THE BOLTZMANN EQUATION, BESOV SPACES, AND OPTIMAL TIME DECAY RATES IN $\mathbb{R}^n_x$

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Abstract. We prove that k-th order derivatives of perturbative classical solutions to the hard and soft potential Boltzmann equation (without the angular cut-off assumption) in the whole space, $\mathbb{R}^n_x$ with $n \geq 3$, converge in large-time to the global Maxwellian with the optimal decay rate of $O\left(t^{-\frac{1}{2}\left(k+\varrho+\frac{n}{2}-\frac{n}{r}\right)}\right)$ in the $L^r_x(L^2_v)$-norm for any $2 \leq r \leq \infty$. These results hold for any $\varrho \in [0, n/2]$ as long as initially $\|f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} < \infty$. In the hard potential case, we prove faster decay results in the sense that if $\|\mathbf{P} f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} < \infty$ and $\|\{\mathbf{I}-\mathbf{P}\} f_0\|_{\dot{B}_2^{-\varrho+1,\infty} L^2_v} < \infty$ for $\varrho \in (n/2, (n+2)/2]$ then the solution decays to zero in $L^2_v(L^2_x)$ with the optimal large time decay rate of $O\left(t^{-\frac{1}{2}\varrho}\right)$.

Contents

1. Introduction and the main result

The study of optimal large time decay rates in the whole space for perturbative solutions to non-linear dissipative partial differential equations with degenerate structure has received a substantial amount of attention in recent times, for example [6–11, 18, 25, 28, 29, 33, 34, 36]. For equations in which $L^2(\mathbb{R}^n_x)$ based norms can be propagated by the solution, it is common to make a smallness assumption on the $L^1(\mathbb{R}^n_x)$ norm of the initial data and combine this with $L^2(\mathbb{R}^n_x)$ type estimates in order to obtain large time decay estimates. However it is often the case that propagating bounds on $L^1(\mathbb{R}^n_x)$ norms is difficult along the time evolution. This can cause severe difficulties in applications because one could improve existing

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theories by showing that an $L^1(\mathbb{R}^n_x)$ type norm is small or bounded after a finite but large time $T > 0$, and then applying the aforementioned decay theory. To overcome these types of difficulties, it is of great interest to prove decay rates in an $L^2(\mathbb{R}^n_x)$ based space which is larger than $L^1(\mathbb{R}^n_x)$. In this paper we accomplish this task for the non-cutoff Boltzmann equation in the homogeneous Besov-Lipschitz space $\dot{B}^{-\varrho,\infty}_2 \supset L^p(\mathbb{R}^n_x)$ where for $p \in [1,2]$ we use $\varrho = \frac{n}{p} - \frac{n}{2}$. We remark that these spaces can be thought of as a physical choice since it is possible to obtain the $L^1(\mathbb{R}^n_x)$ embedding. In the hard-potential case, we prove faster decay results in the more singular spaces $\dot{B}^{-\varrho,\infty}_2$ for $\varrho \in (\frac{n}{2}, \frac{n+2}{2}]$. We anticipate that our methods are applicable to a much wider class of degenerately dissipative equations.

For the non-cutoff Boltzmann equation, particularly for the soft potential case, there are two competing degenerate effects; so that this equation can be thought of as "doubly degenerate". Firstly, for both the hard and the soft potentials, there is a degeneracy in the whole space because the macroscopic part of the solution is not a part of the dissipation. Second there is a further degeneracy, for the soft potentials, due to the weak velocity decay in the dissipation. As described below, we develop new methods to overcome the combination of these difficulties in $\dot{B}^{-\varrho,\infty}_2 L^2_v$.

We study solutions to the *Boltzmann equation*, which is given by

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = \mathcal{Q}(F,F), \quad F(0,x,v) = F_0(x,v). \tag{1.1}$$

Here the unknown is $F = F(t,x,v) \geq 0$, which for $t \geq 0$ physically represents the density of particles in phase space. The spatial coordinates are $x \in \mathbb{R}^n_x$, and velocities are $v \in \mathbb{R}^n_v$ with $n \geq 3$. The *Boltzmann collision operator*, $\mathcal{Q}$, is a bilinear operator which acts only on the velocity variables, v, instantaneously in (t,x) as

$$\mathcal{Q}(G,F)(v) \overset{\text{def}}{=} \int_{\mathbb{R}^n} dv_* \int_{\mathbb{S}^{n-1}} d\sigma \; B(v-v_*,\sigma) \left[G'_* F' - G_* F\right].$$

We use the standard shorthand $F = F(v)$, $G_* = G(v_*)$, $F' = F(v')$, $G'_* = G(v'_*)$. In this expression, v, v_* and v', v'_* are the velocities of a pair of particles before and after collision. They are connected through the formulas

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \qquad \sigma \in \mathbb{S}^{n-1}.$$

We will discuss below in more detail the *Boltzmann collision kernel*, $B(v-v_*,\sigma)$.

We will study the linearization of (1.1) around the Maxwellian equilibrium states

$$F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v), \tag{1.2}$$

where without loss of generality the Maxwellian is given by

$$\mu(v) \overset{\text{def}}{=} (2\pi)^{-n/2} e^{-|v|^2/2}.$$

We use the homogeneous mixed Besov space $\dot{B}^{\varrho,\infty}_2 L^2_v$ with norm

$$\|g\|_{\dot{B}^{\varrho,\infty}_2 L^2_v} \overset{\text{def}}{=} \sup_{j\in\mathbb{Z}} \left(2^{\varrho j} \|\Delta_j g\|_{L^2(\mathbb{R}^n_x \times \mathbb{R}^n_v)}\right), \quad \varrho \in \mathbb{R}. \tag{1.3}$$

Here Δ_j are the standard Littlewood-Paley projections onto frequencies of order 2^j (in the spatial, x, variable only); they are defined in Section 4.2. We provide a discussion of more general Besov spaces in Section 4.2. We suppose once and for all that K is an integer satisfying $K \geq 2K^*_n$, where $K^*_n \overset{\text{def}}{=} \lfloor \frac{n}{2} + 1 \rfloor$ is the smallest integer which is strictly greater than $\frac{n}{2}$. The rest of our notation is defined in Section 1.2 just below. Our main theorems are stated as follows:

Theorem 1.1. *Suppose that $\epsilon_{K,\ell}$ from (1.25) is sufficiently small with $\ell \geq \ell_0$, where ℓ_0 is given by (1.30) below. Consider the global solution $f(t,x,v)$ to the Boltzmann equation from Theorem 1.4 with initial data $f_0(x,v)$.*

Fix $\varrho \in (0, n/2]$ and we consider any $k \in \{0,1,\dots,K-1\}$. Suppose additionally that $\|f_0\|_{\dot{B}_2^{-\varrho,\infty}L^2_v} < \infty$. Then uniformly for $t \geq 0$ we have

$$\|f(t)\|^2_{\dot{B}_2^{m,\infty}L^2_v} \lesssim (1+t)^{-(m+\varrho)}, \quad \forall m \in [-\varrho, k]. \tag{1.4}$$

Furthermore

$$\sum_{k \leq |\alpha| \leq K} \|\partial^\alpha f(t)\|^2_{L^2_x L^2_v} \lesssim (1+t)^{-(k+\varrho)}. \tag{1.5}$$

Notice we have established the optimal L^p_x decay rate for $p \in [1,2)$ for all of the derivatives of order $k \in \{0,1,\dots,K-1\}$ in the larger space $\dot{B}_2^{-\varrho,\infty}$ with $\varrho \in [0, n/2]$ and $\varrho = \frac{n}{p} - \frac{n}{2}$ we only need to assume that initially $\|f_0\|_{\dot{B}_2^{-\varrho,\infty}L^2_v} < \infty$.

Corollary 1.2. *Fix any $2 \leq r \leq \infty$ and $k \in \{0,1,\dots,K-1\}$ satisfying the inequality $k < K - 1 - \frac{n}{2} + \frac{n}{r}$.*[1] *Suppose all the conditions from Theorem 1.1 hold. Then for $\varrho \in (0, n/2]$ with $\|f_0\|_{\dot{B}_2^{-\varrho,\infty}L^2_v} < \infty$, we have the following estimate*

$$\sum_{|\alpha|=k} \|\partial^\alpha f(t)\|^2_{L^r_x L^2_v} \lesssim (1+t)^{-k-\varrho-\frac{n}{2}+\frac{n}{r}},$$

which holds uniformly over $t \geq 0$.

Regarding Corollary 1.2, one can, in principle, use the methods described in the proof to obtain decay estimates in the stronger norm $L^2_v L^r_x$. In order to do this, we would have to reverse the order of the norms in the interpolation estimates of Section 4 (which is possible). We do not currently pursue this issue.

We can furthermore analyze the full energy functional defined in (1.23). In the hard potential case (1.8), we have faster decay results.

Theorem 1.3. *Suppose that $\epsilon_{K,\ell}$ from (1.25) is sufficiently small with $\ell \geq \ell_0$, where ℓ_0 is given by (1.30) below. Consider the global solution $f(t,x,v)$ to the Boltzmann equation from Theorem 1.4 with initial data $f_0(x,v)$.*

Fix $\varrho \in (0, n/2]$, suppose additionally that $\|f_0\|_{\dot{B}_2^{-\varrho,\infty}L^2_v} < \infty$. Then

$$\mathcal{E}_{K,\ell}(t) \lesssim (1+t)^{-\varrho}, \tag{1.6}$$

which holds uniformly for $t \geq 0$. Here $\mathcal{E}_{K,\ell}(t)$ is the full instant energy functional given as in (1.23) and (1.26) below. Furthermore, in the hard potential case (1.8), for $\varrho \in (n/2, (n+2)/2]$, and $\mathbf{P}$ defined in (1.13) below, if

$$\|\mathbf{P}f_0\|_{\dot{B}_2^{-\varrho,\infty}L^2_v} + \|\{\mathbf{I}-\mathbf{P}\}f_0\|_{\dot{B}_2^{-\varrho+1,\infty}L^2_v} < \infty,$$

then the solution also uniformly satisfies (1.6) with this ϱ.

[1] We remark that the proof of Corollary 1.2 easily shows that if $2 \leq r < \infty$ then we can allow $k \leq K - 1 - \frac{n}{2} + \frac{n}{r}$, and we only need to restrict to $k < K - 1 - \frac{n}{2}$ when $r = \infty$.

When we say that these large time decay rates are "optimal" we mean that they are the same as those for the linear Boltzmann equation (5.1), as seen in Theorem 5.1. The optimal rates in $L^r_x L^2_v$, from Corollary 1.2 also hold for (5.1). These rates for 0-th order derivatives also coincide with classical time-decay results for the Boltzmann equation [33,34] with angular cut-off studied using spectral analysis.

These are also consistent with the classical optimal large time decay rates for the Heat equation; see for instance [21]. In particular it is well known that if $g_0(x)$ is a tempered distribution vanishing at infinity and satisfying $\|g_0\|_{\dot{B}_2^{-\varrho,\infty}(\mathbb{R}^n_x)} < \infty$, then one further has

$$\|g_0\|_{\dot{B}_2^{-\varrho,\infty}(\mathbb{R}^n_x)} \approx \left\| t^{\varrho/2} \left\| e^{t\Delta} g_0 \right\|_{L^2(\mathbb{R}^n_x)} \right\|_{L^\infty_t((0,\infty))}, \quad \text{for any } \varrho > 0.$$

See for instance [21, Theorem 5.4] where further references and more general results can be found. Notice that the faster decay rates of higher derivatives for solutions to the Heat equation can be easily obtained in the same way.

Notice that for the Heat equation, and for the linear Boltzmann equation (5.1) in Theorem 5.1, these decay results using initial conditions in the negative regularity Besov spaces (of order "$-\varrho$") hold for any $\varrho > 0$. For the non-linear problem the restriction of $0 < \varrho \le (n+2)/2$ is also encountered in the large time optimal decay rates for the incompressible Navier-Stokes system; see [21]. For incompressible Navier-Stokes, it appears that we may not hope to go beyond $\varrho = (n+2)/2$ without choosing special initial data [21]. Thus the range $0 < \varrho \le (n+2)/2$ seems to us to represent a satisfying theory of decay rates in these spaces.

We are furthermore concerned in this paper primarily with obtaining the optimal large time convergence rates. In that light we are not as concerned with optimizing the assumptions that we use on the regularity ($K \ge 2K^*_n$) or the number of weights placed on the initial data ($\ell \ge \ell_0$ with ℓ_0 from (1.30) below).

We obtain decay for all derivatives of order $k \in \{0, 1, \dots, K-1\}$, where K is the Sobolev regularity of the initial data in $\epsilon_{K,\ell}$ from (1.25) and the existence theory in Theorem 1.4. Our obstruction to obtaining the higher order decay of the highest order derivative K comes from the estimates of the functionals $\mathcal{I}^k(t)$ in Lemma 2.2, which fatally contain error terms including derivatives of order $k+1$ when controlling derivative energy estimates of order k.

In the rest of this section, we will finish introducing the full model (1.1), including the collision kernel, and then we discuss its geometric fractionally diffusive behavior. The Boltzmann collision kernel, $B(v - v_*, \sigma)$, will physically depend upon the *relative velocity* $|v - v_*|$ and on the *deviation angle* θ through the formula $\cos\theta = (v - v_*) \cdot \sigma / |v - v_*|$ where, without restriction, we can suppose by symmetry that $B(v - v_*, \sigma)$ is supported on $\cos\theta \ge 0$.

