

Extension of Caputo evolution equations with time-nonlocal initial condition

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Presenting the work...



T. (2018).

Stochastic classical solutions for space-time fractional evolution equations on bounded domain.

To appear in: J Math Anal Appl. arXiv: 1805.02464.

And time permitting



Du, T., Zhou (2018).

Stochastic solutions for time-nonlocal evolution equations.

Submission: Sept. 2018.



Hernández-Hernández, Kolokoltsov, T. (2017).

Generalised fractional evolution equations of Caputo type.

Chaos, Solitons & Fractals, 102 (2017): 184-196.

Main idea

Let $\partial_{t,\infty}^\beta$ be the Marchaud derivative (extension of Caputo derivative, $\beta \in (0, 1)$).

Consider the *extension* of Caputo evolution equations with **time-nonlocal initial condition**

$$\begin{cases} \partial_{t,\infty}^\beta u(t, x) = \Delta u(t, x), & \text{in } (0, T] \times \mathbb{R}^d, \\ u(t, x) = \phi(t, x), & \text{in } (-\infty, 0] \times \mathbb{R}^d. \end{cases} \quad (1)$$

The **stochastic representation** is

$$u(t, x) = \mathbb{E} \left[\phi \left(-W(t), B_{E(t)}^x \right) \right].$$

Here $W(t)$ is the waiting time of $B_{E(t)}^x$ (the fractional kinetic process).

Question: Are time-nonlocal initial conditions meaningful for applications?

Overview

- 1 Marchaud evolution equation
- 2 Stochastic representation
 - Intuition
 - Motivation
- 3 Proof of Theorem
- 4 Generalised Marchaud evolution equations

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Caputo evolution equation (EE)

Consider the **Caputo** evolution equation

$$\begin{cases} \partial_{t,0}^\beta u(t, x) = \Delta u(t, x), & \text{in } (0, T] \times \mathbb{R}^d, \\ u(0, x) = \phi(0, x), & \text{in } \{0\} \times \mathbb{R}^d. \end{cases} \quad (2)$$

where the Caputo derivative $\partial_{t,0}^\beta$, $\beta \in (0, 1)$, is defined as

$$\partial_{t,0}^\beta u(t) := \int_0^t u'(r) \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)}. \quad (3)$$

It is well known that the stochastic solution reads

$u(t, x) = \mathbb{E}[\phi(0, B_{E(t)}^x)]$, where B is a Brownian motion and $E(t)$ is an independent inverse β -stable subordinator [Saichev, Zaslavsky '97], [Beaumer, Meerschaert '01], [Meerschaert, Scheffler '04].

Notable properties of the fractional kinetic $Y_t = B_{E(t)}^x$:

- 1 **Universality:** Y_t is the quenched scaling limit of random conductance models [Barlow, Černý '11]. Note that Y_t is non-Markovian process (with memory), but it is the limit of Markovian processes (without memory).
- 2 **Subdiffusion:** Mean squared displacement $\mathbb{E}[Y_t^2] = t^\beta < t = \mathbb{E}[B_t^2]$.
- 3 **A model for trappings:** the continuous non-Markovian time change $E(t)$ is i.o. constant on time intervals.
- 4 **Universality for $\beta = 1/2$:** Y_t is the intermediate time behaviour of perturbed cellular flows [Hairer et al. '18].
- 5 **Connection to 4th order PDEs:** $Y_t, \beta = 1/2$ is the fundamental solution to $\partial_t u = \Delta^2 u + \Delta \phi(0)/\sqrt{\pi t}$ [Meerschaert, Nane '09] (hence the solution is positivity preserving).

Marchaud to Caputo derivative

Consider the **Marchaud** derivative

$$\partial_{t,\infty}^\beta u(t) := \int_{-\infty}^t u'(r) \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)}. \quad (4)$$

If $u(t) = u(0)$ for all $t < 0$ the **Marchaud** derivative equals the **Caputo** derivative, as

$$\partial_{t,\infty}^\beta u(t) = \int_0^t u'(r) \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)} = \partial_{t,0}^\beta u(t).$$

Probabilistically $-\partial_{t,\infty}^\beta$ is the generator of the inverted β -stable-subordinator $-X_s^\beta$, easily observed from the representation

$$-\partial_{t,\infty}^\beta u(t) = \int_0^\infty (u(t-r) - u(t)) \frac{r^{-1-\beta} dr}{-\Gamma(-\beta)}. \quad (5)$$

