminus \overleftrightarrow{x_i x_k}$ that does not contain x_j . It follows that Γ crosses $\overline{x_i x_k}$ between \hat{x}_i and \hat{x}_k and hence, there exists $x_{j'} \in V \cap U$ with $\hat{x}_i < \hat{x}_{j'} < \hat{x}_k$. Without loss of generality, let $\hat{x}_j < \hat{x}_{j'} < \hat{x}_k$. Therefore, $\widehat{x_i x_j x_{j'}} < \pi/2$ which contracts (2) in Definition 3.1 for the kink formed by x_i, x_j and x_k . \square

Lemma 3.3. *Let $\Gamma = (V, E)$ be a spoked chain with $V = \bigcup_{i=1}^n \{x_i\}$ and $\hat{x}_1 < \dots < \hat{x}_n$. If x_i, x_j and x_k form an intruding kink in Γ , then*

$$\angle(\overleftrightarrow{x_i x_{i+1}}, \overleftrightarrow{x_{k-1} x_k}) < \pi/2.$$

PROOF. Let x_i, x_j and x_k form an intruding kink in Γ . Since the kink is intruding, we know that x_l lies in the interior of the triangle $\{x_0, x_i, x_k\}$, for all $i < l < k$. Suppose the statement of the Lemma is false, that is,

$$\angle(\overleftrightarrow{x_i x_{i+1}}, \overleftrightarrow{x_{k-1} x_k}) \geq \pi/2.$$

This forces either x_{i+1} or x_{k-1} to be in the interior of the triangle $\{x_i, x_j, x_k\}$. Without loss of generality, suppose, in fact, that x_{i+1} is in the interior of that triangle. Therefore, $\widehat{x_{i+1} x_j x_k} < \pi/2$ which by Definition 3.1, contradicts the fact that x_i, x_j and x_k form a kink in Γ . \square

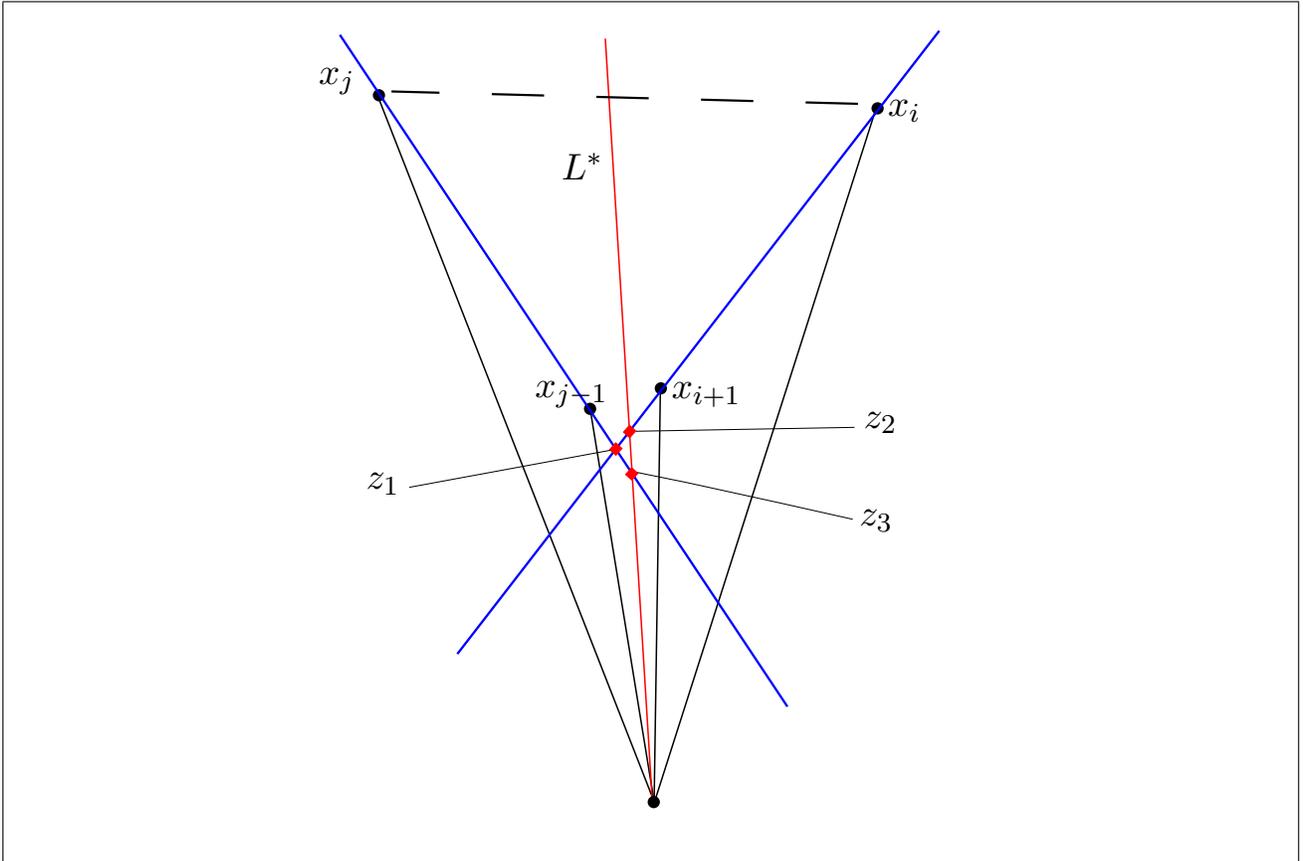


Figure 6: Low bound for the angle $\widehat{x_0 X_{j-1} x_j}$.

3.2. Intermediary Lemmas. Kinks of intruding and protruding nature may occur in $\partial_B \cap Q_1$, although the number of them is bounded above, see Lemma 3.4 and Lemma 3.5 below. We shall separate $\partial_B \cap Q_1$ into a finite number of kinkless pieces, each of which is a spoked chain, which are easier to work with.

Lemma 3.4. *The number of intruding kinks in $\partial_B \cap Q_1$ is bounded above by 2.*

PROOF. We show that the angle between two intruding kinks in a spoked chain is greater than $\pi/4$. Since $\partial_B \cap Q_1$ lies in the quadrant Q_1 , and is a spoked chain by definition, the statements follows immediately. Let $\Gamma = (V, E)$ be a spoked chain and order the elements of V such that $\hat{x}_1 < \dots < \hat{x}_n$. Suppose there is an intruding kink in Γ . By the definition of an intruding kink and Lemma 3.3, there exist $1 \leq i < n - 1$ and $i + 1 < j \leq n$, such that

$$\angle(\overrightarrow{\hat{x}_i x_{i+1}}, \overrightarrow{\hat{x}_{j-1} x_j}) < \pi/2, \quad (3.1)$$

and $\overline{\hat{x}_i x_j}$ lies outside of the induced polygon $P(\Gamma, x_0)$. The straight lines $\overrightarrow{\hat{x}_i x_{i+1}}$ and $\overrightarrow{\hat{x}_{j-1} x_j}$ split the plane into four regions. Since the kink is intruding, the point x_0 must lie in the opposite region to that of the line segment $\overline{\hat{x}_i x_j}$. Let L^* be the radial line of angle $(\hat{x}_{i+1} - \hat{x}_{j-1})/2$ and $z_1 \in \mathbb{R}^2$ be the point of the intersection of $\overrightarrow{\hat{x}_i x_{i+1}}$ and $\overrightarrow{\hat{x}_{j-1} x_j}$ and let z_2, z_3 be the points of intersection of L^* with $\overrightarrow{\hat{x}_i x_{i+1}}$ and $\overrightarrow{\hat{x}_{j-1} x_j}$ respectively, see Figure 6. Then,

$$\widehat{x_i z_1 x_j} + \widehat{x_i z_2 x_0} + \widehat{x_0 z_3 x_j} = 2\pi, \quad (3.2)$$

which implies, together with (3.1), that

$$\max\{\widehat{x_i z_2 x_0}, \widehat{x_0 z_3 x_j}\} \geq \frac{2\pi - \pi/2}{2} = \frac{3\pi}{4}. \quad (3.3)$$