The Collision Kernel. Our assumptions are the following:

- We suppose that $B(v - v_*, \sigma)$ takes product form in its arguments as
$$B(v - v_*, \sigma) = \Phi(|v - v_*|)\, b(\cos\theta).$$
It generally holds that both b and Φ are non-negative functions.
- The angular function $t \mapsto b(t)$ is not locally integrable; for $c_b > 0$ it satisfies

$$\frac{c_b}{\theta^{1+2s}} \le \sin^{n-2}\theta\; b(\cos\theta) \le \frac{1}{c_b \theta^{1+2s}}, \quad s \in (0,1), \quad \forall\, \theta \in \left(0, \frac{\pi}{2}\right]. \tag{1.7}$$

- The kinetic factor $z \mapsto \Phi(|z|)$ satisfies for some $C_\Phi > 0$

$$\Phi(|v - v_*|) = C_\Phi |v - v_*|^\gamma, \quad \gamma \geq -2s. \tag{1.8}$$

In the rest of this paper these will be called "hard potentials."

- Our results will also apply to the more singular situation

$$\Phi(|v - v_*|) = C_\Phi |v - v_*|^\gamma, \quad -2s > \gamma > -n, \quad \gamma + 2s > -\frac{n}{2}. \tag{1.9}$$

These will be called "soft potentials" throughout this paper.

These collision kernels are physically motivated since they can be derived from a spherical intermolecular repulsive potential such as $\phi(r) = r^{-(p-1)}$ with $p \in (2, \infty)$ as shown by Maxwell in 1866. In the physical dimension ($n = 3$), B satisfies the conditions above with $\gamma = (p-5)/(p-1)$ and $s = 1/(p-1)$; see [35].

We linearize the Boltzmann equation (1.1) around (1.2). This grants an equation for the perturbation, $f(t, x, v)$, that is given by

$$\partial_t f + v \cdot \nabla_x f + L(f) = \Gamma(f, f), \quad f(0, x, v) = f_0(x, v), \tag{1.10}$$

where the *linearized Boltzmann operator*, L, is defined as

$$L(g) \overset{\text{def}}{=} -\mu^{-1/2} \mathcal{Q}(\mu, \sqrt{\mu} g) - \mu^{-1/2} \mathcal{Q}(\sqrt{\mu} g, \mu),$$

and the bilinear operator, Γ, is then

$$\Gamma(g, h) \overset{\text{def}}{=} \mu^{-1/2} \mathcal{Q}(\sqrt{\mu} g, \sqrt{\mu} h). \tag{1.11}$$

The $(n+2)$-dimensional null space of L is well known [12]:

$$N(L) \overset{\text{def}}{=} \operatorname{span}\left\{ \sqrt{\mu},\ v_1 \sqrt{\mu},\ \ldots,\ v_n \sqrt{\mu},\ (|v|^2 - n)\sqrt{\mu} \right\}. \tag{1.12}$$

Now, for fixed (t, x), we define the orthogonal projection from L^2_v to $N(L)$ as

$$\mathbf{P} f = a^f(t, x)\sqrt{\mu} + \sum_{i=1}^n b_i^f(t, x) v_i \sqrt{\mu} + c^f(t, x) \frac{1}{\sqrt{2n}} (|v|^2 - n)\sqrt{\mu}, \tag{1.13}$$

where the functions a^f, $b^f \overset{\text{def}}{=} [b_1, \cdots, b_n]$ and c^f are defined by

$$\begin{aligned} a &= \langle \sqrt{\mu}, f \rangle = \langle \sqrt{\mu}, \mathbf{P} f \rangle, \\ b_i &= \langle v_i \sqrt{\mu}, f \rangle = \langle v_i \sqrt{\mu}, \mathbf{P} f \rangle, \\ c &= \frac{1}{\sqrt{2n}} \langle (|v|^2 - n)\sqrt{\mu}, f \rangle = \frac{1}{\sqrt{2n}} \langle (|v|^2 - n)\sqrt{\mu}, \mathbf{P} f \rangle. \end{aligned} \tag{1.14}$$

Then we can write $f = \mathbf{P} f + \{\mathbf{I} - \mathbf{P}\} f$. It is a well-known fact [12] that:

$$\mathbf{P}\Gamma(f, f) = 0. \tag{1.15}$$

We further define $[a, b, c]$ to be the vector with components a, b, c. And $|[a, b, c]|$ is the standard Euclidean length of these vectors.

In the works [13–15], Gressman and the second author have introduced into the Boltzmann theory the following sharp weighted geometric fractional Sobolev norm:

$$|f|^2_{N^{s,\gamma}} \overset{\text{def}}{=} |f|^2_{L^2_{\gamma+2s}} + \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv' \ (\langle v \rangle \langle v' \rangle)^{\frac{\gamma+2s+1}{2}} \frac{|f' - f|^2}{d(v, v')^{n+2s}} \mathbf{1}_{d(v,v') \leq 1}. \tag{1.16}$$

Generally, $\mathbf{1}_A$ is the standard indicator function of the set A. Now this space includes the weighted L^2_ℓ space, for $\ell \in \mathbb{R}$, with norm given by

$$|f|^2_{L^2_\ell} \overset{\text{def}}{=} \int_{\mathbb{R}^n} dv \ \langle v \rangle^\ell \ |f(v)|^2.$$

The weight is $\langle v\rangle \stackrel{\text{def}}{=} \sqrt{1+|v|^2}$. The fractional differentiation effects are measured using the anisotropic metric $d(v,v')$ on the "lifted" paraboloid (in $\mathbb{R}^{n+1}$) as

$$d(v,v') \stackrel{\text{def}}{=} \sqrt{|v-v'|^2 + \frac{1}{4}\left(|v|^2-|v'|^2\right)^2}.$$

This metric encodes the nonlocal anisotropic changes in the power of the weight.

In this space, the linearized collision operator L is non-negative in L^2_v and it is coercive in the sense that there is a constant $\lambda>0$ such that [13, Theorem 8.1]:

$$\langle g, Lg\rangle \geq \lambda |\{\mathbf{I}-\mathbf{P}\}g|^2_{N^{s,\gamma}}. \tag{1.17}$$

The norm $N^{s,\gamma}$ provides a sharp characterization of the linearized collision operator [13, (2.13)]; in earlier work [22] the sharp gain of velocity weight in L^2_v was established for the non-derivative part of (1.16).

1.1. **Discussion of the method.** There have been numerous investigations on the rate of convergence to Maxwellian equilibrium for the nonlinear Boltzmann equation or related kinetic equations in the whole space; see for example [2, 6, 7, 9, 18, 24–27]. Many of the early results are well documented in Glassey [12]. Further detailed discussions of more recent results can be found in [7] and [25]. We point out that this current work was motivated by several recent results [6, 7, 11, 13–15, 18, 25–27].

To establish our main results in Theorem 1.1 we use the following method. Notice that we give two proofs of Theorem 1.1. The first proof, starting from a differential inequality for high order derivatives in (3.2), uses a time weighted energy estimate combined with a new time-regularity comparison via dyadic decomposition to achieve the optimal decay rates in spite of the degenerate dissipation. We alternatively give another proof of Theorem 1.1 which uses a double interpolation, the first interpolation is in terms of spatial-regularity and the second interpolation is in terms of the degenerate velocity components of the solution. In regards to the second proof, we recall here the interpolations used in [18, 26].

To establish the high order differential inequality mentioned above, e.g. (3.2), we develop several product interpolation estimates. The details of the proofs of these estimates are contained in the long Section 4. There is a rather serious added complication in the context of the non cut-off Boltzmann equation because all of the product estimates are in spaces such as $L^p_x N^{s,\gamma}$ with the exotic non-isotropic $N^{s,\gamma}$ as in (1.16). The interesting element of our construction is that we prove a large number of Sobolev type inequalities (in the variable x) in the mixed spaces $L^p_x\mathcal{H}_v$ where $\mathcal{H}_v$ is an arbitrary separable Hilbert space (in the variable v). Then the desired product estimates for $L^p_x N^{s,\gamma}$ follow in the special case when $\mathcal{H}_v = N^{s,\gamma}$.

Additionally, to prove the faster decay rates in Theorem 1.3 we generally use the time weighted estimates combined with the linear decay theory. Notice that the slower decay results in Theorem 1.1 and Corollary 1.2 are proven "Non-Linearly" without the linear decay theory since we can establish a uniform in time upper bound on $\|f(t)\|^2_{B_2^{-\varrho,\infty}L^2_v}$ for $\varrho \in (0,\frac{n}{2}]$ as in (1.4). In the more singular situation when $\varrho \in (\frac{n}{2},\frac{n+2}{2}]$ we are unable to prove these uniform bounds and they may be unavailable. Instead we prove the faster linear decay rates in Theorem 5.6 by using a detailed frequency decomposition of the linearized operator (5.9) in a small frequency ball around the origin, which is motivated by the work of Ellis and Pinsky [11]. Using this refined linear decay theory, we gain an additional order of $t^{-\frac{1}{2}}$ decay on the non-linear term because it is purely microscopic. We also need to iterate this non-linear decay analysis a finite number of times in order to overcome degeneracies in the time integral estimates and obtain the optimal decay rates.

We will explain other difficulties when they are encountered at the appropriate places throughout the course of the paper. In the next sub-section we will give a detailed description of the remaining notation, as well as stating the relevant existence result from Theorem 1.4.

1.2. **Notation, and the existence result.** For any $m \geq 0$, we use H^m to denote the usual Sobolev spaces $H^m(\mathbb{R}^n_x \times \mathbb{R}^n_v)$, $H^m(\mathbb{R}^n_x)$, or $H^m(\mathbb{R}^n_v)$, respectively, where for example

$$H^m_\ell(\mathbb{R}^n_v) \stackrel{\text{def}}{=} \left\{ f \in L^2_\ell(\mathbb{R}^n_v) : |f|^2_{H^m_\ell(\mathbb{R}^n)} \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} dv \, \langle v \rangle^\ell \left| (I - \Delta_v)^{m/2} f(v) \right|^2 < \infty \right\}.$$

Then let us denote $H^m_0 = H^m$. Then sharp comparisons of $|\cdot|_{N^{s,\gamma}}$ to the weighted isotropic Sobolev spaces are established by the inequalities:

$$|g|^2_{L^2_{\gamma+2s}(\mathbb{R}^n_v)} + |g|^2_{H^s_\gamma(\mathbb{R}^n_v)} \lesssim |g|^2_{N^{s,\gamma}} \lesssim |g|^2_{H^s_{\gamma+2s}(\mathbb{R}^n_v)}. \tag{1.18}$$

See [13, Eq. (2.15)].

For Banach spaces X and Y, we write $X(\mathbb{R}^n_v) = X_v$ and $Y(\mathbb{R}^n_x) = Y_x$. Specifically, we use L^p_x, L^p_v and $L^p_{x,v}$ to denote $L^p(\mathbb{R}^n_x)$, $L^p(\mathbb{R}^n_v)$ and $L^p(\mathbb{R}^n_x \times \mathbb{R}^n_v)$ with $1 \leq p \leq \infty$ respectively. There should be no confusion between L^2_x, L^2_v and L^2_ℓ, etc, since x and v are never used to denote a weight. Given the spaces as above, we define the following ordered mixed spaces:

$$\|h\|_{Y_x X_v} \stackrel{\text{def}}{=} \left\| \, |h|_{X(\mathbb{R}^n_v)} \, \right\|_{Y(\mathbb{R}^n_x)}, \tag{1.19}$$

$$\|h\|_{X_v Y_x} \stackrel{\text{def}}{=} \left| \, \|h\|_{Y(\mathbb{R}^n_x)} \, \right|_{X(\mathbb{R}^n_v)}, \tag{1.20}$$

Thus for example, as in Corollary 1.2, the space $L^p_v L^r_x$ may be different from $L^r_x L^p_v$. Generally a norm with only one line $|\cdot|_{X_v}$ denotes that it is only in the "v" variable, however a norm with two sets of lines $\|\cdot\|$ is either in both variables "(x, v)" or only in the "x" variable (and there should be no confusion between these cases). We remark that this is a slight departure from the notation used in the second authors previous papers, e.g. [13–15, 25]; in this paper it is necessary to distinguish between the ordering of the evaluation of the norms.

Recalling the notations surrounding (1.3), and in Section 4.2, we will also use the following mixed Besov space semi-norm as

$$\|g\|_{\dot{B}^{\varrho,q}_p X_v} \stackrel{\text{def}}{=} \left\| \left(2^{\varrho j} \|\Delta_j g\|_{L^p_x X_v} \right)_j \right\|_{\ell^q_j}, \quad p, q \in [1, \infty], \quad \varrho \in \mathbb{R}, \tag{1.21}$$

where for a sequence, $(a_j)_{j \in \mathbb{Z}}$, we use the standard ℓ^q_j norm as

$$\| (a_j)_j \|_{\ell^q_j} \stackrel{\text{def}}{=} \left(\sum_{j \in \mathbb{Z}} |a_j|^q \right)^{1/q} \quad \text{for } 1 \leq q < \infty, \text{ and} \quad \| (a_j)_j \|_{\ell^\infty_j} \stackrel{\text{def}}{=} \sup_{j \in \mathbb{Z}} |a_j|.$$

Thus $\dot{B}^{s,q}_p$ is the homogeneous Besov space in the variable x. We always use the Besov space and the frequency projection Δ_j only in the spatial variable $x \in \mathbb{R}^n_x$. Notice that in the special case of Besov semi-norms, $\|\cdot\|_{\dot{B}^{\varrho,q}_p X_v}$, we do not follow the Banach space ordering convention described above. As we will see, this will add an additional complication in the interpolation estimates we want to use.