Marchaud to Caputo evolution equation

Consider the **Marchaud** evolution equation

$$\begin{cases} \partial_{t,\infty}^\beta u(t, x) = \Delta u(t, x), & \text{in } (0, T] \times \mathbb{R}^d, \\ u(t, x) = \phi(t, x), & \text{in } (-\infty, 0] \times \mathbb{R}^d. \end{cases} \quad (6)$$

Then, if $\phi(t) = \phi(0)$ for all $t < 0$, then $\partial_{t,\infty}^\beta = \partial_{t,0}^\beta$ the EE (6) becomes the standard **Caputo** EE.

- The Marchaud EE (6) is the natural fractional counterpart the time-nonlocal evolution equations proposed in [Chen, Du, Li, Zhou '17] and [Du, Yang, Zhou '17]. In [Allen '17] uniqueness of weak solutions is considered.
- With a little extra work, existence/regularity results follow from results about inhomogeneous Caputo EEs, such as [Allen, Caffarelli, Vasseur '16], [Baeumer, Kurita, Meerschaert '05].

The Theorem

Here is a rough statement of the main result.

Theorem

Assuming certain regularity on ϕ , there exists a unique classical solution to the Marchaud EE

$$\begin{cases} \partial_{t,\infty}^\beta u(t, x) = \Delta u(t, x), & \text{in } (0, T] \times \mathbb{R}^d, \\ u(t, x) = \phi(t, x), & \text{in } (-\infty, 0] \times \mathbb{R}^d. \end{cases} \quad (7)$$

Moreover, the solution allows the stochastic representation

$$u(t, x) = \mathbb{E} \left[\phi \left(-W(t), B_{E(t)}^x \right) \right], \quad (8)$$

where $W(t)$ is the waiting/trapping time of the fractional kinetic process $B_{E(t)}^x$.

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Stochastic representation: Intuition

Denote by $Y_t^x = B_{E(t)}^x$ the fractional kinetic process.

The solution

$$\mathbb{E} \left[\phi \left(-W(t), Y_t^x \right) \right]$$

weights the initial condition with respect to the duration of the holding time $W(t)$ of the process Y_t .

Example

The initial condition $\phi(t, x) = \mathbf{1}_{(-\infty, -1]}(t) \tilde{\phi}(x)$, results in

$$\mathbb{E} \left[\tilde{\phi} \left(Y_t^x \right) \mid Y_t^x \text{ is trapped for more than 1 time-unit} \right],$$

We will now plot $t \mapsto (-W(t), Y_t) \in (-\infty, 0] \times \mathbb{R}^2$, where the values of $-W(t)$ are thought of as the depth underneath a surface $\{0\} \times \mathbb{R}^2$ (and not the past).

Brownian motion on the sea surface: $(0, B_t) \in \{0\} \times \mathbb{R}^2$

Fractional kinetic on the sea surface: $(0, Y_t) \in \{0\} \times \mathbb{R}^2$

Fractional kinetic in the sea: $(-W(t), Y_t) \in (-\infty, 0] \times \mathbb{R}^2$

Key remark

Probabilistically $-\partial_{t,\infty}^\beta$ is the generator of the inverted β -stable-subordinator $-X_s^\beta$, easily observed from the representation

$$-\partial_{t,\infty}^\beta u(t) = \int_0^\infty (u(t-r) - u(t)) \frac{r^{-1-\beta} dr}{-\Gamma(-\beta)}. \quad (9)$$

Stochastic representation: Motivation

$$\begin{cases} \mathcal{G}u = 0, & \text{in } \Omega, \\ u = \phi, & \text{in } \partial\Omega, \end{cases} \quad \mathcal{G} \text{ Markovian generator of } G_s$$

should be solved by $u(\omega) = \mathbb{E} \left[\phi \left(G_{\tau_{\partial\Omega}(\omega)}^\omega \right) \right]$, where

$\tau_{\partial\Omega}(\omega) := \inf\{s : G_s^\omega \notin \Omega\}$.