Without loss of generality, let $\widehat{x_0 z_3 x_j} \geq 3\pi/4$. Because x_{j-1} lies on the line segment $\overline{z_3 x_j}$, it follows that

$$\widehat{x_0 x_{j-1} x_j} \geq \widehat{x_0 z_3 x_j} \geq 3\pi/4.$$

Suppose now that there is another intruding kink in Γ , formed by the points x_k, x_l and x_m for $j < k < l < m \leq n$. Then, by Lemma 3.3, we have that

$$\angle(\overrightarrow{\hat{x}_k x_{k+1}}, \overrightarrow{\hat{x}_{m-1} x_m}) < \pi/2.$$

Let t_{k+1} be the tangent to $B(\tau(x_0, x_k, x_{k+1}))$ at x_{k+1} , see Figure 7. Then, by noting that

$$|V \cap B(\tau(x_0, x_{j-1}, x_j))| = 0,$$

which is a consequence of the properties of the Delaunay tessellation (quadratic position), it follows that

$$\angle(t_{k+1}, \overrightarrow{\hat{x}_0 x_{k+1}}) \leq \angle(\overrightarrow{\hat{x}_k x_{k+1}}, \overrightarrow{\hat{x}_0 x_{k+1}}) \leq \angle(\overrightarrow{\hat{x}_k x_{k+1}}, \overrightarrow{\hat{x}_{m-1} x_m}) < \pi/2. \quad (3.4)$$

here, the first inequality follows directly from the definition of the tangent and the second inequality is a consequence of the fact that $\hat{x}_{k+1} < \hat{x}_{m-1} < \hat{x}_m$. For $1 \leq r < n$, let x_{r+1}^* denote the centre of the circumcircle of the triangle $\tau(x_0, x_r, x_{r+1}) \in \text{Del}_3(V \cup \{x_0\})$. Since the points $\{x_0, x_{k+1}^*, x_{k+1}\}$ form an isosceles, see Figure 7, we can conclude that

$$\hat{x}_{k+1} - \hat{x}_{k+1}^* = \pi/2 - \angle(t_{k+1}, \overrightarrow{\hat{x}_0 x_{k+1}}) > 0. \quad (3.5)$$

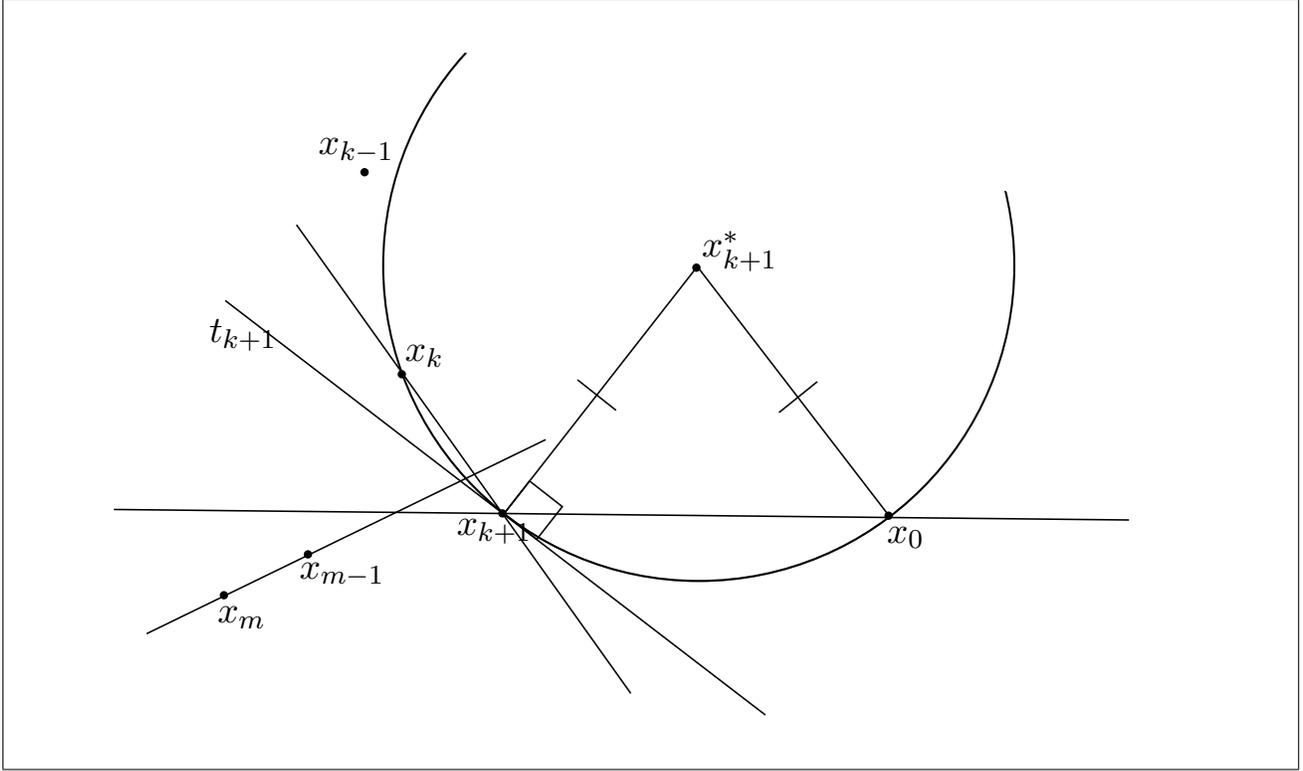


Figure 7: The intruding kink formed by x_k, x_l and x_m .

Let y be the antipodal point to x_0 on the circumscribed ball of the triangle $\tau(x_0, x_{j-1}, x_j)$ in \mathbb{R}^2 . Since, $|x_0 - y|$ is equal to the diameter of that circle, it follows that $\widehat{x_0 x_j y} = \pi/2$, see Figure 8. The points x_0, x_{j-1}, x_j and y form a cyclic quadrilateral. Using $\widehat{x_0 x_{j-1} x_j} \geq \widehat{x_0 z_3 x_j} \geq 3\pi/4$ from above, and the fact that opposite angles of a cyclic quadrilateral add up to π , we see that $\widehat{x_0 y x_j} \leq \pi/4$. Hence, by (3.5)

$$\hat{x}_{k+1} - \hat{x}_j > \hat{x}_{k+1}^* - \hat{x}_j \geq \hat{x}_j^* - \hat{x}_j = \widehat{y x_0 x_j} = \pi - \widehat{x_0 x_j y} - \widehat{x_0 y x_j} \geq \pi/4,$$

where the second inequality is due to a further property of the Delaunay structure that we show in Appendix D. This implies that the angle between intruding kinks must be greater than $\pi/4$. \square

Lemma 3.5. *There are no protruding kinks in $\partial_B \cap Q_1$.*

PROOF. We order the elements of $V_B = \bigcup_{i=1}^n \{x_i\}$ such that $\hat{x}_1 < \dots < \hat{x}_n$. Suppose we have a protruding kink, then we have for some $1 \leq i < j < k \leq n$,

$$\widehat{x_i x_j x_k} < \pi/2.$$

The pair $\{x_i, x_k\}$ does not form an edge of E_B , therefore, by the properties of The Delaunay graph, x_j lies inside the circumcircle $B(\tau(x_0, x_i, x_k))$ of the triangle $\tau(x_0, x_i, x_k)$. The line segment $\overline{x_i x_k}$ is a chord which splits the ball $B(\tau(x_0, x_i, x_k))$ into two regions. Since we have a protruding kink, $\overline{x_i x_k}$ lies inside the induced polygon $P(\partial_B \cap Q_1; x_0)$, and so x_j does not lie in the same region as x_0 . Therefore by the above inequality, it follows that $\widehat{x_i x_0 x_k} \geq \pi/2$, and hence, there are no protruding kinks in $\partial_B \cap Q_1$. \square