We use $\langle \cdot, \cdot \rangle$ to denote the inner product over the Hilbert space L^2_v, i.e.

$$\langle g, h \rangle = \int_{\mathbb{R}^n} g(v)\overline{h(v)}\, dv, \quad g = g(v),\ h = h(v) \in L^2_v.$$

Analogously $(\cdot, \cdot)$ denotes the inner product over $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_v)$.

We also define the weight function as follows

$$w(v) \stackrel{\text{def}}{=} \langle v \rangle\,. \tag{1.22}$$

Again our notation for (1.22) is different from the notation in the second authors previous papers [13,25] when $\gamma+2s<0$ from (1.9). We then consider the weighted anisotropic derivative space as in (1.16):

$$|h|^2_{N^{s,\gamma}_\ell} \stackrel{\text{def}}{=} |w^\ell h|^2_{L^2_{\gamma+2s}} + \int_{\mathbb{R}^n} dv\ \langle v\rangle^{\gamma+2s+1}\, w^{2\ell}(v) \int_{\mathbb{R}^n} dv'\ \frac{|h'-h|^2}{d(v,v')^{n+2s}} \mathbf{1}_{d(v,v')\le 1}.$$

Note that $|h|_{N^{s,\gamma}} = |h|_{N^{s,\gamma}_0}$. Throughout the paper, we will use $\mathcal{H}_v$ to denote a separable Hilbert space in the v variable. In particular, one can take $\mathcal{H}_v$ to be L^2_v, $L^2_{\gamma+2s}$, H^s_v and $\mathbb{R}$ with the absolute value. Moreover, one can take $\mathcal{H}_v = N^{s,\gamma}_\ell$. That $N^{s,\gamma}_\ell$ is separable follows from the equivalence of norms given in [13, Eq. (7.7)].

For multi-indices, we denote

$$\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} \partial^{\beta_1}_{v_1} \cdots \partial^{\beta_n}_{v_n}, \quad \alpha = [\alpha_1, \dots, \alpha_n], \quad \beta = [\beta_1, \dots, \beta_n].$$

The length of α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and the length of β is $|\beta| = \beta_1 + \cdots + \beta_n$.

Given a solution, $f(t,x,v)$, to the Boltzmann equation (1.10), we define an instant energy functional to be a continuous function, $\mathcal{E}_{K,\ell}(t)$, which satisfies

$$\mathcal{E}_{K,\ell}(t) \approx \sum_{|\alpha|+|\beta|\le K} \|w^{\ell-|\beta|\rho} \partial^\alpha_\beta f(t)\|^2_{L^2_x L^2_v}. \tag{1.23}$$

Above and below: $\ell \ge 0$. We also define the dissipation rate $\mathcal{D}_{K,\ell}(t)$ as

$$\mathcal{D}_{K,\ell}(t) \stackrel{\text{def}}{=} \sum_{1\le|\alpha|\le K} \|\partial^\alpha f(t)\|^2_{L^2_x N^{s,\gamma}_\ell} + \sum_{|\alpha|+|\beta|\le K} \|\partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f(t)\|^2_{L^2_x N^{s,\gamma}_{\ell-|\beta|\rho}}. \tag{1.24}$$

Here we also use $\rho = 1$ under (1.8) and otherwise we use $\rho = -\gamma - 2s > 0$ for the soft potentials (1.9). (This ρ is to correct for our change in the definition of the weight (1.22) from previous papers such as [13,25].) We do not explicitly use these functionals in our proofs herein. Initially we define

$$\epsilon_{K,\ell} \stackrel{\text{def}}{=} \mathcal{E}_{K,\ell}(0). \tag{1.25}$$

We can now state the following existence result, and Lyapunov inequalities:

Theorem 1.4. [13,25]. *Fix $\ell \ge 0$ and $f_0(x,v)$. There are $\mathcal{E}_{K,\ell}(t)$, $\mathcal{D}_{K,\ell}(t)$ such that if $\epsilon_{K,\ell}$ is sufficiently small, then the Cauchy problem to the Boltzmann equation* (1.10) *admits a unique global solution $f(t,x,v)$ satisfying the Lyapunov inequality*

$$\frac{d}{dt}\mathcal{E}_{K,\ell}(t) + \lambda \mathcal{D}_{K,\ell}(t) \le 0, \quad \forall t \ge 0. \tag{1.26}$$

Here $\lambda > 0$ may depend on ℓ.

This Theorem 1.4 is the building block for our decay results stated earlier in Theorem 1.1 and Corollary 1.2. In those statements we used a sufficient number of weights, which we now define precisely. We define the quantity ℓ_0^d by:

$$\ell_0^d \stackrel{\text{def}}{=} \begin{cases} \frac{n}{2} + \lceil \frac{n}{2} \rceil - 3, & \text{for } n \text{ odd}, \\ n + 2\lceil \frac{n}{2} \rceil - 7, & \text{for } n \text{ even}. \end{cases} \tag{1.27}$$

Furthermore, let us define the quantity M by:

$$M \stackrel{\text{def}}{=} \max\left\{ \left(\frac{2K}{n-2} - 1 \right), \ell_0^d \right\} \tag{1.28}$$

These choices come from (4.28), (4.29), (4.30), (4.37) and (4.42) respectively.

Here and in several places in the rest of the paper, we will use the notation

$$\tilde{j} \stackrel{\text{def}}{=} \lceil n/2 \rceil - 1, \tag{1.29}$$

which denotes the largest integer that is strictly less than $\frac{n}{2}$. We recall that in general $m = \lceil a \rceil$ is the smallest integer satisfying $m \geq a$ and $m' = \lfloor a \rfloor$ is the largest integer satisfying $m' \leq a$.

Now we define the following weight:

$$\ell_0 \stackrel{\text{def}}{=} \begin{cases} \max\left\{ \frac{\gamma+2s}{2} + 1, 2, 2(\gamma+2s) \right\}, & \text{hard potentials: (1.8)}, \\ -\frac{(\gamma+2s)}{2} \max\left\{ (M+1), (\frac{n}{2} + K) \right\}, & \text{soft potentials: (1.9)}. \end{cases} \tag{1.30}$$

Above and below we are using M from (1.28). For future reference, we will also define the following related weight:

$$\ell_0' \stackrel{\text{def}}{=} \begin{cases} 0, & \text{for the hard potentials: (1.8)}, \\ -\frac{(\gamma+2s)}{2} M, & \text{for the soft potentials: (1.9)}. \end{cases} \tag{1.31}$$

We note that in the rest of this article, we will implicitly assume sometimes without mention that $\ell \geq \ell_0$. We further define

$$\ell_0^1 \stackrel{\text{def}}{=} \frac{(\gamma+2s)^+}{2}. \tag{1.32}$$

Here we recall the general notation $(a)^+ = \max\{a, 0\}$.

Throughout this paper we let C denote some positive (generally large) inessential constant and λ denotes some positive (generally small) inessential constant, where both C and λ may change values from line to line. Furthermore $A \lesssim B$ means $A \leq CB$, and $A \gtrsim B$ means $B \lesssim A$. In addition, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

1.3. **Organization of the paper.** In Section 2 we will prove several non-linear energy estimates for the solutions to the non cut-off Boltzmann equation (1.10) from Theorem 1.4 which will be used in the proofs of Theorem 1.1 and Corollary 1.2. After that in Section 3 we give two proofs of our main Theorem 1.1; one proof uses time-velocity and time-frequency splitting methods simultaneously, and the other proof uses interpolation. We also prove Corollary 1.2 and Theorem 1.3. Then in Section 4, in part because of the exotic nature of some of our spaces (1.16), we prove a collection of functional interpolation inequalities in separable Hilbert spaces. In the first part of Section 5 we prove large time decay rates of the linear Boltzmann equation (5.1) in Besov spaces using dyadic time-frequency splittings and a pointwise time-frequency differential inequality from [25]. Finally, in the second part of Section 5 we prove faster decay rates for the linearized problem in the hard potential case (1.8) under the additional assumption that the initial data is purely microscopic as in (1.13). Our analysis in this part is based on a precise understanding of the spectrum of the spatial Fourier transform of the linearized operator for frequencies near zero.

2. Non-linear energy estimates

In this section, we will prove some non-linear differential and integral inequalities for the solutions of the Boltzmann equation (1.10) in Theorem 1.4. Our strategy will be to use product estimates from [13, 25], as well as the functional interpolation inequalities from Section 4. The vector-valued functions we study, among others, take values in the non-isotropic Sobolev spaces in v, which were previously used in [13]. In proving these estimates, one encounters the difficulty that the macroscopic part doesn't appear in the coercivity estimate (1.17), and hence these terms have to be taken care of separately. All of these issues are addressed in Sub-section 2.1. Moreover, we can apply the Littlewood-Paley projection operators defined in Section 4.2 to obtain energy estimates for solutions of (1.10) in functional Besov spaces. The latter question is studied in sub-Section 2.2.

2.1. **Derivative estimates.** This sub-section is devoted to proving two energy estimates for solutions to the Boltzmann equation (1.10). In Proposition 2.1 we prove Lyapunov inequalities for the Sobolev norms of fixed order $k \in \{0, 1, \dots, K\}$. Then in Lemma 2.2 we prove a differential inequality which includes the macroscopic components (1.14) of the dissipation (1.24).

Proposition 2.1. *Suppose that $\epsilon_{K,\ell}$ from (1.25) is sufficiently small with $\ell \geq \ell_0$ for ℓ_0 as in (1.30). Let $f(t,x,v)$ be the solution to the Boltzmann equation (1.10) from Theorem 1.4. Fix $k \in \{0, 1, \dots, K\}$; then the following inequality holds*

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k}\|\partial^\alpha f\|^2_{L^2_x L^2_v} + \lambda \sum_{|\alpha|=k}\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(t)\|^2_{L^2_x N^{s,\gamma}} \lesssim \epsilon_{K,\ell} \sum_{j_* \leq |\alpha| \leq K}\|\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}},$$

where $j_* \overset{def}{=} \min\{k+1, K\}$.

Proof of Proposition 2.1. We take ∂^α derivatives of (1.10), multiply by $\partial^\alpha f$, integrate over $\mathbb{R}^n_x \times \mathbb{R}^n_v$, and use (1.17) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|^2_{L^2_{x,v}} + \lambda\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(t)\|^2_{L^2_x N^{s,\gamma}} \lesssim \sum_{\alpha_1 \leq \alpha}\left|\left(\Gamma(\partial^{\alpha_1} f, \partial^{\alpha-\alpha_1} f), \{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\right)\right|.$$

The right hand side in the above is obtained by using that $\mathbf{P}$ and ∂^α commute, and (1.15). In the rest of this proof we will focus on estimating the non-linear term.

We write $f = \mathbf{P}f + \{\mathbf{I}-\mathbf{P}\}f$ and we expand the non-linear term as:

$$\Gamma(f,f) = \Gamma(\mathbf{P}f, \mathbf{P}f) + \Gamma(\{\mathbf{I}-\mathbf{P}\}f, \mathbf{P}f) + \Gamma(f, \{\mathbf{I}-\mathbf{P}\}f).$$

We thus expand

$$\left(\Gamma(\partial^{\alpha_1} f, \partial^{\alpha-\alpha_1} f), \{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\right) = A_1 + A_2 + A_3.$$

where:

$$
\begin{aligned}
A_1 &\stackrel{\text{def}}{=} \left(\Gamma(\mathbf{P}\partial^{\alpha_1} f, \mathbf{P}\partial^{\alpha-\alpha_1} f), \partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right),\\
A_2 &\stackrel{\text{def}}{=} \left(\Gamma(\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha_1} f, \mathbf{P}\partial^{\alpha-\alpha_1} f), \partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right),\\
A_3 &\stackrel{\text{def}}{=} \left(\Gamma(\partial^{\alpha_1} f, \{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1} f), \{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\right).
\end{aligned}
\tag{2.1}
$$

We will estimate each of these terms individually.