Now set $\mathcal{G} \equiv (-\partial_{t,\infty}^\beta + \Delta)$, $\Omega \equiv (0, T] \times \mathbb{R}^d$, and

$\partial\Omega \equiv (-\infty, 0] \times \mathbb{R}^d$. Then

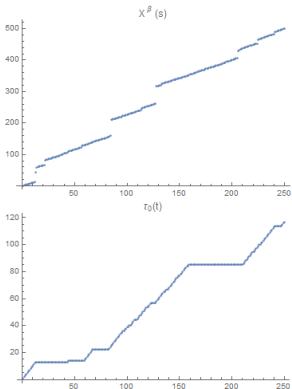
$$G_s^\omega = (t - X_s^\beta, B_s^x), \quad t - X^\beta \perp B^x, \quad \omega = (t, x)$$

$$\tau_{\partial\Omega}(\omega) = \tau_0(t) := \inf\{s : t - X_s^\beta \leq 0\} = \inf\{s : t < X_s^\beta\} =: E(t)$$

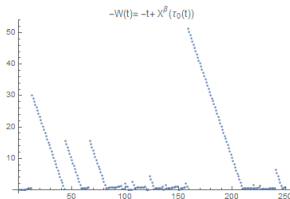
$$u(t, x) = \mathbb{E} \left[\phi \left(t - X_{\tau_0(t)}^\beta, B_{\tau_0(t)}^x \right) \right] = \mathbb{E} \left[\phi \left(-W(t), B_{E(t)}^x \right) \right],$$

where $W(t)$ is the waiting time of $B_{\tau_0(t)}^x$.

Stochastic representation: Motivation $X_{\tau_0(t)}^\beta - t = W(t)$



(a) X^β and $t \mapsto \tau_0(t) = E(t)$



(b) $t \mapsto W(t)$

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Definition of classical solution for Marchaud EE

Definition

- 1 $u \in C_{b,\partial\Omega}((-\infty, T] \times \Omega) \cap C^{1,2}((0, T) \times \Omega)$,
- 2 $\partial_t u \in L^1((0, T] \times \Omega)$
- 3 $u(t, x) \rightarrow \phi(0, x)$, as $t \downarrow 0$, for each $x \in \Omega$, and
- 4 u satisfies

$$\begin{cases} \partial_{t,\infty}^\beta u(t, x) = -(-\Delta^{\frac{\alpha}{2}})u(t, x), & \text{in } (0, T] \times \Omega, \\ u(t, x) = \phi(t, x), & \text{in } (-\infty, 0) \times \Omega, \\ u(t, x) = 0, & \text{in } (0, T] \times \Omega^c, \end{cases} \quad (10)$$

for a given time-nonlocal initial condition ϕ , where $-(-\Delta^{\frac{\alpha}{2}})$, $\alpha \in (0, 2)$ is the fractional Laplacian.

Theorem's statement

Let B^x be the rotationally symmetric α -stable Lévy process killed on exiting Ω , $\alpha \in (0, 2)$.

Theorem (T. '18)

Let $\Omega \subset \mathbb{R}^d$ be a regular set. Assume that $\phi \in C_{b, \partial\Omega}^1((-\infty, 0]; \text{Dom}((-\Delta^{\frac{\alpha}{2}})^k))$, for some $k > -1 + (3d + 4)/(2\alpha)$, and $\partial_t \phi$ is Lipschitz at 0.

Then

$$u(t, x) = \mathbb{E} \left[\phi \left(-W(t), B_{\tau_0}^x(t) \right) \right]$$

is the unique classical solution to the **Marchaud** EE (10).

The heat kernel is

$$H_{\beta, \alpha}^{t, x}(r, y) = \int_0^t \frac{-\Gamma(-\beta)^{-1}}{(z-r)^{1+\beta}} \left(\int_0^\infty p_s^\Omega(x, y) p_s^\beta(t-z) ds \right) dz,$$

where $p_s^\Omega(x)$ is the law of B_s^x and p_s^β is the law of X_s^β .

Proof: Rewrite Marchaud EE as an inhomogeneous Caputo EE

Observe that if u equals ϕ for $t \leq 0$, then for $t > 0$

$$\begin{aligned} \partial_{t,\infty}^\beta u(t) &= \int_0^t u'(r) \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)} - \int_{-\infty}^0 \phi'(r) \frac{(t-r)^{-\beta} dr}{-\Gamma(1-\beta)} \\ &= \partial_{t,0}^\beta u(t) - f_\phi(t), \end{aligned}$$

and so we solve the inhomogeneous Caputo EE

$$\begin{cases} \partial_{t,0}^\beta u(t, x) = \Delta u(t, x) + f_\phi(t, x), & \text{in } (0, T] \times \mathbb{R}^d, \\ u(t, x) = \phi(0, x), & \text{in } \{0\} \times \mathbb{R}^d. \end{cases}$$

(In short: (Caputo, IC = $\phi(0)$, FT = f_ϕ).)