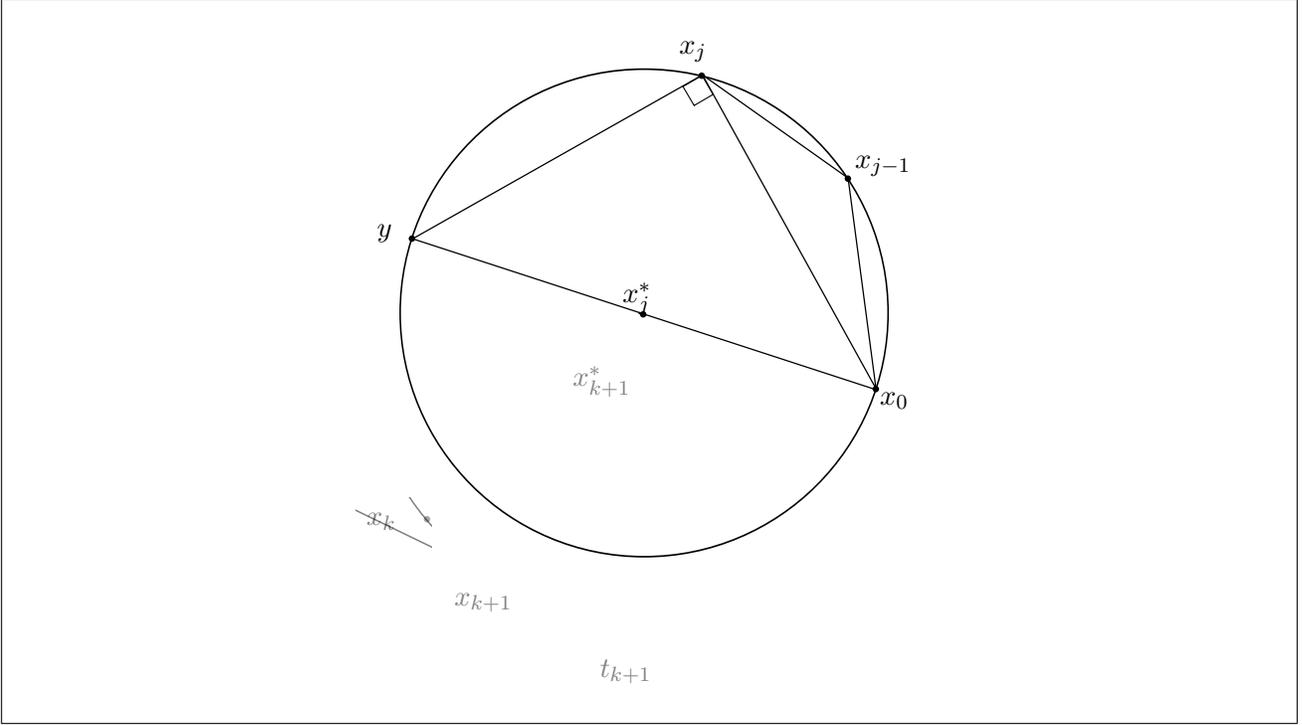


Figure 8: Lower bound for the angle between kinks of type 2.

3.3. Edge drawing. Before we can finish the proof of Theorem 2.1 in Section 3.4 below we need two more results to gain some control over the edge drawing mechanism in $E_{x_0, \zeta}^{(\text{ext})}$. In the following we choose the radius $R > 0$ of the ball B such that $V_B \subset V_{x_0, \zeta}$ almost surely. Denote by $\tilde{\nu}_\zeta$ the edge drawing mechanism with probability

$$\tilde{p}_2(\eta_{xy}) = \frac{\mathbb{1}\{|x - y| \leq R\}}{\frac{q}{\beta}|x - y|^4 + 1}, \quad (3.6)$$

and denote $\tilde{\nu}_\zeta^{(\text{ext})}$ the corresponding edge drawing mechanism on $E_{x_0, \zeta}^{(\text{ext})}$ only.

Lemma 3.6. *For all $q \geq 1$, $\Lambda \in \mathbb{R}^2$ and all $\zeta \in \Omega_{\Lambda, \omega}$,*

$$\mu_{\Lambda, \zeta}^{(q)} \succcurlyeq \tilde{\nu}_\zeta.$$

PROOF. It suffices to show that for all edges $\eta \in \text{Del}_2(\zeta)$,

$$\frac{p(\eta)}{q(1 - p(\eta))} \geq \frac{\tilde{p}_2(\eta)}{(1 - \tilde{p}_2(\eta))}. \quad (3.7)$$

Recall from (2.1) that

$$p(\eta) = \frac{1}{\frac{1}{\beta}\ell(\eta)^4 + 1}.$$

Thus, if $\ell(\eta) > R$, then $\tilde{p}_2(\eta) = 0$ and the inequality (3.7) trivially holds. Suppose that $\ell(\eta) < R$, then, in fact, we also have

$$\frac{p(\eta)}{q(1 - p(\eta))} = \frac{\tilde{p}_2(\eta)}{(1 - \tilde{p}_2(\eta))}.$$

Henceforth, (3.7) holds for all $\eta \in \text{Del}_2(\zeta)$. \square

Note that E_B is not necessarily a subset of $E_{x_0, \zeta}^{(\text{ext})}$, in fact, they belong to different Delaunay tessellations

$$E_B \subset \text{Del}_2(V_B \cup \{x_0\}) \text{ and } E_{x_0, \zeta}^{(\text{ext})} \subset \text{Del}_2(\zeta \cup \{x_0\}).$$

We therefore introduce another edge drawing mechanism, but this time on E_B . Let μ^* denote the distribution of the random edge configurations $\{\eta \in E_B : v(\eta) = 1\}$, where $((v(\eta))_{\eta \in E_B})$ are independent Bernoulli random variables with probability

$$\mathbb{P}(v(\eta) = 1) = p^*(\eta) = \frac{\mathbb{1}\{\ell(\eta) \leq \frac{2}{\pi} \wedge R\} \mathbb{1}\{|\hat{x} - \hat{y}| \leq \frac{\pi}{2}\}}{\frac{q}{\beta} \left(\frac{\pi}{2} \ell(\eta)\right)^4 + 1} \mathbb{1}_{E_B}(\eta), \quad \text{for } \eta = \eta_{xy}. \quad (3.8)$$

We now compare the probability that two points are connected (i.e. built an open edge) with respect to $\tilde{\nu}_\zeta$ and with respect to μ^* .

Lemma 3.7. *Pick $\zeta \in \Omega_{\Lambda, \omega}$. Let $\eta_{xy} \in E_B$ and let $x \leftrightarrow y$ denote the event that x and y lie in the same connected component of (ζ, E) , where E is a \tilde{p}_2 -thinning of the edge set $E_{x_0, \zeta}^{(\text{ext})}$. Then,*

$$\tilde{\nu}_\zeta^{(\text{ext})}(x \leftrightarrow y) \geq p^*(\eta_{xy}). \quad (3.9)$$

PROOF. By the definition of p^* , (3.9) follows for $x, y \in V_B$ with $|x - y| > \frac{2}{\pi} \wedge R$ or with $|\hat{x} - \hat{y}| > \frac{\pi}{2}$. Therefore, we assume that $|x - y| \leq \frac{2}{\pi} \wedge R$ and $|\hat{x} - \hat{y}| \leq \frac{\pi}{2}$.