The desired estimate for the term A_3 then follows from [13, Eq (6.6)]. In other words, using [13, Eq (6.6)] for any small $\delta > 0$ we have

$$
\begin{aligned}
|A_3| &= \left|\left(\Gamma(\partial^{\alpha_1} f, \{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1} f), \{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\right)\right|\\
&\lesssim \int_{\mathbb{R}^n} dx\ |\partial^{\alpha_1} f|_{L^2_v} |\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1} f|_{N^{s,\gamma}} |\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f|_{N^{s,\gamma}}\\
&\lesssim C_\delta \left\| |\partial^{\alpha_1} f|_{L^2_v} |\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1} f|_{N^{s,\gamma}} \right\|^2_{L^2_x} + \delta \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}}.
\end{aligned}
\tag{2.2}
$$

For the first term and the second term in (2.1), we notice from (1.13) that

$$
\Gamma(f, \mathbf{P}f) = \sum_{i=1}^{n+2} \psi_i(t,x)\Gamma(f, \chi_i),
$$

where the $\psi_i(t,x)$ are the elements from (1.14) and the $\chi_i(v)$ are the smooth rapidly decaying velocity basis vectors in (1.12). Thus from [13, Proposition 6.1]:

$$
\begin{aligned}
|A_2| &= \left|\left(\Gamma(\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha_1} f, \mathbf{P}\partial^{\alpha-\alpha_1} f), \partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\right|\\
&\lesssim \int_{\mathbb{R}^n} dx\ |\partial^{\alpha_1}\{\mathbf{I}-\mathbf{P}\}f|_{L^2_{\gamma+2s-(n-1)}} |\partial^{\alpha-\alpha_1}[a,b,c]|\ |\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f|_{L^2_{\gamma+2s-(n-1)}}\\
&\lesssim C_\delta \left\| |\partial^{\alpha_1}\{\mathbf{I}-\mathbf{P}\}f|_{L^2_{\gamma+2s-(n-1)}} |\partial^{\alpha-\alpha_1}[a,b,c]| \right\|^2_{L^2_x} + \delta \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}}.
\end{aligned}
\tag{2.3}
$$

Here $|[a,b,c]|$ is the Euclidean absolute value norm of the coefficients from (1.14). The last case to consider is A_1. From [13, Proposition 6.1] again

$$
\begin{aligned}
|A_1| &= \left|\left(\Gamma(\mathbf{P}\partial^{\alpha_1} f, \mathbf{P}\partial^{\alpha-\alpha_1} f), \partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\right|\\
&\lesssim \int_{\mathbb{R}^n} dx\ |\partial^{\alpha_1}[a,b,c]|\ |\partial^{\alpha-\alpha_1}[a,b,c]|\ |\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f|_{L^2_{\gamma+2s-(n-1)}}\\
&\lesssim C_\delta \left\|\ |\partial^{\alpha_1}[a,b,c]|\ |\partial^{\alpha-\alpha_1}[a,b,c]|\ \right\|^2_{L^2_x} + \delta \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}}.
\end{aligned}
\tag{2.4}
$$

Collecting these estimates in (2.2), (2.3), (2.4), and summing over $|\alpha| = k$ we obtain

$$
\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=k} \|\partial^\alpha f\|^2_{L^2_{x,v}} + \lambda \sum_{|\alpha|=k} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(t)\|^2_{L^2_x N^{s,\gamma}} \lesssim B.
$$

Here $B = B_1 + B_2 + B_3$ with

$$
\begin{aligned}
B_1 &\stackrel{\text{def}}{=} \sum_{|\alpha|=k}\sum_{\alpha_1\le\alpha} \left\|\ |\partial^{\alpha_1}[a,b,c]|\ |\partial^{\alpha-\alpha_1}[a,b,c]|\ \right\|^2_{L^2_x},\\
B_2 &\stackrel{\text{def}}{=} \sum_{|\alpha|=k}\sum_{\alpha_1\le\alpha} \left\| |\partial^{\alpha_1}\{\mathbf{I}-\mathbf{P}\}f|_{L^2_{\gamma+2s-(n-1)}} |\partial^{\alpha-\alpha_1}[a,b,c]| \right\|^2_{L^2_x},\\
B_3 &\stackrel{\text{def}}{=} \sum_{|\alpha|=k}\sum_{\alpha_1\le\alpha} \left\| |\partial^{\alpha_1} f|_{L^2_v} |\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1} f|_{N^{s,\gamma}} \right\|^2_{L^2_x}.
\end{aligned}
\tag{2.5}
$$

Thus we have reduced the proof of Proposition 2.1 to proving that

$$B \lesssim \epsilon_{K,\ell} \sum_{j_* \le |\alpha| \le K} \|\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}}. \tag{2.6}$$

The rest of our proof will be devoted to establishing (2.6). We will prove (2.6) for each of the terms in (2.5) individually.

In order to bound B_1 and B_2, we will use interpolation. Suppose that $\mathcal{H}_v$ is a separable Hilbert space of functions in v as in Section 1.2. Furthermore, suppose that $|\alpha| = k \in \{0, 1, \dots, K\}$, $\alpha_1 \le \alpha$. Then, there exist $p, q \ge 2$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ such that we have:

$$\|\partial^{\alpha-\alpha_1} f\|_{L^q_x \mathcal{H}_v} \|\partial^{\alpha_1} f\|_{L^p_x \mathcal{H}_v} \lesssim \|f\|_{H^{K^*_n}_x \mathcal{H}_v} \sum_{|\alpha'| = \min\{k+1, K\}} \|\partial^{\alpha'} f\|_{L^2_x \mathcal{H}_v}. \tag{2.7}$$

The inequality (2.7) follows from the results of Lemma 4.13, Lemma 4.14, and Lemma 4.15, all of which are proved in Section 4.

We begin by looking at the term B_1 from (2.5). We note that by (1.14), by the exponential decay in v of $\sqrt{\mu}$, and by the Cauchy-Schwarz inequality:

$$|\partial^{\alpha_1}[a, b, c]| \lesssim |w^{-j} \partial^{\alpha_1} f|_{L^2_v}, \tag{2.8}$$

which holds for all $j \ge 0$.

In particular, we can take $j = 0$ in the case of hard potentials and $j = \frac{-\gamma - 2s}{2}$ in the case of soft potentials, hence we can deduce from (2.8) that

$$|\partial^{\alpha_1}[a, b, c]| \lesssim |\partial^{\alpha_1} f|_{L^2_{\gamma+2s}}. \tag{2.9}$$

We also note that $|\mathbf{P}\partial^{\alpha_1} f|_{L^2_{\gamma+2s}} \lesssim |\partial^{\alpha_1}[a, b, c]|$ and so (2.9) implies:

$$|\mathbf{P}\partial^{\alpha_1} f|_{L^2_{\gamma+2s}} \lesssim |\partial^{\alpha_1} f|_{L^2_{\gamma+2s}}. \tag{2.10}$$

Let us take $\mathcal{H}_v = L^2_{\gamma+2s}$ in (2.7). Then, for the p, q which one obtains, since $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, we can use Hölder's inequality in x to estimate:

$$\Big\| \, |\partial^{\alpha_1}[a, b, c]| \, |\partial^{\alpha-\alpha_1}[a, b, c]| \, \Big\|^2_{L^2_x} \lesssim \|\partial^{\alpha_1} f\|^2_{L^q_x L^2_{\gamma+2s}} \|\partial^{\alpha-\alpha_1} f\|^2_{L^p_x L^2_{\gamma+2s}}. \tag{2.11}$$

The fact that B_1 is bounded by the right-hand side in (2.6) follows directly from (2.11) and (2.7) with $\mathcal{H}_v = L^2_{\gamma+2s}$ if we use (1.26) and (1.25). More precisely, we observe that, by construction:

$$\|f(t)\|^2_{H^{K^*_n}_x L^2_{\gamma+2s}} \approx \sum_{|\alpha| \le K^*_n} \|w^{\frac{\gamma+2s}{2}} \partial^\alpha f(t)\|^2_{L^2_x L^2_v} \lesssim \mathcal{E}_{K,\ell}(t) \lesssim \epsilon_{K,\ell},$$

which holds since $\ell \ge \frac{\gamma+2s}{2}$ in the hard potential case, and since $\ell \ge 0$ in the soft potential case by our choice of ℓ, and because of (1.26) and (1.25).

For the term B_2 in (2.5), we argue similarly. We let p, q be as in the bound for B_1. Using (2.9), the fact that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and Hölder's inequality in x, we have

$$\begin{aligned} \Big\| |\partial^{\alpha_1}\{\mathbf{I} - \mathbf{P}\} f|_{L^2_{\gamma+2s-(n-1)}} |\partial^{\alpha-\alpha_1}[a, b, c]| \Big\|^2_{L^2_x} & \\ \lesssim \|\partial^{\alpha_1} f\|^2_{L^q_x L^2_{\gamma+2s}} \|\partial^{\alpha-\alpha_1} f\|^2_{L^p_x L^2_{\gamma+2s}}&. \end{aligned} \tag{2.12}$$

Here we used that $\gamma + 2s - (n-1) \le \gamma + 2s$ and that $\{\mathbf{I} - \mathbf{P}\}$ is a bounded linear operator on $L^2_{\gamma+2s}$ which commutes with ∂^{α_1}. The boundedness property follows from the fact that $\mathbf{P}$ is bounded on $L^2_{\gamma+2s}$ by (2.10). Then, exactly as before, (2.6) follows from (2.12) and (2.7) using also (1.26) and (1.25).

For the last term B_3 in (2.5), also when $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, we have

$$\left\| |\partial^{\alpha_1} f|_{L^2_v} |\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha-\alpha_1} f|_{N^{s,\gamma}} \right\|^2_{L^2_x} \lesssim \|\partial^{\alpha_1} f\|^2_{L^q_x L^2_v} \|\partial^{\alpha-\alpha_1} f\|^2_{L^p_x N^{s,\gamma}}. \tag{2.13}$$

In this case, for the upper bound in (2.13) we note that there exist $p,q\geq 2$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ for which the following estimate holds:

$$\begin{aligned} &\|\partial^{\alpha_1} f\|_{L^q_x L^2_v} \|\partial^{\alpha-\alpha_1} f\|_{L^p_x N^{s,\gamma}} \\ &\qquad \lesssim \left(\|f\|_{H^{K^*_n}_x N^{s,\gamma}_{\ell'_0}} + \|w^{\ell'_0} f\|_{H^K_x L^2_v} \right) \sum_{j_*\leq |\alpha'|\leq K} \|\partial^{\alpha'} f\|_{L^2_x N^{s,\gamma}}. \end{aligned} \tag{2.14}$$

Again $j_* \overset{\text{def}}{=} \min\{k+1, K\}$ and ℓ'_0 is from (1.31). Inequality (2.14) follows from Lemma 4.13, Lemma 4.14, and Lemma 4.16 which are proved in Section 4.

Using the fact that $K\geq 2K^*_n \geq K^*_n+2$, (1.18), and that $s\in(0,1)$, we obtain

$$\begin{aligned} \|f\|^2_{H^{K^*_n}_x N^{s,\gamma}_{\ell'_0}} &\lesssim \|f\|^2_{H^{K-2}_x N^{s,\gamma}_{\ell'_0}} \lesssim \|w^{\ell'_0} f\|^2_{H^{K-2}_x H^s_{\gamma+2s}} \\ &\lesssim \|w^{\ell'_0} f\|^2_{H^{K-2}(\mathbb{R}^n_x) H^1_{\gamma+2s}(\mathbb{R}^n_v)}. \end{aligned} \tag{2.15}$$

Our goal is to show that the above expression is $\lesssim \epsilon_{K,\ell}$. In order to do this, we will use our assumption that $\ell\geq \ell_0$ with ℓ_0 in (1.30) and consider the hard potential case (1.8) and soft potential case (1.9) separately.

In the hard potential case (1.8), recall that $\rho=1$ in (1.23) and $\ell'_0=0$ in (1.31). Using $\mathcal{E}_{K,\ell}(t)$ and $\epsilon_{K,\ell}$ from (1.23) and (1.25) and Theorem 1.4 we have

$$\|f\|^2_{H^{K-2}_x H^1_{\gamma+2s}} \lesssim \sum_{|\alpha|\leq K-2, |\beta|\leq 1} \|w^{\frac{1}{2}(\gamma+2s)} \partial^\alpha_\beta f\|^2_{L^2_x L^2_v} \lesssim \mathcal{E}_{K,\ell}(t) \lesssim \epsilon_{K,\ell},$$

provided that $\ell-1\geq \frac{1}{2}(\gamma+2s)$, which holds since $\ell\geq\ell_0$ using (1.30). The desired bound on B_3 from (2.6) then follows from (2.13), (2.14), and this analysis.

In the soft potential case (1.9), we recall $\rho=-\gamma-2s$ and we note that:

$$\begin{aligned} \|w^{\ell'_0} f\|^2_{H^{K-2}_x H^1_{\gamma+2s}} &\approx \sum_{|\alpha|\leq K-2, |\beta|\leq 1} \|w^{\ell'_0+\frac{1}{2}(\gamma+2s)} \partial^\alpha_\beta f\|^2_{L^2_x L^2_v} \\ &\lesssim \sum_{|\beta|\leq 1, |\alpha|\leq K-2} \|w^{\ell-|\beta|\rho} \partial^\alpha_\beta f\|^2_{L^2_x L^2_v} \lesssim \mathcal{E}_{K,\ell}(t) \lesssim \epsilon_{K,\ell}, \end{aligned}$$

whenever $\ell-\rho\geq \ell'_0-\frac{\rho}{2}$, or equivalently that $\ell\geq \ell'_0-\frac{1}{2}(\gamma+2s)$, which holds since $\ell_0\geq \ell'_0-\frac{1}{2}(\gamma+2s)$. We thus deduce again that B_3 satisfies the bound in (2.6). □

We recall that the proof of Proposition 2.1 relied on the use of (1.17) and hence we only obtained the microscopic terms $\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f(t)\|^2_{L^2_x N^{s,\gamma}}$ on the left-hand side of the inequality. In order to control the macroscopic terms, we use an interaction functional approach to prove the following bound:

Lemma 2.2. *Under the conditions from Theorem 1.4, there exists continuous functionals $\mathcal{I}^k(t)$, for any $k\in\{0,1,\dots,K-1\}$, such that*

$$-\frac{d\mathcal{I}^k}{dt} + \lambda \sum_{|\alpha|=k+1} \|\partial^\alpha [a,b,c]\|^2_{L^2_x} \lesssim \sum_{k\leq|\alpha|\leq k+1} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_{\gamma+2s}}. \tag{2.16}$$

The functional $\mathcal{I}^k(t)$ furthermore satisfies for any $m \geq 0$ the uniform estimates

$$\left|\mathcal{I}^k(t)\right| \lesssim \sum_{k \leq |\alpha| \leq k+1} \|w^{-m}\partial^\alpha f(t)\|^2_{L^2_x L^2_v}. \tag{2.17}$$

Additionally, we will give the precise definition of $\mathcal{I}^k(t)$ below.