Proof: Obtain the stochastic representation (1)

The stochastic representation for the inhomogeneous EE (Caputo, IC = $\phi(0)$, FT = f_ϕ) is expected to be

$$u(t, x) = \mathbb{E} \left[\phi \left(0, B_{\tau_0(t)}^x \right) \right] + \mathbb{E} \left[\int_0^{\tau_0(t)} f_\phi \left(t - X_s^\beta, B_s^x \right) ds \right].$$

Now note that for ϕ extended to $\phi(0)$ on $(0, T]$

$$-f_\phi(t) = \int_{-\infty}^t \phi'(r) \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)} = -\partial_{t,\infty}^\beta \phi(t),$$

and by Dynkin formula

$$\mathbb{E} \left[\phi \left(0, B_{\tau_0(t)}^x \right) \right] = \phi(0, x) + \mathbb{E} \left[\int_0^{\tau_0(t)} \Delta \phi \left(t - X_s^\beta, B_s^x \right) ds \right].$$

Proof: Obtain the stochastic representation (2)

Recombining and by Dynkin formula the solution to (Caputo, IC = $\phi(0)$, FT = f_ϕ)

$$\begin{aligned} u(t, x) &= \mathbb{E} \left[\phi \left(0, B_{\tau_0(t)}^x \right) \right] + \mathbb{E} \left[\int_0^{\tau_0(t)} f_\phi \left(t - X_s^\beta, B_s^x \right) ds \right] \\ &= \phi(0, x) + \mathbb{E} \left[\int_0^{\tau_0(t)} \left(-\partial_{t,\infty}^\beta + \Delta \right) \phi \left(t - X_s^\beta, B_s^x \right) ds \right] \\ &= \mathbb{E} \left[\phi \left(t - X_{\tau_0(t)}^\beta, B_{\tau_0(t)}^x \right) \right] = u(t, x), \end{aligned}$$

the solution to (Marchaud, IC = ϕ).

Proof: Small summary

- 1 Solutions to (Marchaud, IC = ϕ) = solutions to (Caputo, IC = $\phi(0)$, FT = f_ϕ).
- 2 Feynman-Kac for (Marchaud, IC = ϕ) = Feynman-Kac for (Caputo, IC = $\phi(0)$, FT = f_ϕ).

Theorem (T. '18)

And so, as the unique classical solution to (Caputo, IC = $\phi(0)$, FT = f) is

$$u(t, x) = \mathbb{E} \left[\phi \left(0, B_{\tau_0(t)}^x \right) \right] + \mathbb{E} \left[\int_0^{\tau_0(t)} f \left(t - X_s^\beta, B_s^x \right) ds \right],$$

if $\phi(0) \in \text{Dom}((-\Delta^{\frac{\alpha}{2}})^k)$, $f \in C^1([0, T]; \text{Dom}((-\Delta^{\frac{\alpha}{2}})^k))$, for some $k > -1 + (3d + 4)/(2\alpha)$,

simply select ϕ such that $f_\phi \in C^1([0, T]; \text{Dom}((-\Delta^{\frac{\alpha}{2}})^k))$.

Proof: Plan for (Caputo EE, IC= $\phi(0)$, FT= f)

- 1 **Prove that the candidate stochastic representation is a weak solution:**
using BVP point of view in the motivation slide, not discussed.
- 2 **Prove smoothness of the candidate stochastic representation:**
extends [Chen, Meerschaert, Nane '12] using separation of variables.
- 3 **Uniqueness of classical solution:**
easy by separation of variables, not discussed.