Case I: If $\eta_{xy} \in E_{x_0, \zeta}^{(\text{ext})}$, we get

$$\tilde{\nu}_\zeta^{(\text{ext})}(x \leftrightarrow y) \geq \tilde{p}_2(\eta_{xy}) = \frac{1}{\frac{q}{\beta} \ell(\eta_{xy})^4 + 1} \geq \frac{1}{\frac{q}{\beta} \left(\frac{\pi}{2} \ell(\eta_{xy})\right)^4 + 1} \geq p^*(\eta_{xy}).$$

Case II: If $\eta_{xy} \notin E_{x_0, \zeta}^{(\text{ext})}$, the proof is no longer straightforward and will take some care. Since $\eta_{xy} \notin E_{x_0, \zeta}^{(\text{ext})}$, and $x, y \in V_B$ there exists $z \in \zeta \cap V_B^c$, such that $\eta_{zx_0} \in \text{Del}_2(\zeta \cup \{x_0\})$. This implies that $z \in V_{x_0, \zeta} \setminus V_B$ and $\hat{x} < \hat{z} < \hat{y}$. We now check whether $\eta_{xz}, \eta_{zy} \in E_{x_0, \zeta}^{(\text{ext})}$. If they are not, we find more points $z \in V_{x_0, \zeta} \setminus V_B$ with $\hat{x} < \hat{z} < \hat{y}$. Therefore, there exists a sequence (finite) $z_1, \dots, z_n \in V_{x_0, \zeta} \setminus V_B$ with $\hat{x} < \hat{z}_1 < \dots < \hat{z}_n < \hat{y}$ such that

$$\eta_{xz_1}, \eta_{z_1, z_2}, \dots, \eta_{z_n y} \in E_{x_0, \zeta}^{(\text{ext})}.$$

The event that each of these edges is open implies the event that x and y belong to the same connected component of open edges, hence

$$\tilde{\nu}_\zeta^{(\text{ext})}(x \leftrightarrow y) \geq \tilde{p}_2(\eta_{xz_1}) \tilde{p}_2(\eta_{z_1 z_2}) \cdots \tilde{p}_2(\eta_{z_n y}). \quad (3.10)$$

For any two points $x_1, x_2 \in \zeta$ with $\hat{x}_1 < \hat{x}_2$, define $C_{x_1, x_2}^{x_0}$ to be the arc on the circumcircle $B(\tau(x_1, x_2, x_0))$ of the triangle $\tau(x_1, x_2, x_0)$ between x_1 and x_2 , and define U_{x_1, x_2} to be the subset of \mathbb{R}^2 bounded by this arc $C_{x_1, x_2}^{x_0}$ and $\overline{x_1 x_2}$, that is, the convex hull of $C_{x_1, x_2}^{x_0}$. Let $n = \#\{z \in V_{x_0, \zeta} : \hat{x} < \hat{z} < \hat{y}\}$. We claim that

$$L(C_{x, z_1}^{x_0}) + \dots + L(C_{z_n, y}^{x_0}) \leq L(C_{x, y}^{x_0}), \quad n \in \mathbb{N}, \quad (3.11)$$

when $L(C)$ denotes the length of the arc C . We will prove the claim (3.11) below after we show how this claim implies the proof of the statement in the lemma. By our assumption that $|x - y| \leq \frac{2}{\pi} \wedge R$ and $|\hat{x} - \hat{y}| \leq \frac{\pi}{2}$, it follows that $L(C_{x, y}^{x_0}) \leq 1$, and by (3.11) we also have

$$L(C_{x, z_1}^{x_0}) + \dots + L(C_{z_n, y}^{x_0}) \leq 1.$$

Obviously, this shows that

$$|x - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n| + |z_n - y| \leq 1. \quad (3.12)$$

Now choose $\beta > 0$ such that $q/\beta < 1$. Then for $a, b \in \mathbb{R}$ with $0 \leq a \leq b \leq 1$, we have the following simple fact

$$\left(\frac{1}{\frac{q}{\beta}a^4 + 1}\right)\left(\frac{1}{\frac{q}{\beta}b^4 + 1}\right) = \frac{1}{\frac{q}{\beta}(\frac{q}{\beta}a^4b^4 + a^4 + b^4) + 1} \geq \frac{1}{\frac{q}{\beta}(a+b)^4 + 1}, \quad (3.13)$$

where the inequality follows because $q/\beta < 1$ and the given constraints on a and b . Hence, using (3.12), we obtain

$$\begin{aligned} \tilde{p}_2(\eta_{xz_1})\tilde{p}_2(\eta_{z_1z_2}) \cdots \tilde{p}_2(\eta_{z_ny}) &\geq \left(\frac{1}{\frac{q}{\beta}|x - z_1|^4 + 1}\right) \cdots \left(\frac{1}{\frac{q}{\beta}|z_n - y|^4 + 1}\right) \\ &\geq \frac{1}{\frac{q}{\beta}(|x - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n| + |z_n - y|)^4 + 1} \\ &\geq \frac{1}{\frac{q}{\beta}L(C_{x,y}^{x_0})^4 + 1} \geq \frac{1}{\frac{q}{\beta}\left(\frac{\pi}{2}|x - y|\right)^4 + 1} = p^*(\eta_{xy}), \end{aligned} \quad (3.14)$$

where the second inequality results from repeated use of relation (3.13) with $a = |z_i - z_{i+1}|$ and $b = |z_j - z_{j+1}|$. With this we conclude with statement in the lemma. We now finish the whole proof by verifying the claim (3.11): Suppose there exists $z \in V_{x,\zeta} \setminus V_B$ such that $\eta_{xz}, \eta_{zy} \in E_{x_0,\zeta}^{(\text{ext})}$. Since $z \notin B$, it must be true that $z \in U_{xy}$. Therefore, by a direct application of Theorem D.2, we have

$$L(C_{x,z}^{x_0}) + L(C_{z,y}^{x_0}) \leq L(C_{x,y}^{x_0}),$$

and the claim follows for $n = 1$. We shall proceed by induction with respect to $n \in \mathbb{N}$. Assume the claim holds for $n = k - 1$. There exist $z_1, \dots, z_k \in V_{x_0,\zeta} \setminus V_B$ such that $\hat{x}_1 < \cdots < \hat{x}_k$ and $\eta_{x,z_1}, \dots, \eta_{z_k y} \in E_{x_0,\zeta}^{(\text{ext})}$. Let

$$i = \operatorname{argmax}_{1 \leq j \leq k} |z_j - \overline{xy}|.$$

It follows that $z_i \in U_{z_{i-1}, z_{i+1}}$ where, for convenience, we write $z_0 = x$ and $z_{k+1} = y$. By Theorem D.2 again,

$$L(C_{z_{i-1}, z_i}^{x_0}) + L(C_{z_i, z_{i+1}}^{x_0}) \leq L(C_{z_{i-1}, z_{i+1}}^{x_0}). \quad (3.15)$$

By making a notation change due to $z'_j = z_j$ for $1 \leq j < i$ and $z'_j = z_{j+1}$ for $i \leq j \leq k - 1$, it follows from the previous inequality that

$$L(C_{x, z_1}^{x_0}) + \cdots + L(C_{z_k, y}^{x_0}) \leq L(C_{x, z'_1}^{x_0}) + \cdots + L(C_{z'_{k-1}, y}^{x_0}),$$

and hence, by our assumption for $n = k - 1$, we conclude with the statement of the claim (3.11) by induction. \square

Lemma 3.8. *Let $\delta > 0$ and $\Gamma = (V, E)$ be a spoked chain with $V \subset Q_1$. If Γ does not contain a kink, then the number of edges in E with length greater than 2δ is at most $6\left(\frac{R}{\delta}\right)^2$.*