This lemma was essentially already proven in [13], except that some of the estimates therein were too crude as written in the statements of the theorems and lemmas. This type of estimate is well known, and we refer to for example [5, 13, 16–18, 23, 25, 28, 29] for previous developments. We will explain carefully the main differences between this estimate, and what is done in [13].

Proof of Lemma 2.2. We define $\mathcal{I}^k(t)$ as follows

$$\mathcal{I}^k(t) = \sum_{|\alpha|=k} \left\{\mathcal{I}^\alpha_a(t) + \mathcal{I}^\alpha_b(t) + \mathcal{I}^\alpha_c(t)\right\},$$

where each of the functionals above are defined as

$$\begin{aligned}
\mathcal{I}^\alpha_a(t) &\overset{\text{def}}{=} \int_{\mathbb{R}^n} dx\ (\nabla \cdot \partial^\alpha b)\, \partial^\alpha a(t,x) + \sum_{i=1}^n \int_{\mathbb{R}^n} dx\ \partial_{x_i}\partial^\alpha r_{bi}\ \partial^\alpha a(t,x), \\
\mathcal{I}^\alpha_b(t) &\overset{\text{def}}{=} -\sum_{j\neq i} \int_{\mathbb{R}^n} dx\ \partial_{x_j}\partial^\alpha r_{ij}\partial^\alpha b_i, \\
\mathcal{I}^\alpha_c(t) &\overset{\text{def}}{=} -\int_{\mathbb{R}^n} dx\ \partial^\alpha r_c(t,x)\ \cdot \nabla_x \partial^\alpha c(t,x).
\end{aligned}$$

Here the r_{bi}, r_{ij}, r_c are all of the form $\langle\{\mathbf{I}-\mathbf{P}\}f, e_k\rangle$ where each e_k is some fixed linear combination of the following basis:

$$\left(v_i|v|^2\sqrt{\mu}\right)_{1\leq i\leq n},\ \left(v_i^2\sqrt{\mu}\right)_{1\leq i\leq n},\ \left(v_i v_j\sqrt{\mu}\right)_{1\leq i<j\leq n},\ \left(v_i\sqrt{\mu}\right)_{1\leq i\leq n},\ \sqrt{\mu}. \tag{2.18}$$

Notice that (2.17) follows directly from the definition of $\mathcal{I}^k(t)$.

Our goal is then to establish (2.16). Now following the proof of [13, Theorem 8.4] we directly obtain

$$\begin{aligned}
-\frac{d\mathcal{I}^k}{dt} + \lambda \sum_{|\alpha|=k+1} \|\partial^\alpha[a,b,c]\|^2_{L^2_x} \lesssim &\sum_{k\leq|\alpha|\leq k+1} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_{\gamma+2s}} \\
&+ \sum_{|\alpha|=k} \|\langle\partial^\alpha\Gamma(f,f), e\rangle\|^2_{L^2_x}.
\end{aligned} \tag{2.19}$$

Here we have trivially sharpened our use of [13, Lemma 8.6] in the proof of [13, Theorem 8.4] from what is stated in [13, Lemma 8.6] to instead what is proved in [13, Lemma 8.6]. Namely, we keep a more precise track of the order of derivatives occurring in the upper bound. The "e" above is another fixed linear combination of the basis (2.18).

We then *claim* that we have the following estimate

$$\begin{aligned}
&\sum_{|\alpha|=k} \|\langle\partial^\alpha\Gamma(f,f), e\rangle\|^2_{L^2_x} \\
&\qquad \lesssim \epsilon_{K,\ell} \sum_{|\alpha'|=k+1} \left(\left\|\partial^{\alpha'}[a,b,c]\right\|^2_{L^2_x} + \|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha'} f\|^2_{L^2_x L^2_{\gamma+2s}}\right).
\end{aligned} \tag{2.20}$$

Now (2.19) combined with (2.20) establishes (2.16) since $\epsilon_{K,\ell}$ is sufficiently small. Thus our goal is reduced to establishing (2.20).

We now use equation (6.12) of [13, Proposition 6.1] to obtain that

$$\sum_{|\alpha|=k} \|\langle \partial^\alpha \Gamma(f,f), e\rangle\|_{L^2_x} \lesssim \sum_{|\alpha|=k} \sum_{\alpha_1 \le \alpha} \left\| \, |\partial^{\alpha_1} f|_{L^2_{-m}} \, |\partial^{\alpha-\alpha_1} f|_{L^2_{-m}} \, \right\|_{L^2_x},$$

which holds for any large $m \ge 0$. Here, we are also using the fact that e satisfies the property (4.1) in [13] which we need in order to apply [13, Proposition 6.1]. More precisely, we are using the fact that $|e| \lesssim \exp(-\lambda|v|^2)$, which follows since e is a linear combination of functions satisfying this bound.

Then for all $p, q \ge 2$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ we have

$$\left\| \, |\partial^{\alpha_1} f|_{L^2_{-m}} \, |\partial^{\alpha-\alpha_1} f|_{L^2_{-m}} \, \right\|_{L^2_x} \lesssim \|\partial^{\alpha_1} f\|_{L^q_x L^2_{\gamma+2s}} \left\|\partial^{\alpha-\alpha_1} f\right\|_{L^p_x L^2_{\gamma+2s}}.$$

In the hard potential case, we take $m \ge 0$ and in the soft potential case, we take $m \ge \frac{-\gamma-2s}{2}$. We do a micro-macro decomposition, as in (1.13), to see that

$$\begin{aligned}
\text{(2.21)} \quad & \|\partial^{\alpha_1} f\|_{L^q_x L^2_{\gamma+2s}} \left\|\partial^{\alpha-\alpha_1} f\right\|_{L^p_x L^2_{\gamma+2s}} \\
&\lesssim \|\partial^{\alpha_1}[a,b,c]\|_{L^q_x} \left\|\partial^{\alpha-\alpha_1}[a,b,c]\right\|_{L^p_x} \\
&+ \|\partial^{\alpha_1}[a,b,c]\|_{L^q_x} \left\|\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\} f\right\|_{L^p_x L^2_{\gamma+2s}} \\
&+ \|\partial^{\alpha_1}\{\mathbf{I}-\mathbf{P}\} f\|_{L^q_x L^2_{\gamma+2s}} \left\|\partial^{\alpha-\alpha_1}[a,b,c]\right\|_{L^p_x} \\
&\quad + \|\partial^{\alpha_1}\{\mathbf{I}-\mathbf{P}\} f\|_{L^q_x L^2_{\gamma+2s}} \left\|\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\} f\right\|_{L^p_x L^2_{\gamma+2s}}.
\end{aligned}$$

To estimate these terms we will use (2.22) just below.

For any $k \in \{0, 1, \dots, K-1\}$ such that $|\alpha| = k$ and $\alpha_1 \le \alpha$ there exists $p, q \ge 2$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ such that we have

$$\begin{aligned}
\text{(2.22)} \quad \|\partial^{\alpha-\alpha_1} f\|_{L^q_x \mathcal{H}_v} \|\partial^{\alpha_1} g\|_{L^p_x \mathcal{H}'_v} &\lesssim \|f\|_{H^{K-2}_x \mathcal{H}_v} \sum_{|\alpha'|=k+1} \|\partial^{\alpha'} g\|_{L^2_x \mathcal{H}'_v} \\
&\quad + \|g\|_{H^{K-2}_x \mathcal{H}'_v} \sum_{|\alpha'|=k+1} \|\partial^{\alpha'} f\|_{L^2_x \mathcal{H}_v}.
\end{aligned}$$

This holds for any separable Hilbert spaces $\mathcal{H}_v$ and $\mathcal{H}'_v$ such as those from Section 1.2. The bound (2.22) follows from Lemma 4.13, Lemma 4.14, and Lemma 4.15, which are proved in Section 4. We are also using the fact that $K-2 \ge K^*_n$. Notice further that in the above lemma $\mathcal{H}_v$ and $\mathcal{H}'_v$ could be $\mathbb{R}$ with norm given by absolute values, as in for instance $\|\partial^{\alpha_1}[a,b,c]\|_{L^q_x}$. To finish this off, we notice that all of the terms in the upper bounds of (2.21) can be bounded above by the norms in the lower bound of the inequality of (2.22) with $\|\cdot\|_{\mathcal{H}_v}$ and $\|\cdot\|_{\mathcal{H}'_v}$ either given by absolute values, $|\cdot|$, or by $|\cdot|_{L^2_{\gamma+2s}}$. Thus using (2.22) with the appropriate p, q as in (2.22), and using Theorem 1.4, in (2.21) we notice that (2.20) holds true. Here, we use the fact that $\ell \ge \frac{\gamma+2s}{2}$ by (1.30). □

2.2. **Estimates in the homogeneous Besov space.** In this sub-section, we assume that the initial data f_0 is sufficiently regular and we prove the following integral inequality for the functional Besov norms of the solution f to (1.10).

Proposition 2.3. *Consider $f(t,x,v)$, the solution to the Boltzmann equation obtained in Theorem 1.4, with initial data $f_0(x,v)$ satisfying $\|f_0\|_{\dot{B}_2^{-\varrho,\infty}L^2_v} < \infty$ with $\varrho \in (0, n/2]$. Let $\epsilon_{K,\ell}$ from (1.25) be small with $\ell \geq \ell^1_0$ with ℓ^1_0 from (1.32). Then*

$$\|f(t)\|^2_{\dot{B}_2^{-\varrho,\infty}L^2_v} \lesssim \|f_0\|^2_{\dot{B}_2^{-\varrho,\infty}L^2_v} + \epsilon_{K,\ell}\int_0^t ds \sum_{|\beta|\leq 2} \|\partial_\beta\{\mathbf{I}-\mathbf{P}\}f(s)\|^2_{L^2_x L^2_{\gamma+2s}}$$
$$+ \int_0^t ds \, \|\mathbf{P}f(s)\|^2_{L^2_x L^2_v} \sum_{|\alpha|=\lfloor \frac{n}{2}-\varrho \rfloor} \|\partial^\alpha \mathbf{P}f(s)\|^{2(1-\theta)}_{L^2_x L^2_v} \sum_{|\alpha'|=\lfloor \frac{n}{2}-\varrho+1 \rfloor} \left\|\partial^{\alpha'}\mathbf{P}f(s)\right\|^{2\theta}_{L^2_x L^2_v}, \tag{2.23}$$

where $\theta = \frac{n}{2} - \varrho - \lfloor \frac{n}{2} - \varrho \rfloor \in [0,1)$.

We note that the integrand in the first integral is related to the dissipation rate (1.24), whose time integral we know is finite by Theorem 1.4. This will be a crucial observation in the following section, when we prove uniform a priori bounds on the macroscopic part in a functional Besov space. The precise bound is given in (3.5).