(2) Homogeneous term as in [Chen, Meerschaert, Nane '12]

Denote by $\{\lambda_n, \varphi_n : n \geq 1\}$ the eigenvalue/eigenfunctions of the restricted fractional Laplacian $(-\Delta^{\alpha/2})_{\Omega}$. Then

$$\begin{aligned}\mathbb{E} \left[\phi \left(0, B_{\tau_0(t)}^x \right) \right] &= \int_0^\infty \mathbb{E}[\phi(B_s^x)] d_s \mathbb{P}[\tau_0(t) \leq s] \\ &= \sum_{n \geq 1} \langle \phi, \varphi_n \rangle \varphi_n(x) \int_0^\infty e^{-\lambda_n s} d_s \mathbb{P}[\tau_0(t) \leq s] \\ &= \sum_{n \geq 1} \langle \phi, \varphi_n \rangle \varphi_n(x) \mathbb{E}[e^{-\lambda_n \tau_0(t)}],\end{aligned}$$

where $\mathbb{E}[e^{-\lambda_n \tau_0(t)}] = E_\beta(\lambda_n t^\beta) := \sum_{m \geq 0} \frac{(-\lambda_n t^\beta)^m}{\Gamma(m\beta + 1)}$, the Mittag-Leffler function that solves the homogeneous Caputo IVP

$$\partial_{t,0}^\beta g(t) = -\lambda_n g(t), \quad g(0) = 1.$$

(2) Inhomogeneous term

For the inhomogeneous term we compute

$$\begin{aligned}
 & \mathbb{E} \left[\int_0^{\tau_0(t)} f \left(t - X_s^\beta, B_s^x \right) ds \right] \\
 &= \sum_{n \geq 0} \varphi_n(x) \mathbb{E} \left[\int_0^{\tau_0(t)} e^{-\lambda_n s} \langle f \left(t - X_s^\beta \right), \varphi_n \rangle ds \right] \\
 &= \sum_{n \geq 0} \varphi_n(x) E_{\beta, \lambda_n} \star \langle f(\cdot), \varphi_n \rangle(t),
 \end{aligned}$$

where the Mittag-Leffler convolution

$$E_{\beta, \lambda} \star \langle f, \varphi_n \rangle(t) \equiv -\lambda_n^{-1} \int_0^t \langle f(r), \varphi_n \rangle \partial_t E_\beta(-\lambda_n(t-r)^\beta) dr$$

is the solution to the inhomogeneous Caputo IVP

$$\partial_{t,0}^\beta g(t) = -\lambda_n g(t) + \langle f, \varphi_n \rangle, \quad g(0) = 0.$$

(2) Inhomogeneous term

Convergence of the first derivative in time of the series depends on bounds on the function

$$\partial_t E_{\beta, \lambda} \star f(t) = \partial_t \int_0^t f(r) (t-r)^{\beta-1} \beta E'_{\beta}(-\lambda(t-r)^{\beta}) dr.$$

If f is $C^1([0, T])$ we can hit f with ∂_t to access the bound

$$|\partial_t E_{\beta, \lambda} \star f(t)| \leq \frac{c}{\lambda} \left(\|f'\|_{\infty} + f(0) \frac{\lambda t^{\beta-1}}{1 + \lambda t^{\beta}} \right).$$

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Generalised Marchaud evolution equations

Perform the natural probabilistic generalisation

$\partial_{t,\infty}^\beta u(t) \mapsto \partial_{t,\infty}^{(\nu)} u(t) := \int_0^\infty (u(t) - u(t-r)) \nu(t, dr)$, and consider

$$\begin{cases} \partial_{t,\infty}^{(\nu)} u(t, x) = \Delta u(t, x), & \text{in } (0, T] \times \mathbb{R}^d, \\ u(t, x) = \phi(t, x), & \text{in } (-\delta, 0] \times \mathbb{R}^d, \end{cases} \quad (11)$$

where δ is the length of the support of the Lévy-type kernel ν .

A (simplified) theorem reads

Theorem (Du, T., Zhi '18)

Suppose that $\nu(t, dr) \equiv \nu(r)dr$, with $\int_0^\infty \nu(r)dr = \infty$ and let

$\phi \in L^\infty(-\infty, 0; H^1(\mathbb{R}^d))$. Then $u(t, x) = \mathbb{E} \left[\phi \left(-X_{\tau_0(t)}^{t,(\nu)}, B_{\tau_0(t)}^x \right) \right]$

is a weak solution to (11).

Summary

- 1 **Marchaud-type fractional derivatives allow to meaningfully define time-nonlocal initial conditions for EEs (extending Caputo-type EEs).**
- 2 **The stochastic representation for the solution provides intuition for the time-nonlocal initial condition, as the trapping time of the anomalous diffusion weights the initial condition.**
- 3 **Marchaud-type EEs can be solved in terms of inhomogeneous Caputo-type EEs.**

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Thank you!