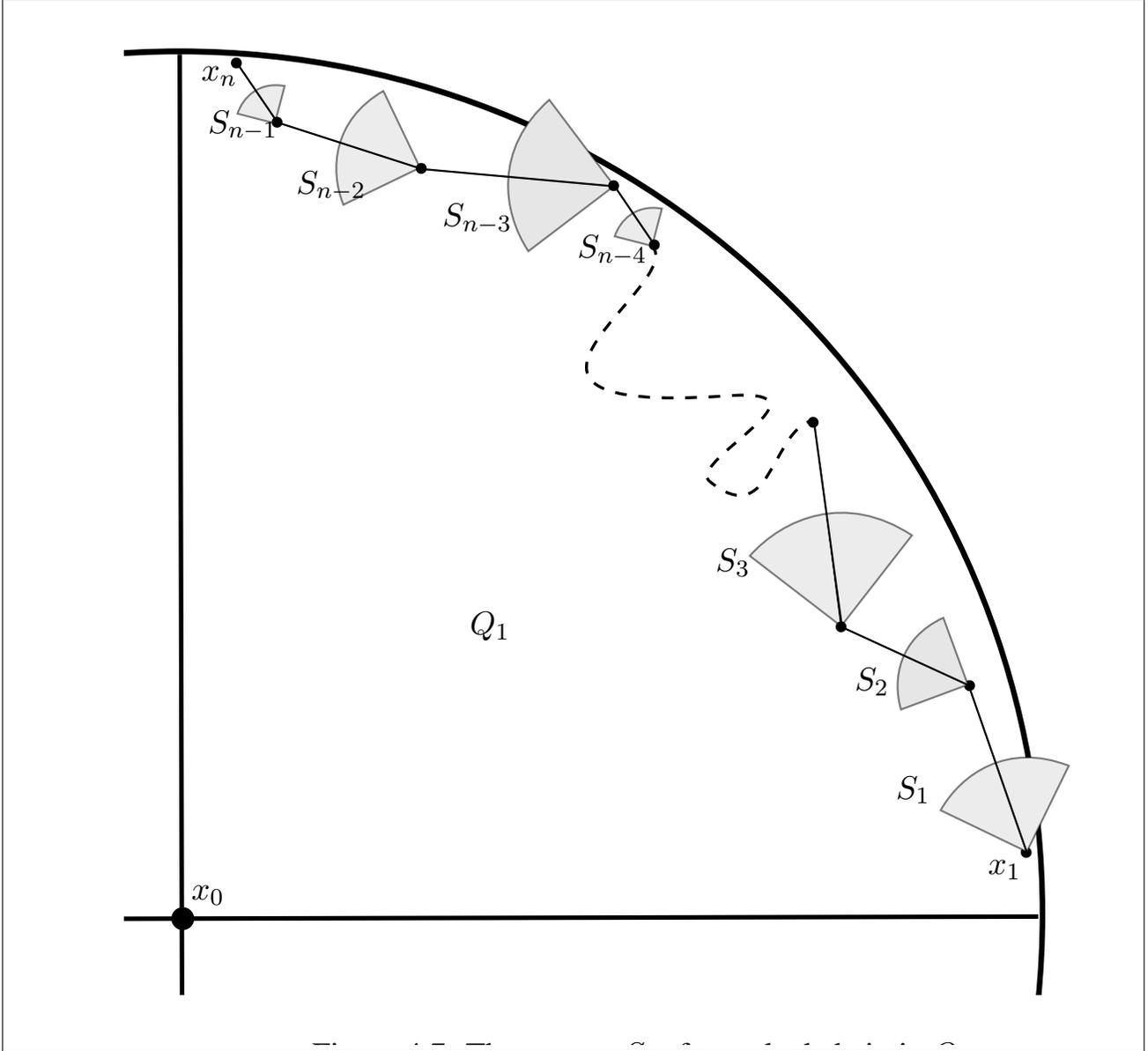


Figure 9: The sectors S_i of a spoked chain in Q_1 .

PROOF. Let $V = \bigcup_{i=1}^n \{x_i\}$ with $\hat{x}_1 < \dots < \hat{x}_n$. For $1 \leq i < n$, let $D_i \subset \mathbb{R}^2$ be the disc of radius $|x_i - x_{i+1}|/2$ centred at x_i . Let $S_i \subset \mathbb{R}^2$ be the sector of D_i with interior angle $\pi/2$ and line of symmetry $\overline{x_i x_{i+1}}$. We claim the following,

$$S_i \cap S_j = \emptyset \text{ for } i \neq j,$$

$$\bigcup_{i=1}^{n-1} S_i \subset Q_1 \oplus \frac{R}{\sqrt{2}}. \tag{3.16}$$

Assume our claim (3.16) is true, the sum of the areas of the sectors S_i must not exceed the area of $Q_1 \oplus \frac{R}{\sqrt{2}}$ which is less than $\frac{3}{2}\pi R^2$, see Figure 9. Now each edge $\eta \in E$ of length greater than 2δ contributes a sector of area greater than $\pi/4\delta^2$, therefore, the maximum number of

such edges in Γ is simply

$$\frac{3/2\pi R^2}{\pi/4\delta^2} = 6\left(\frac{R}{\delta}\right)^2,$$

which gives the result. We are left to prove our claim (3.16) above. Pick $x_i \in V$ and let ℓ_1 be the image of the line $\overleftrightarrow{x_i x_{i+1}}$ under a rotation with an angle $\pi/2$, centred at point x_{i+1} . There are exactly two connected components of $\mathbb{R}^2 \setminus \ell_1$. Let U denote the component that contains x_i . Now suppose $x_k \in U$ for some $i+1 < k \leq n$. This implies that $\widehat{x_i x_{i+1} x_k} < \pi/2$. Then, by Definition 3.1, this contradicts the fact that Γ does not contain a kink. Therefore, $x_k \in U^c$ for all $i+1 < k \leq n$. Let ℓ_2 and ℓ_3 be the images of the half line $\overleftarrow{x_i x_{i+1}}$ under rotations, centred at x_{i+1} , of angles $\pi/4$ and $-\pi/4$ respectively, see Figure 10.