Proof of Proposition 2.3. The operators Δ_j are defined in Section 4.2. We take Δ_j of (1.10), multiply (1.10) by $\Delta_j f$, integrate over $\mathbb{R}^n_x \times \mathbb{R}^n_v$ and use (1.17) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j f\|^2_{L^2_{x,v}} + \lambda\|\{\mathbf{I}-\mathbf{P}\}\Delta_j f\|^2_{L^2_x N^{s,\gamma}} \lesssim |(\Delta_j \Gamma(f,f), \{\mathbf{I}-\mathbf{P}\}\Delta_j f)|.$$

Here, we used (1.15) and the fact that $\mathbf{P}$ and Δ_j commute. We estimate the upper bound directly as:

$$|(\Delta_j\Gamma(f,f),\{\mathbf{I}-\mathbf{P}\}\Delta_j f)| \leq C_\lambda \|w^{\frac{-\gamma-2s}{2}}\Delta_j\Gamma(f,f)\|^2_{L^2_x L^2_v} + \frac{\lambda}{2}\|\{\mathbf{I}-\mathbf{P}\}\Delta_j f\|^2_{L^2_x L^2_{\gamma+2s}}.$$

As in (4.2), using the Bernstein inequality, one obtains

$$\|w^{\frac{-\gamma-2s}{2}}\Delta_j\Gamma(f,f)\|^2_{L^2_x L^2_v} \lesssim 2^{2\varrho j}\|w^{\frac{-\gamma-2s}{2}}\Gamma(f,f)\|^2_{L^p_x L^2_v},$$

where $\varrho = \frac{n}{p} - \frac{n}{2}$ for $p \in [1,2]$. Equivalently $p = \frac{2n}{n+2\varrho}$ for $\varrho \in [0, \frac{n}{2}]$. We need to estimate $\|w^{\frac{-\gamma-2s}{2}}\Gamma(f,f)\|^2_{L^p_x L^2_v}$. Let us use the estimate from [25, Proposition 3.1, Eq (3.20)] to obtain

$$\|w^{\frac{-\gamma-2s}{2}}\Gamma(f,f)\|^2_{L^p_x L^2_v} \lesssim \left\| \left|w^{\frac{(\gamma+2s)^+}{2}}f\right|_{L^2_v} \left|w^{\frac{(\gamma+2s)^+}{2}}f\right|_{H^i_v} \right\|^2_{L^p_x} \tag{2.24}$$

Here $i = 1$ if $s \in (0, 1/2)$ and $i = 2$ if $s \in [1/2, 1)$ as in (1.7) and $(\gamma+2s)^+$ is the positive part of $\gamma + 2s$. More precisely, we recall that [25, Proposition 3.1, Eq (3.20)] states that for real numbers $b^+, b^-, b' \geq 0$ with $b^- \geq b'$, and $b = b^+ - b^-$, one has the following uniform estimate:

$$|w^b\Gamma(f,g)|_{L^2_v} \lesssim |w^{b^+-b'}f|_{L^2_{\gamma+2s}} |w^{b+b'}g|_{H^i_{(\gamma+2s)+(\gamma+2s)^+}} \tag{2.25}$$

where $i = 1$ for $s \in (0, \frac{1}{2})$ and $i = 2$ for $s \in [\frac{1}{2}, 1)$. Then to obtain (2.24) we use (2.25). In the hard potential case (1.8), we take $b^+ = 0, b^- = \frac{\gamma+2s}{2}$ and $b' = 0$. In the soft potential case (1.9), we take $b^+ = -\frac{\gamma+2s}{2}, b^- = 0$, and $b' = 0$.

In the following we will prove upper bounds for the estimate in (2.24). We do a micro-macro decomposition, e.g. (1.13), of f to further bound (2.24) as

$$
\begin{aligned}
\left\| \left| w^{\frac{(\gamma+2s)^+}{2}} f \right|_{L^2_v} \left| w^{\frac{(\gamma+2s)^+}{2}} f \right|_{H^i} \right\|_{L^p_x}^2 \lesssim & \left\| \, |[a,b,c]|^2 \, \right\|_{L^p_x}^2 \\
& + \left\| \left| w^{\frac{(\gamma+2s)^+}{2}} \{\mathbf{I}-\mathbf{P}\} f \right|_{L^2_v} |[a,b,c]| \right\|_{L^p_x}^2 \\
& + \left\| \, |[a,b,c]| \left| w^{\frac{(\gamma+2s)^+}{2}} \{\mathbf{I}-\mathbf{P}\} f \right|_{H^i} \right\|_{L^p_x}^2 \\
& + \left\| \left| w^{\frac{(\gamma+2s)^+}{2}} \{\mathbf{I}-\mathbf{P}\} f \right|_{L^2_v} \left| w^{\frac{(\gamma+2s)^+}{2}} \{\mathbf{I}-\mathbf{P}\} f \right|_{H^i} \right\|_{L^p_x}^2 .
\end{aligned}
\tag{2.26}
$$

We will now estimate each of the terms in the upper bound of (2.26) separately. The main difficulty in estimating these terms arises from the macroscopic parts of the solution. Notice that for the first upper bound in (2.26) we have the equality

$$
\left\| \, |[a,b,c]|^2 \, \right\|_{L^p_x}^2 = \| [a,b,c] \|_{L^{2p}_x}^4 .
$$

If $p=1$, which is the case when $\varrho = \frac{n}{2}$, then a major difficulty is that this term can not be further estimated from above in terms of the dissipation from (1.24). We first note that, by arguing as in the proof of (2.8), we can deduce that:

$$
|\partial^{\alpha_1}[a,b,c]| \lesssim |\partial^{\alpha_1} \mathbf{P} f|_{L^2_v}
\tag{2.27}
$$

Since $\frac{1}{2}+\frac{\varrho}{n}=\frac{1}{p}$, use Hölder and (2.27) with $\alpha_1=0$ to obtain

$$
\left\| \, |[a,b,c]|^2 \, \right\|_{L^p_x}^2 \lesssim \| \mathbf{P} f \|_{L^2_x L^2_v}^2 \| \mathbf{P} f \|_{L^{\frac{n}{\varrho}}_x L^2_v}^2 .
\tag{2.28}
$$

Since $\varrho \in (0, \frac{n}{2}]$ then $\frac{n}{\varrho} \in [2,\infty)$.

Let us first consider the subcase when $\frac{n}{2}-\varrho \notin \mathbb{Z}$. For the term in $L^{\frac{n}{\varrho}}_x L^2_v$ we use (4.10) and Lemma 4.14 to obtain

$$
\| \mathbf{P} f \|_{L^{\frac{n}{\varrho}}_x L^2_v}^2 \lesssim \left(\sum_{|\alpha|=m} \| \partial^\alpha \mathbf{P} f \|_{L^2_x L^2_v} \right)^{2(1-\theta)} \left(\sum_{|\alpha|=j} \| \partial^\alpha \mathbf{P} f \|_{L^2_x L^2_v} \right)^{2\theta}
\tag{2.29}
$$

where now $\theta = \frac{\frac{n}{2}-\varrho-m}{j-m}$ and m and j, with $m \neq j$ are to be chosen. We generally choose $j \overset{\text{def}}{=} \lfloor \frac{n}{2} - \varrho + 1 \rfloor$ and choose $m \overset{\text{def}}{=} \lfloor \frac{n}{2} - \varrho \rfloor$. Then $j-m=1$ and $\theta = \frac{n}{2} - \varrho - \lfloor \frac{n}{2} - \varrho \rfloor \in (0,1)$, since by assumption $\frac{n}{2} - \varrho \notin \mathbb{Z}$.

If $\frac{n}{2} - \varrho \in \mathbb{Z}$, we can use (4.11) to deduce that (2.29) holds with $\theta = 0$ and $m = \frac{n}{2} - \varrho$. This completes our estimate for the first term in (2.26).

The remaining terms can be estimated quickly. Indeed since $\frac{1}{2} + \frac{\varrho}{n} = \frac{1}{p}$ the last three terms in (2.26) are all bounded above by a constant multiple of

$$\begin{aligned}(2.30)\quad \|w^{\frac{(\gamma+2s)^+}{2}} f\|^2_{L_x^{\frac{n}{\varrho}} L_v^2} \sum_{|\beta|\le 2} \|w^{\frac{(\gamma+2s)^+}{2}} \partial_\beta \{\mathbf{I}-\mathbf{P}\} f\|^2_{L^2_{x,v}} \\ \lesssim \epsilon_{K,\ell} \sum_{|\beta|\le 2} \|w^\ell \partial_\beta \{\mathbf{I}-\mathbf{P}\} f\|^2_{L^2_x L^2_{\gamma+2s}}.\end{aligned}$$

We used the functional Sobolev embedding (4.13) to obtain the last inequality above. More precisely, we note that: $\|w^{\frac{(\gamma+2s)^+}{2}} f\|_{L_x^{\frac{n}{\varrho}} L_v^2} \lesssim \|w^{\frac{(\gamma+2s)^+}{2}} f\|_{H^m_x L^2_v}$ for $m = \lfloor \frac{n}{2} - \varrho + 1 \rfloor \le K$. Hence, since $\ell \ge \frac{(\gamma+2s)^+}{2}$, it follows from Theorem 1.4 that $\|w^{\frac{(\gamma+2s)^+}{2}} f\|^2_{L_x^{\frac{n}{\varrho}} L_v^2} \lesssim \epsilon_{K,\ell}$. The bound (2.30) now follows. □

3. Large-time non-linear decay in Besov spaces

Our goal in this section is to prove the main result. We will do this by combining the differential inequalities from the previous section to prove a stronger differential inequality for a quantity which is equivalent to the appropriate Sobolev norm of the solution f to (1.10). More precisely, for a fixed $k \in \{0, 1, \dots, K-1\}$ and for a small $\kappa > 0$, we define $\mathcal{E}^k(t)$ to be the following continuous energy functional

$$\mathcal{E}^k(t) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{k\le|\alpha|\le K} \|\partial^\alpha f(t)\|^2_{L^2_x L^2_v} - \kappa \sum_{k\le j\le K-1} \mathcal{I}^j(t).$$

Then by (2.17) for $\kappa > 0$ sufficiently small we have

$$\mathcal{E}^k(t) \approx \sum_{k\le|\alpha|\le K} \|\partial^\alpha f(t)\|^2_{L^2_x L^2_v}. \tag{3.1}$$

Our goal will be to prove a strong differential inequality for $\mathcal{E}^k(t)$.

We will give two proofs of the main result. They will differ in the techniques which we will use in order to prove them, but they will also differ in the main differential inequality which we prove for $\mathcal{E}^k(t)$. The first proof below uses a new dyadic time-regularity summation argument to close the time decay estimates. The second proof uses a double interpolation separately in spatial regularity and in velocity. For the interpolation steps, we use techniques from Section 4.

The main differential inequality obtained from the first proof is given in (3.13) and the one which follows from the second proof is given in (3.17). Both of them give the same high Sobolev norm estimate $\mathcal{E}^k(t) \lesssim (1+t)^{-(k+\varrho)}$. In order to derive both differential inequalities, we need to assume the a priori uniform bound (3.5) on the macroscopic terms in the functional Sobolev space. The latter bound is proved by using the integral inequality in Besov space given in Proposition 2.3.

In Sub-section 3.1, we prove some preliminary differential inequalities which follow from the results of the previous section. In Sub-section 3.2, we give the first proof of Theorem 1.1 under the assumption of the a priori bound (3.5). In the following sub-section, we give the second proof. Furthermore, in Sub-section 3.4, we verify the a priori bound (3.5). In Sub-section 3.5, we collect all the estimates and use an interpolation argument to deduce the bounds which are claimed in Theorem 1.1 and in Corollary 1.2. We note that the bounds we prove are, in fact, slightly stronger than the ones in the statement; see (3.23). Finally, in Sub-section 3.6, we prove Theorem 1.3 by using the improved linear decay estimates from Section 5.

3.1. **Some preliminary differential inequalities.** In this sub-section, we collect the main estimates from Section 2 in order to deduce one differential inequality for $\mathcal{E}^k(t)$. Namely, using Lemma 2.2 with Proposition 2.1, we deduce the following instantaneous differential inequality for some $\eta > 0$:

$$\frac{d\mathcal{E}^k}{dt}(t) + \eta\left(\mathcal{D}^h_{k+1}(t) + \mathcal{D}^m_k(t)\right) \leq 0. \tag{3.2}$$

where the hydrodynamic part of the dissipation, $\mathcal{D}^h_{k+1}(t)$, and the microscopic part of the dissipation, $\mathcal{D}^m_k(t)$, are each defined as

$$\mathcal{D}^h_{k+1}(t) \stackrel{\text{def}}{=} \sum_{k+1\leq|\alpha|\leq K} \|\partial^\alpha[a,b,c]\|^2_{L^2_x}, \quad \mathcal{D}^m_k(t) \stackrel{\text{def}}{=} \sum_{k\leq|\alpha|\leq K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_{\gamma+2s}}.$$

More precisely, we first note that, by Lemma 2.2, Proposition 2.1 and by the definition of $\mathcal{E}^k(t)$, it follows that for some $C_1, C_2 > 0$ we have

$$\begin{aligned}\frac{d\mathcal{E}^k}{dt}(t) &\leq C_1\epsilon_{K,\ell} \sum_{k+1\leq|\alpha|\leq K} \|\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}} - 2\lambda \sum_{k\leq|\alpha|\leq K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}} \\ &\quad + C_2\,\kappa \sum_{k\leq|\alpha|\leq K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_{\gamma+2s}} - \kappa\lambda \sum_{k+1\leq|\alpha|\leq K} \|\partial^\alpha[a,b,c]\|^2_{L^2_x} \\ &\leq (C_1\epsilon_{K,\ell} - 2\lambda + C_2\kappa) \sum_{k\leq|\alpha|\leq K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}} \\ &\quad + C_1\epsilon_{K,\ell} \sum_{k+1\leq|\alpha|\leq K} \|\mathbf{P}\partial^\alpha f\|^2_{L^2_x N^{s,\gamma}} - \kappa\lambda \sum_{k+1\leq|\alpha|\leq K} \|\partial^\alpha[a,b,c]\|^2_{L^2_x}.\end{aligned} \tag{3.3}$$

Now, we note that, for some $C_3 > 0$:

$$\|\mathbf{P}\partial^\alpha f\|_{L^2_x N^{s,\gamma}} \leq C_3 \|\partial^\alpha[a,b,c]\|_{L^2_x} \tag{3.4}$$

To prove this we recall (1.13) and the fact that $\mathbf{P}$ and ∂^α commute. Then (3.4) follows taking $|\cdot|_{N^{s,\gamma}}$, using the triangle inequality, and taking $\|\cdot\|_{L^2_x}$.