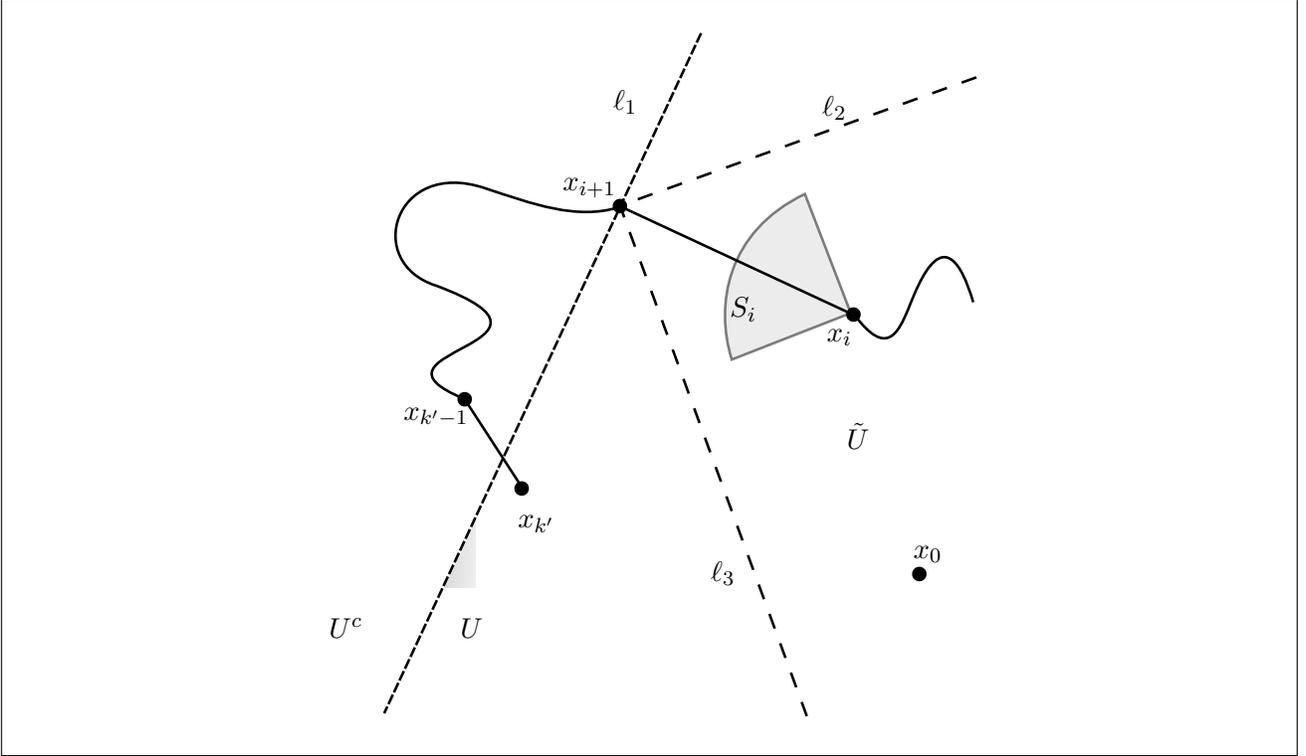


Figure 10: The point x'_k is the first time after x_{i+1} that the chain enters U .

Again, there are two connected components of $\mathbb{R}^2 \setminus (\ell_2 \cup \ell_3)$. Let \tilde{U} denote the one that contains x_i . The first part of the claim (3.16) follows by noticing that $S_i \subset \tilde{U}$ and $S_k \subset \tilde{U}^c$ for all $i+1 < k \leq n$. The second of the claim (3.16) is easily verified when we consider $S_i \subset D_i$ for all $1 \leq i < n$ and the maximal radius for D_i is half the maximal edge length, which, considering we are restricted to Q_1 , is $\sqrt{2}R$. □

3.4. Final proof of Theorem 2.1. Recall that we split $B = B_R(x_0) \subset \mathbb{R}^2$ into four quadrants, $Q_i \subset \mathbb{R}^2, i = 1, 2, 3, 4$. Now $\partial_B \cap Q_1$ contains all vertices and edges of ∂_B that lie wholly in Q_1 . By construction, $\partial_B \cap Q_1$ is a spoked chain. It follows from Lemma 3.4 and Lemma 3.5 that there are at most 2 intruding kinks in the spoked chain $\partial_B \cap Q_1$ and no single protruding kink. For each intruding kink x_i, x_j, x_k , we remove the edge $\eta_{x_j x_{j+1}}$ from $\partial_B \cap Q_1$. Since removing an

edge anywhere except from the end of the spoked chain will result in leaving two spoked chains, we are left with at most 3 spoked chains in Q_1 . Importantly, none of these contains an intruding or protruding kink. Let denote $\Gamma = (V^\Gamma, E^\Gamma)$ by one of these kinkless spoked chains in Q_1 . We denote (compare with Theorem 2.1) $N_\Gamma^{(\text{cc})}(\zeta, E)$ to be the number of connected components (clusters) of (ζ, E) that intersect V^Γ . We endeavour to bound the expectation of $N_\Gamma^{(\text{cc})}(\zeta, \cdot)$ with respect to the edge drawing mechanism $\mu_{(\text{ext}),\zeta}^{(g)}$ on $E_{x_0,\zeta}^{(\text{ext})}$ given in (2.4). To conclude the Theorem, we shall use

$$\int N_{x_0}^{(\text{cc})}(\zeta, E) \mu_{(\text{ext}),\zeta}^{(g)}(dE) \leq \int N_\Gamma^{(\text{cc})}(\zeta, E) \mu_{(\text{ext}),\zeta}^{(g)}(dE),$$

where the factor 12 is considering at most three kinkless spoked chains in each of the four quadrants. Order the elements in $V_\Gamma = \bigcup_{i=1}^n \{x_i\}$ such that $\hat{x}_1 < \dots < \hat{x}_n$. Recall that $\{x \leftrightarrow y\}$ denotes the event that x and y belong to the same cluster of (ζ, E) and notice that the following holds,

$$\begin{aligned} \int N_\Gamma^{(\text{cc})}(\zeta, E) \mu_{(\text{ext}),\zeta}^{(g)}(dE) &\leq 1 + \sum_{j=1}^{n-1} (1 - \mu_{(\text{ext}),\zeta}^{(g)}(\{x_j \leftrightarrow x_{j+1}\})) \\ &\leq 1 + \sum_{j=1}^{n-1} (1 - \tilde{\nu}_\zeta^{(\text{ext})}(\{x_j \leftrightarrow x_{j+1}\})) \\ &\leq 1 + \sum_{j=1}^{n-1} (1 - p^*(\eta_{x_j x_{j+1}})) \\ &\leq 1 + \sum_{\eta \in E^\Gamma} (1 - p^*(\eta)). \end{aligned} \tag{3.17}$$

We partition the edge set E^Γ of the spoked chain Γ into subsets of edges according to their lengths. Let

$$\begin{aligned} E_1 &= \{\eta_{xy} \in E^\Gamma : |x - y| > \frac{2}{\pi} \wedge R\}, \\ E_i &= \left\{ \eta_{xy} \in E^\Gamma : \frac{2/\pi \wedge}{i} < |x - y| \leq \frac{2/\pi \wedge R}{i-1} \right\}, \quad i \geq 2, i \in \mathbb{N}. \end{aligned}$$

By recalling that

$$p^*(\eta_{xy}) = \frac{\mathbb{1}\{\ell(\eta) \leq \frac{2}{\pi} \wedge R\} \mathbb{1}\{|\hat{x} - \hat{y}| \leq \frac{\pi}{2}\}}{\frac{g}{\beta} \left(\frac{\pi}{2} \ell(\eta)\right)^4 + 1} \mathbb{1}_{E_B}(\eta)$$

from (3.8), we see that $1 - p^*(\eta) = 1$ for all $\eta \in E_1$. However, since Γ is contained in Q_1 , and henceforth $|\hat{x} - \hat{y}| < \frac{\pi}{2}$, we have

$$1 - p^*(\eta_{xy}) = \frac{1}{\frac{g}{\beta} \left(\frac{\pi}{2} |x - y|\right)^4 + 1}$$

for all $\eta_{xy} \in E_i, i \geq 2$. Let $r := 1 \wedge \frac{R\pi}{2}$. Then, considering $|x - y| \leq \frac{2r}{\pi(i-1)}$ for all $\eta_{xy} \in E_i$ and noticing that $\bigcup_{i=1}^{\infty} E_i = E^\Gamma$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, it follows readily that

$$\begin{aligned} \sum_{\eta \in E^\Gamma} (1 - p^*(\eta)) &= \sum_{i=1}^{\infty} \sum_{\eta \in E_i} (1 - p^*(\eta)) \leq \sum_{\eta \in E_1} 1 + \sum_{i=2}^{\infty} \sum_{\eta \in E_i} \left(\frac{1}{\frac{\beta}{q} \left(\frac{i-1}{r} \right)^4 + 1} \right) \\ &\leq 6R^2 \pi^2 r^{-2} + \sum_{i=2}^{\infty} (6R^2 \pi^2 i^2 r^{-2}) \left(\frac{1}{\frac{\beta}{q} \left(\frac{i-1}{r} \right)^4 + 1} \right) \\ &\leq 6R^2 \pi^2 r^{-2} \left(1 + \frac{q}{\beta} \sum_{i=2}^{\infty} \frac{i^2}{(i-1)^4} \right), \end{aligned}$$

where the second inequality comes from an application of Lemma 3.8 to each of the single edges sets E_i . We use that

$$\sum_{i=2}^{\infty} \frac{i^2}{(i-1)^4} \leq \sum_{i=2}^{\infty} \frac{4}{(i-1)^2} = 4 \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{2}{3} \pi^2. \quad (3.18)$$