One first fixes $\lambda > 0$ which satisfies Lemma 2.2 and Proposition 2.1. Afterwards, one takes $\kappa > 0$ small in order to satisfy (3.1) and for which $-2\lambda + C_2\kappa < 0$. Finally, one chooses $\epsilon_{K,\ell}$ small enough so that $C_1\epsilon_{K,\ell} - 2\lambda + C_2\kappa < 0$ and $C_1 C_3^2 \epsilon_{K,\ell} - \kappa\lambda < 0$. Substituting these estimates into (3.3), we deduce that (3.2) indeed holds.

We now prove Theorem 1.1. As was mentioned above, we will give two proofs. We will suppose in both proofs below that for some $\varrho \in (0, \frac{n}{2}]$ we have the following uniform estimate

$$\|[a,b,c](t)\|_{\dot{B}_2^{-\varrho,\infty}} \leq C_0 < \infty, \quad \forall t \geq 0. \tag{3.5}$$

Let us see how Theorem 1.1 follows if we know this additional bound.

3.2. **First proof of Theorem 1.1.** We recall that the first proof is based on a dyadic time-regularity decomposition.

First proof of Theorem 1.1. We use time-weighted estimates. Fix $s \geq 0$ to be chosen later, and $\varepsilon > 0$ small, we multiply (3.2) by the time weight $(1+\varepsilon t)^s$ to obtain

$$\frac{d}{dt}\left((1+\varepsilon t)^s \mathcal{E}^k(t)\right) + \eta(1+\varepsilon t)^s\left(\mathcal{D}^h_{k+1}(t) + \mathcal{D}^m_k(t)\right) \leq s\varepsilon(1+\varepsilon t)^{s-1}\mathcal{E}^k(t). \tag{3.6}$$

We use (3.1), the decomposition (1.13) with (1.14), and estimates which are analogous to the one used in the proof of (3.4) to obtain

$$\mathcal{E}^k(t) \lesssim \sum_{|\alpha|=k} \|\partial^\alpha [a,b,c]\|^2_{L^2_x}(t) + \mathcal{D}^h_{k+1}(t) + \sum_{k\le|\alpha|\le K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_v}(t). \tag{3.7}$$

We will handle each of the terms in the upper bound of (3.7) separately.

To handle the first term in the upper bound of (3.7) we notice that

$$\sum_{|\alpha|=m} \|\partial^\alpha [a,b,c]\|^2_{L^2_x}(t) \approx \sum_{j\in\mathbb{Z}} 2^{2mj} \|\Delta_j [a,b,c]\|^2_{L^2_x}(t), \quad \forall m = 0,1,2,\dots \tag{3.8}$$

Recall Δ_j are the Littlewood-Paley projections onto frequencies 2^j which are defined in Section 4.2. Notice that (3.8) is a consequence of Theorem 4.10 combined with Lemma 4.13 and (4.3). Now (3.8) implies that we have

$$\begin{aligned}(1+\varepsilon t)^{-1} \sum_{2^j\sqrt{1+\varepsilon t}\ge 1} 2^{2kj} \|\Delta_j [a,b,c]\|^2_{L^2_x}(t) &\lesssim \sum_{2^j\sqrt{1+\varepsilon t}\ge 1} 2^{2(k+1)j} \|\Delta_j [a,b,c]\|^2_{L^2_x}(t) \\ &\lesssim \sum_{|\alpha|=k+1} \|\partial^\alpha [a,b,c]\|^2_{L^2_x}(t).\end{aligned}$$

Crucially the implied constants are uniform in $\sqrt{1+\varepsilon t}$. Using (3.8) we have shown

$$\begin{aligned}(1+\varepsilon t)^{-1} \sum_{|\alpha|=k} \|\partial^\alpha [a,b,c]\|^2_{L^2_x}(t) &\lesssim \sum_{|\alpha|=k+1} \|\partial^\alpha [a,b,c]\|^2_{L^2_x}(t) \\ &\quad + (1+\varepsilon t)^{-1} \sum_{2^j\sqrt{1+\varepsilon t}\le 1} 2^{2kj} \|\Delta_j [a,b,c]\|^2_{L^2_x}(t).\end{aligned} \tag{3.9}$$

Furthermore, we have the direct estimate

$$\begin{aligned}&\sum_{2^j\sqrt{1+\varepsilon t}\le 1} 2^{2kj} \|\Delta_j [a,b,c](t)\|^2_{L^2_x} \\ &\qquad \lesssim \|[a,b,c](t)\|^2_{\dot{B}_2^{-\varrho,\infty}} \sum_{2^j\sqrt{1+\varepsilon t}\le 1} 2^{2(k+\varrho)j} \\ &\qquad \lesssim \|[a,b,c](t)\|^2_{\dot{B}_2^{-\varrho,\infty}} (1+\varepsilon t)^{-(k+\varrho)}.\end{aligned}$$

In the last inequality we have used the following calculation

$$\begin{aligned}\sum_{2^j\sqrt{1+\varepsilon t}\le 1} 2^{2(k+\varrho)j} &= (1+\varepsilon t)^{-(k+\varrho)} \sum_{2^j\sqrt{1+\varepsilon t}\le 1} \left(2^j\sqrt{1+\varepsilon t}\right)^{2(k+\varrho)} \\ &\lesssim (1+\varepsilon t)^{-(k+\varrho)},\end{aligned}$$

where the implied uniform constant in the upper bound is independent of the size of $\sqrt{1+\varepsilon t}$. Collecting these estimates, including (3.9), we obtain

$$\begin{aligned}(1+\varepsilon t)^{-1} \sum_{|\alpha|=k} \|\partial^\alpha [a,b,c]\|^2_{L^2_x}(t) &\lesssim \sum_{|\alpha|=k+1} \|\partial^\alpha [a,b,c]\|^2_{L^2_x}(t) \\ &\quad + \|[a,b,c](t)\|^2_{\dot{B}_2^{-\varrho,\infty}} (1+\varepsilon t)^{-1-(k+\varrho)}.\end{aligned}$$

which by (3.5) is:

$$\lesssim \sum_{|\alpha|=k+1} \|\partial^\alpha [a,b,c]\|^2_{L^2_x}(t) + C_0^2(1+\varepsilon t)^{-1-(k+\varrho)}. \tag{3.10}$$

This will be our main estimate for the first term in the upper bound of (3.7).

For the third term in (3.7) we will use a time-velocity splitting. Recalling (1.22), we define the time-velocity splitting sets by

$$E(t) = \{(1+\varepsilon t)^{-1} \le w^{\gamma+2s}(v)\}, \quad E^c(t) = \{1 < w^{-\gamma-2s}(v)(1+\varepsilon t)^{-1}\}. \tag{3.11}$$

With this splitting, we have the following estimate

$$\begin{aligned}(1+\varepsilon t)^{-1} \sum_{k\le|\alpha|\le K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_v} &\lesssim \sum_{k\le|\alpha|\le K} \|\mathbf{1}_E\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_{\gamma+2s}} \\ &+ A(1+\varepsilon t)^{-1-(k+\varrho)} \sum_{k\le|\alpha|\le K} \|w^{\ell_0}\mathbf{1}_{E^c}\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_v} \\ &\lesssim \sum_{k\le|\alpha|\le K} \|\mathbf{1}_E\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|^2_{L^2_x L^2_{\gamma+2s}} + A(1+\varepsilon t)^{-1-(k+\varrho)}\epsilon_{K,\ell}.\end{aligned} \tag{3.12}$$

where $\mathbf{1}_E$ is the usual indicator function of the set E from (3.11). Using E^c the last estimate above holds for $A=1$ since $\ell_0 \ge -(\gamma+2s)(k+\varrho)/2 \ge 0$ in the case of the soft potentials (1.9) using (1.30). For the hard potentials (1.8), we notice that $\mathbf{1}_E \equiv 1$ (always) and we then have the above estimate with $A=0$.

We use (3.6), (3.7), (3.10), and (3.12) to deduce that, for $\varepsilon>0$ sufficiently small

$$\begin{aligned}\frac{d}{dt}\left((1+\varepsilon t)^s \mathcal{E}^k(t)\right) + \lambda(1+\varepsilon t)^s \left(\mathcal{D}^h_{k+1}(t) + \mathcal{D}^m_k(t)\right) \\ \lesssim (1+\varepsilon t)^{s-1-(k+\varrho)}\left(C_0^2 + \epsilon_{K,\ell}\right).\end{aligned} \tag{3.13}$$

Choosing $s = k+\varrho+\delta$ for a small $\delta\in(0,1)$ and integrating this in time:

$$\begin{aligned}\mathcal{E}^k(t) &\lesssim \left(C_0^2+\epsilon_{K,\ell}\right)(1+\varepsilon t)^{-k-\varrho-\delta}\int_0^t du\ (1+\varepsilon u)^{\delta-1} \\ &\quad + (1+\varepsilon t)^{-k-\varrho-\delta}\mathcal{E}^k(0) \\ &\lesssim (1+t)^{-k-\varrho-\delta}(1+t)^{\delta} \approx (1+t)^{-(k+\varrho)}.\end{aligned} \tag{3.14}$$

The constant is uniform in $t\ge 0$. Then (3.14) holds as long as we have (3.5). We note that (1.5) immediately follows from (3.1) and (3.14). □

3.3. **Second proof of Theorem 1.1.** The first proof of Theorem 1.1 used dyadic decomposition. We give another proof of this theorem which uses instead interpolation as in the following lemma:

Lemma 3.1. *Suppose $k\ge 0$ and $\varrho>0$. We have the interpolation estimate*

$$\sum_{|\alpha|=k} \|\partial^\alpha g\|^2_{L^2_x} \lesssim \left(\sum_{|\alpha|=k+1} \|\partial^\alpha g\|^2_{L^2_x}\right)^{\theta} \|g\|^{2(1-\theta)}_{\dot{B}_2^{-\varrho,\infty}},$$

where $\theta = \frac{\varrho+k}{\varrho+k+1}$.

Lemma 3.1 will be proven in Lemma 4.5 of Section 4. We now prove again Theorem 1.1, assuming Lemma 3.1 and using Proposition 2.3 as follows.

Second proof of Theorem 1.1. From Lemma 3.1 and (3.5), we obtain that

$$\sum_{|\alpha|=k} \|\partial^\alpha[a,b,c]\|_{L^2_x}^2(t) \le C \left(\sum_{|\alpha|=k+1} \|\partial^\alpha[a,b,c]\|_{L^2_x}^2(t) \right)^{\frac{\varrho+k}{\varrho+k+1}}$$

At the same time from Theorem 1.4 we have

$$\sum_{k+1\le|\alpha|\le K} \|\partial^\alpha[a,b,c]\|_{L^2_x}^2(t) \lesssim \mathcal{E}_{K,\ell}(t) \lesssim \epsilon_{K,\ell}.$$

Collecting these estimates, since $1/\theta > 1$ we obtain

$$\left(\sum_{k\le|\alpha|\le K} \|\partial^\alpha[a,b,c]\|_{L^2_x}^2(t) \right)^{1+\frac{1}{\varrho+k}} \lesssim \sum_{k+1\le|\alpha|\le K} \|\partial^\alpha[a,b,c]\|_{L^2_x}^2(t). \tag{3.15}$$

This is the main estimate that we will use for the macroscopic part of the dissipation.

To handle the microscopic part of the dissipation, we first consider the soft potential case when $\gamma+2s<0$ in (1.9). We use the weighted interpolation estimate

$$\sum_{k\le|\alpha|\le K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_{L^2_x L^2_v}^2 \lesssim \left(\sum_{k\le|\alpha|\le K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_{L^2_x L^2_{\gamma+2s}}^2 \right)^{\theta'} \times \left(\sum_{k\le|\alpha|\le K} \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\theta'}{1-\theta'}}\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_{L^2_x L^2_v}^2 \right)^{(1-\theta')}.$$

This follows directly from the Hölder inequality, and holds for any $\theta' \in (0,1)$. However from the previous interpolation we choose $\theta' = \theta = \frac{\varrho+k}{\varrho+k+1}$. Then we have

$$\left(\frac{-\gamma-2s}{2}\right)\frac{\theta'}{1-\theta'} = \left(\frac{-\gamma-2s}{2}\right)(\varrho+k) \overset{\mathrm{def}}{=} \ell_1.$$

Then again if we apply Theorem 1.4 with any $\ell \ge \ell_1$ we observe that

$$\sum_{k\le|\alpha|\le K} \|w^{\ell_1}\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_{L^2_x L^2_v}^2 \lesssim \mathcal{E}_{K,\ell}(t) \lesssim \epsilon_{K,\ell}.$$

Since $\ell \ge \ell_0$ from (1.30), then one indeed has $\ell \ge \ell_0 \ge \ell_1$. The estimates in this paragraph then imply that

$$\left(\sum_{k\le|\alpha|\le K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_{L^2_x L^2_v}^2 \right)^{1+\frac{1}{\varrho+k}} \lesssim \sum_{k\le|\alpha|\le K} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_{L^2_x L^2_{\gamma+2s}}^2. \tag{3.16}$$

This is the main estimate that we will use for the velocity degenerate microscopic part of the dissipation. Notice that (3.16) is also true for the hard potentials (1.8) with $\ell_1 = 0$; in this case the proof of (3.16) does not require interpolation.