Combining all our previous steps we obtain from (3.17) with the two previous bounds that

$$\begin{aligned} \int N_\Gamma^{(\text{cc})}(\zeta, E) \mu_{(\text{ext}), \zeta}^{(q)}(dE) &\leq 1 + 6R^2 \pi^2 r^{-2} \left(1 + \frac{2q\pi^2}{3\beta} \right) \\ &= 1 + 6(4 \vee \pi^2) \left(1 + \frac{2q\pi^2}{2\beta} \right). \end{aligned}$$

We finish the proof of Theorem 2.1 by setting $\alpha := 12(6(4 \vee \pi^2)[1 + \frac{2q\pi^2}{3\beta}])$.

□

4. PROOFS

This section is devoted to the remaining open proofs of our results. We first establish the existence of a Gibbs measure in our models.

4.1. Existence of Gibbs measures. To show the existence of Gibbs measures (Proposition 1.2) for our Delaunay Potts model we will show that it satisfies adapted versions of the conditions necessary for the existence outlined in [DDG12]. The potential ϕ depends solely on the Delaunay hyperedges $\text{Del}_2(\zeta)$, of a marked configuration ζ . Every marked hyperedge $\eta \in \text{Del}_2(\zeta)$ has the so-called finite horizon $\overline{B}(\eta, \zeta)$ where $B(\eta, \zeta)$ is the open ball with $\partial B(\eta, \zeta) \cap \zeta = \eta$ that contains no points of ζ . The finite-horizon property of a general hyperedge potential $\varphi: \Omega_f \times \Omega \rightarrow \mathbb{R}$ says that for each pair (η, ζ) with $\eta \in \text{Del}_m(\zeta)$ there exists some $\Delta \in \mathbb{R}^2$ such that for the pair $(\eta, \tilde{\zeta})$ with $\eta \in \text{Del}_2(\tilde{\zeta})$ we have that $\varphi(\eta, \zeta) = \varphi(\eta, \tilde{\zeta})$ when $\tilde{\zeta} = \zeta$ on $\Delta \equiv \overline{B}(\eta, \zeta)$. As our potential depends solely on the single hyperedge it shares this finite-horizon property. Thus it satisfies the range condition (R) in [DDG12], see [DDG12, Proposition 4.1 & 4.3]. The second requirement is the usual stability condition (S). A hyperedge potential φ is called stable if there is a lower bound for the Hamiltonian for any $\Lambda \in \mathbb{R}^2$, and, as $\phi(\ell) \geq 0$ for all $\ell \geq 0$ the stability condition (S) is satisfied. The third condition to be checked is a partial complementary upper bound for the Hamiltonian in any $\Lambda \in \mathbb{R}^2$. This is a bit more involved and we shall first define appropriate configurations, the so-called pseudo-periodic marked configurations. We consider the partition of \mathbb{R}^2 as given in Appendix A. Note

that in Appendix A we have introduced a length scale $\ell > 0$ which is not necessary for the existence proof as we could put $\ell = 1$. We let $B(0, r)$ be an open ball of radius $r \leq \rho_0 \ell$ where we choose $\rho_0 \in (0, 1/2)$ sufficiently small such that $B(0, r) \subset \Delta_{0,0}$. Note that

$$B^r := \{\zeta \in \Omega_{\Delta_{0,0}} : \zeta = \{x\} \text{ for some } x \in B(0, r)\}$$

is a measurable set of $\Omega_{\Delta_{0,0}} \setminus \{\emptyset\}$. Then

$$\Gamma^r = \{\omega \in \Omega : \theta_{Mz}(\omega_{\Delta_{k,l}}) \in B^r \text{ for all } (k, l) \in \mathbb{Z}^2\}$$

is the set of pseudo-periodic configurations (A.2). Note that these configurations are not marked. The reason is that when a point is shifted its mark remains unchanged. Thus we define the set of pseudo-periodic marked configurations as

$$\mathbf{\Gamma}^r = \{\boldsymbol{\omega} = (\omega^{(1)}, \dots, \omega^{(q)}) : \omega^{(i)} \in \Gamma^r \text{ for all } i \in E\}.$$

The required control of the Hamiltonian from above will be achieved by the following properties. As our hyperedge potential depends only on the single hyperedge the so-called uniform confinement (see [DDG12]) is trivially satisfied. In addition we need the uniform summability, that is,

$$c_r := \sup_{\zeta \in \Gamma^r} \sum_{\eta \in \text{Del}_2(\zeta) : \eta \cap \Delta \neq \emptyset} \frac{\varphi(\eta, \zeta)}{\#\hat{\eta}} < \infty,$$

where $\hat{\eta} = \{(k, l) \in \mathbb{Z}^2 : \eta \cap \Delta_{k,l} \neq \emptyset\}$ and where $\Delta = \Delta_{0,0}$. Note that the length $\ell(\eta)$ of any $\eta \in \text{Del}_2(\zeta) \cap \Delta$ when ζ is any pseudo-periodic configuration satisfies $\ell(1 - 2\rho_0) \leq \ell(\eta) \leq \ell(1 + 2\rho_0)$. There are at most six edges from the centre ball in $\Delta = \Delta_{0,0}$ and each Delaunay edge touches exactly two cells and thus $\hat{\eta} = 2$. We obtain an upper bound for each edge by considering the shortest possible length for each edge, that is,

$$c_r = 3 \log \left(\frac{(\ell(1 - 2\rho_0))^4 + \beta}{(\ell(1 - 2\rho_0))^4} \right) < \infty.$$

We need furthermore the so-called weak non-rigidity, that is $\mathbf{\Pi}_{\Delta_{0,0}}(\mathbf{\Gamma}^r) = qe^{-|\Delta_{0,0}|z} > 0$. Using [DDG12, Theorem 3.3] and [DDG12, Corollary 3.4] we obtain all the statements in Proposition 1.2. \square

4.2. Breaking of the symmetry of the mark distribution. In this section we complete the proof of Theorem 1.4 by analysing the Gibbs distributions $\gamma_{\Lambda, \omega}$ in the limit $\Lambda \uparrow \mathbb{R}^2$. We pick an admissible boundary condition $\omega \in \Omega_{\Lambda_n}^*$ and let $\boldsymbol{\omega} = (\omega \setminus \Lambda_n, \emptyset, \dots, \emptyset)$ be the admissible monochromatic boundary condition such that $\boldsymbol{\omega} \in \Omega_{\Lambda_n}^*$. We write γ_n for $\gamma_{\Lambda_n, \omega}$ for ease of notation and we let P_n be the probability measure on Ω relative to which the marked configurations in distinct parallelotopes $\Lambda_n + (2n+1)M(k, l)$, $(k, l) \in \mathbb{Z}^2$, are independent with identical distribution γ_n . As we are dealing with a cell structure for the partition of \mathbb{R}^2 we confine ourself first to lattice shifts when we employ spatial averaging. Thus,

$$\bar{P}_n = \frac{1}{2n+1} \sum_{(k,l) \in \{-n, \dots, n\}^2} P_n \circ \theta_{M(k,l)}^{-1}.$$

By the periodicity of P_n the measure \bar{P}_n is \mathbb{Z}^2 -shift-invariant. The proof in [DDG12, Chapter 5] shows that $(\bar{P}_n)_{n \geq 1}$ has a subsequence which converges with respect to the topology of local convergence to some $\hat{P} \in \mathcal{M}_1(\Omega)$. As outlined in [DDG12] it is difficult to show that \hat{P} is concentrated on admissible configurations. As \hat{P} is non-degenerate the proof in [DDG12,

Chapter 5] shows that $P = \widehat{P}(\cdot | \{\emptyset\}^c)$ is a Gibbs measure with $P(\{\emptyset\}) = 0$. In order to obtain an \mathbb{R}^2 -shift-invariant Gibbs measure one needs to apply another averaging,

$$P^{(1)} = \int_{\Delta_{0,0}} P \circ \theta_{Mx}^{-1} dx.$$

Applying Propositions 2.9 and 2.2, we see that for $\Delta = \Delta_{0,0}$,

$$\begin{aligned} \int (qN_{\Delta,1} - N_{\Delta}) d\bar{P}_n &\geq \frac{(q-1)}{2n+1} \sum_{(k,l) \in \{-n, \dots, n\}^2} \int N_{\Delta_{k,l} \leftrightarrow \infty} dC_{\Lambda_n, \omega} \\ &\geq (q-1)\varepsilon. \end{aligned}$$

Thus

$$\int (qN_{\Delta,1} - N_{\Delta}) dP^{(1)} > 0,$$

and using the symmetry of $P^{(1)}$ we observe the following break of symmetry in the expected density of particles of type 1 and of any other different type, i.e.,

$$\rho_1(P^{(1)}) > \rho_2(P^{(1)}) = \dots = \rho_q(P^{(1)}),$$

where $\rho_s(P^{(1)}) = 1/|\Delta| \mathbb{E}_{P^{(1)}}[N_{\Delta,s}]$, $s \in E$. We conclude with our statement as in [GH96] by showing that the matrix

$$(\rho_s(P^{(t)}))_{s,t \in E}$$

is regular, where $P^{(t)}$ is obtained from $P^{(1)}$ by swapping the role of 1 and t .

APPENDIX

A. PSEUDO-PERIODIC CONFIGURATIONS

We define pseudo-periodic configurations as in [DDG12]. We obtain a partition of \mathbb{R}^2 which is adapted to the Delaunay tessellation. Pick a length scale $\ell > 0$ and consider the matrix

$$M = (M_1 \quad M_2) = \begin{pmatrix} \ell & \ell/2 \\ 0 & \sqrt{3}/2\ell \end{pmatrix}.$$

Note that $|M_i| = \ell$, $i = 1, 2$, and $\angle(M_1, M_2) = \pi/3$. For each $(k, l) \in \mathbb{Z}^2$ we define the cell

$$\Delta_{k,l} = \{Mx \in \mathbb{R}^2 : x - (k, l) \in [-1/2, 1/2)^2\}. \quad (\text{A.1})$$

These cells together constitute a periodic partition of \mathbb{R}^2 into parallelotopes. Let B be a measurable set of $\Omega_{\Delta_{0,0}} \setminus \{\emptyset\}$ and

$$\Gamma = \{\omega \in \Omega : \theta_{Mz}(\omega_{\Delta_{k,l}}) \in B \text{ for all } (k, l) \in \mathbb{Z}^2\} \quad (\text{A.2})$$

the set of all configurations whose restriction to an arbitrary cell, when shifted back to $\Delta_{0,0}$, belongs to B . Elements of Γ are called **pseudo-periodic** configurations. We define marked pseudo-periodic configurations in an analogous way.

B. TOPOLOGY OF LOCAL CONVERGENCE

We write $\mathcal{M}_1^\theta(\Omega)$ (resp. $\mathcal{M}_1^\theta(\Omega)$) for the set of all shift-invariant probability measures on (Ω, \mathcal{F}) (resp. (Ω, \mathcal{F})). A measurable function $f: \Omega \rightarrow \mathbb{R}$ is called local and tame if

$$f(\omega) = f(\omega_\Lambda) \quad \text{and} \quad |f(\omega)| \leq aN_\Lambda(\omega) + b$$

for all $\omega \in \Omega$ and some $\Lambda \Subset \mathbb{R}^2$ and suitable constants $a, b \geq 0$. Let \mathcal{L} be the set of all local and tame functions. The topology of local convergence, or \mathcal{L} -topology, on $\mathcal{M}_1^\theta(\Omega)$ is then defined as the weak* topology induced by \mathcal{L} , i.e., as the smallest topology for which the mappings $P \mapsto \int f dP$ with $f \in \mathcal{L}$ are continuous.

C. MIXED SITE-BOND PERCOLATION

Given a graph $G = (V, E)$, let \mathbb{P}_p be the probability measure on configurations of open and closed sites of G . Each site of G is open with probability p and closed with probability $1 - p$. Similarly, let $\tilde{\mathbb{P}}_p$ be the probability measure on configurations of open and closed edges of G . Each edge of G is open with probability p and closed with probability $1 - p$. For $x_0 \in V$ and a subset of vertices $X \subset V$, let

$$\begin{aligned} \sigma(p, x_0, X, G) &= \mathbb{P}_p(\exists \text{ a path } x_0 = v_0, e_1, v_1, \dots, e_n, v_n \text{ with } v_n \in X \text{ and all vertices are open}), \\ \beta(p, x_0, X, G) &= \tilde{\mathbb{P}}_p(\exists \text{ a path } x_0 = v_0, e_1, v_1, \dots, e_n, v_n \text{ with } v_n \in X \text{ and all edges are open}). \end{aligned}$$

It is known since [Kes82] that site percolation implies bond percolation, that is, for any $0 \leq p \leq 1$,

$$\sigma(p, x_0, X, G) \leq \beta(p, x_0, X, G). \tag{C.1}$$

In mixed site-bond percolation, both edges and vertices may be open or closed, possibly with different probabilities. Each edge or bond is open independently of anything else with probability p' and each site is open independently of anything else with probability p . The edges and sites that are not open, along with the edges to or from these sites, are closed. We shall consider oaths of open sites and open edges. For $x_0 \in V$ and a subset of vertices $X \subset V$, let

$$\begin{aligned} \gamma(p, p', x_0, X, G) &= \mathbb{P}_{pp'}(\exists \text{ a path } x_0 = v_0, e_1, v_1, \dots, e_n, v_n \text{ with } v_n \in X \\ &\quad \text{and all vertices and all edges are open}). \end{aligned}$$

Let G' be the reduced graph where each edge and site of G is removed independently with probability $1 - p'$ and $1 - p$ respectively. By taking the expectation on both sides of inequality (C.1), on G' , with respect to \mathbb{P}_δ and $\tilde{\mathbb{P}}_\delta$, we arrive at the mixed site-bond percolation result of Hammersley, a generalisation of the work of McDiarmid, see [Ham80]. That is, for $\delta, p, p' \in [0, 1]$ one gets that

$$\gamma(\delta p, p', x_0, X, G) \leq \gamma(p, \delta p', x_0, X, G). \tag{C.2}$$

By setting $\delta = p$ and $p' = 1$ in (C.2), and noticing that $\gamma(p^2, 1, x_0, X, G) = \sigma(p^2, X_0, X, G)$, we arrive at

$$\sigma(p^2, x_0, X, G) \leq \gamma(p, p, x_0, X, G), \tag{C.3}$$

and hence

$$\theta_{(\text{mixed})}(p, p) \geq \theta_{(\text{site})}(p^2), \tag{C.4}$$

where $\theta_{(\text{mixed})}(p, p')$ is the mixed site-bond percolation probability with parameters p and p' , and $\theta_{(\text{site})}(p)$ is the site percolation probability with parameter p .

ACKNOWLEDGMENTS

S.A. thanks D. Dereudre for helpful discussions and exchange of ideas visiting Lille.

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