Now we collect (3.16) and (3.15) into (3.2) to obtain that for some $C>0$

$$\frac{d}{dt}\mathcal{E}^k(t) + C\left(\mathcal{E}^k(t)\right)^{1+\frac{1}{\varrho+k}} \le 0. \tag{3.17}$$

Then by a standard argument with this differential inequality we obtain that

$$\mathcal{E}^k(t) \lesssim (1+t)^{-(\varrho+k)}, \tag{3.18}$$

uniformly in $t \geq 0$ which holds as long as we have (3.5). $\square$

3.4. **Proof of the a priori bound** (3.5)**.** In this sub-section, we verify the a priori bound (3.5) which was crucial in the two proofs of Theorem 1.1. In order to prove this bound, we will use Proposition 2.3. As was noted earlier, the first integral on the right-hand side of the inequality obtained in Proposition 2.3 is bounded by the integral of the dissipation (1.24), whereas for the second integral, we have to work harder. As we will see in the proof, the case when $\varrho \in (0, \frac{n-2}{2})$ can still be estimated by using the integral of the dissipation, whereas the case $\varrho \in [\frac{n-2}{2}, \frac{n}{2}]$ is more difficult, and it requires an additional interpolation step. We are interested in obtaining the endpoint case $\rho = \frac{n}{2}$ since $\dot{B}_2^{-\frac{n}{2},\infty} \supset L^1(\mathbb{R}^n_x)$.

Proof of (3.5). We will estimate the last two terms in the upper bound of (2.23). In particular, for any solution $f(t)$ as in Theorem 1.4, we will prove that

$$\|f(t)\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} \leq C_0 < \infty, \quad \forall t \geq 0, \quad \text{if} \quad \|f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} < \infty, \tag{3.19}$$

which is stronger than (3.5) since $\mathbf{P}$ is a projection on L^2_v and the Littlewood-Paley projections Δ_j commute with $\mathbf{P}$. In other words, we are using that for all j:

$$2^{-\varrho j}\|\Delta_j f\|_{L^2_x L^2_v} \geq 2^{-\varrho j}\|\Delta_j \mathbf{P} f\|_{L^2_x L^2_v} \approx 2^{-\varrho j}|\Delta_j [a,b,c]|_{L^2_v}$$

where we are also using (1.14). By taking suprema in j, it follows that the condition (3.19) is indeed stronger than (3.5).

For the second term in (2.23), for some finite constant $C > 0$, from (1.26) we have that

$$\begin{aligned} \int_0^t ds \sum_{|\beta|\leq 2} \|\partial_\beta \{\mathbf{I}-\mathbf{P}\} f(s)\|^2_{L^2_x L^2_{\gamma+2s}} \\ \leq \int_0^t ds \sum_{|\beta|\leq 2} \|\partial_\beta \{\mathbf{I}-\mathbf{P}\} f(s)\|^2_{L^2_x N^{s,\gamma}_{\ell-2\rho}} \leq C, \end{aligned} \tag{3.20}$$

provided that $\ell - 2\rho \geq 0$. This condition is satisfied because $\ell \geq \ell_0$ for ℓ_0 from (1.30). Namely, for the hard potentials (1.8), we have $\ell_0 \geq 2 = 2\rho$. In the soft potential case (1.9), we also know that $\ell_0 \geq 2(-\gamma - 2s) = 2\rho$ from (1.30).

To estimate the last term in the upper bound of (2.23) we first suppose that $\varrho \in (0, \frac{n-2}{2})$; then (3.5) will follow directly from (2.23) when combined with the time integrated Lyapunov inequality (1.26). In particular $\varrho \in (0, \frac{n-2}{2})$ implies that $\lfloor \frac{n}{2} - \varrho \rfloor \geq 1$ and also $2 \leq \lfloor \frac{n}{2} - \varrho + 1 \rfloor \leq K^*_n$. Then we have

$$\begin{aligned} \int_0^t ds\, \|\mathbf{P} f(s)\|^2_{L^2_x L^2_v} \sum_{|\alpha| = \lfloor \frac{n}{2} - \varrho \rfloor} \|\partial^\alpha \mathbf{P} f(s)\|^{2(1-\theta)}_{L^2_x L^2_v} \sum_{|\alpha'| = \lfloor \frac{n}{2} - \varrho + 1 \rfloor} \left\|\partial^{\alpha'} \mathbf{P} f(s)\right\|^{2\theta}_{L^2_x L^2_v} \\ \lesssim \epsilon_{K,\ell} \int_0^t ds \sum_{1 \leq |\alpha| \leq K^*_n} \|\partial^\alpha \mathbf{P} f(s)\|^2_{L^2_x L^2_v} \lesssim 1. \end{aligned}$$

Recall from (2.23) that $\theta \in [0, 1)$. In the above we have used the time integrated Lyapunov inequality (1.26), (1.24) and the fact that $\mathbf{P}$ is a bounded operator on $L^2_x L^2_v$ which commutes with differentiation in x. Then when $\varrho \in (0, \frac{n-2}{2})$ and $\|f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} < \infty$, (3.5) follows from (2.23) and these last few calculations.

The next case we consider is when $\varrho \in (\frac{n-2}{2}, \frac{n}{2})$ and $\|f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} + \|f_0\|_{L^2_x L^2_v} < \infty$ (as we will see below, the case $\varrho = \frac{n-2}{2}$ is a little bit different). In this situation we obtain that $\|f_0\|_{\dot{B}_2^{-\varrho',\infty} L^2_v} < \infty$ for any $\varrho' \in (0, \varrho)$ by interpolation:

$$\|f_0\|_{\dot{B}_2^{-\varrho',\infty} L^2_v} \le \|f_0\|^{\varrho'/\varrho}_{\dot{B}_2^{-\varrho,\infty} L^2_v} \|f_0\|^{(\varrho-\varrho')/\varrho}_{L^2_x L^2_v}. \tag{3.21}$$

Then (3.21) follows from Lemma 4.8 with $\mathcal{H}_v = L^2_v$ and $L^2_x L^2_v \approx \dot{B}^{0,2}_2 L^2_v \subset \dot{B}^{0,\infty}_2 L^2_v$. We conclude that (3.18) holds for any $\varrho' \in (0, \frac{n-2}{2})$ and any $k \in \{0, 1, \ldots, K-1\}$.

Then, by using (2.23), (3.20), (3.1) and (3.18) for ϱ', it follows that

$$\begin{aligned}(3.22)\quad \|f(t)\|^2_{\dot{B}_2^{-\varrho,\infty} L^2_v} &\lesssim \|f_0\|^2_{\dot{B}_2^{-\varrho,\infty} L^2_v} + 1 \\ &+ \int_0^t ds\ \ (1+s)^{-\varrho'} (1+s)^{-\left(\lfloor \frac{n}{2} - \varrho \rfloor + \varrho'\right)(1-\theta)} (1+s))^{-\left(\lfloor \frac{n}{2} - \varrho + 1 \rfloor + \varrho'\right)\theta} \\ &\le C_0 < \infty.\end{aligned}$$

Here from (2.23) we use $\theta = \frac{n}{2} - \varrho - \lfloor \frac{n}{2} - \varrho \rfloor$. Given any $\varrho \in (\frac{n-2}{2}, \frac{n}{2})$ we choose $\varrho' \overset{\text{def}}{=} \frac{n-2}{2} - \epsilon' \in (0, \frac{n-2}{2})$ for any sufficiently small $\epsilon' \in (0, \frac{\theta}{2})$. This then guarantees that the upper bound for (3.22) is finite since $n \ge 3$. More precisely, we want to guarantee that $-2\varrho' - \lfloor \frac{n}{2} - \varrho \rfloor - \theta = -2\varrho' - \theta < -1$, which can be shown to follow from from $2\epsilon - \theta < n - 3$. Then the above choice of ϵ is sufficient.

Furthermore, if $\varrho = \frac{n-2}{2}$, we note that $\frac{n}{2} - \varrho = 1$, so the above construction gives $\theta = 0$. We take $\varrho' \in (0, \frac{n-2}{2})$ and we note that the integral in (3.22) becomes:

$$\int_0^t (1+s)^{-2\varrho'-1} ds$$

which is uniformly bounded in t since $-2\varrho' - 1 < -1$.

Lastly if $\varrho = \frac{n}{2}$ and $\|f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} < \infty$ we again use the estimates (3.21) and (3.22) with $\theta = 0$ and $\varrho' \overset{\text{def}}{=} \frac{n}{2} - \epsilon \in (\frac{n-2}{2}, \frac{n}{2})$ for some sufficiently small $\epsilon \in (0, 1/2)$. Then $-2\varrho' < -1$ in (3.22) and this establishes (3.5) for any $\varrho \in [0, \frac{n}{2}]$. $\square$

3.5. **Conclusion of the proof; the interpolation step.** In this sub-section, we prove the exact statement of the result in Theorem 1.1. For completeness, we explain now how to deduce (1.4) from (1.5) or (3.18). In particular from (3.1), (3.18) and the fact that $\dot{H}^k_{x,v} \approx \dot{B}^{k,2}_2 L^2_v$, we obtain that

$$\|f(t)\|^2_{\dot{B}_2^{k,2} L^2_v} \lesssim (1+t)^{-(k+\varrho)},$$

for $k \in \{0, 1, \ldots, K-1\}$. We furthermore use the above, (3.19), and (4.15) with $\mathcal{H}_v = L^2_v$ to deduce that

$$\|f(t)\|^2_{\dot{B}_2^{a,s} L^2_v} \lesssim (1+t)^{-(a+\varrho)}, \quad \forall s \ge 2\left(\frac{k+\varrho}{a+\varrho}\right), \quad \forall a \in [-\varrho, k], \tag{3.23}$$

which is stronger than the bound in (1.4). These estimates hold uniformly in $t \ge 0$.

Specifically to obtain (3.23) we used the interpolation result (4.15) with $\ell = a$, $k = -\varrho$, $m = k$, $q' = 2\left(\frac{k+\varrho}{a+\varrho}\right)$, $r' = \infty$, $p' = 2$ and $q = r = p = 2$. We also noted the following embeddings $\dot{B}^{a,q'}_2 L^2_v \subset \dot{B}^{a,s}_2 L^2_v \subset \dot{B}^{a,\infty}_2 L^2_v$.

Finally, we prove Corollary 1.2:

Proof of Corollary 1.2. Fix some $2 \le r \le \infty$. We will use the interpolation estimate

$$\sum_{|\alpha|=k} \|\partial^\alpha f(t)\|^2_{L^r_x L^2_v} \lesssim \|f(t)\|^{2(1-\theta)}_{L^2_x L^2_v} \sum_{|\alpha|=K-1} \|\partial^\alpha f(t)\|^{2\theta}_{L^2_x L^2_v}, \ \theta = \frac{k + \frac{n}{2} - \frac{n}{r}}{K-1}. \tag{3.24}$$

This follows from (4.10) (with $m = 0$, $\varrho = K - 1$, and $\mathcal{H}_v = L^2_v$) combined with Lemma 4.13 and Lemma 4.14. The interpolation (3.24) holds (using (4.10)) when $K - 1 > k + \frac{n}{2} - \frac{n}{r}$ (which explains the corresponding restriction in Corollary 1.2). Then Corollary 1.2 follows by applying (1.5) to the upper bounds of (3.24). □

3.6. **The proof of the faster time-decay rates in Theorem 1.3.** Let us now prove Theorem 1.3. In order to prove this theorem, we will need to use linear decay estimates given by Proposition 3.2 and Proposition 3.3 below. As in the first proof of Theorem 1.1, we will use a dyadic decomposition. Consider the linearized Boltzmann equation with a microscopic source $g = g(t, x, v)$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + Lf = g, \\ f|_{t=0} = f_0. \end{cases} \tag{3.25}$$

For the nonlinear system (1.10), the non-homogeneous source term is given by

$$g = \Gamma(f, f) = \{\mathbf{I} - \mathbf{P}\}\Gamma(f, f). \tag{3.26}$$

Solutions of (3.25) formally take the following form

$$f(t) = \mathbb{A}(t) f_0 + \int_0^t ds\ \mathbb{A}(t-s)\ g(s), \quad \mathbb{A}(t) \overset{\text{def}}{=} e^{-t\mathbf{B}}, \quad \mathbf{B} \overset{\text{def}}{=} L + v \cdot \nabla_x. \tag{3.27}$$

Here $\mathbb{A}(t)$ is the linear solution operator corresponding to (3.25) with $g = 0$.

Our goal in this section will be to prove Theorem 1.3. This theorem is more subtle that the previous decay theorems because of the more severe singularity in for example the space $\dot{B}_2^{-(n+2)/2,\infty}$. In this situation we do not have uniform in time bounds such as either (3.19) or even (3.5). Therefore the previous methods are difficult to apply, and instead we will use linear decay estimates.

Proposition 3.2. *Suppose $m, \varrho \in \mathbb{R}$ with $m + \varrho > 0$ and $\ell \in \mathbb{R}$. Then*

$$\|w^\ell \mathbb{A}(t) f_0\|_{\dot{H}^m_x L^2_v} \lesssim (1+t)^{-\frac{\varrho+m}{2}} \|w^{\ell+\sigma} f_0\|_{\dot{H}^m_x L^2_v \cap \dot{B}_2^{-\varrho,\infty} L^2_v}.$$

This holds when $\|w^{\ell+\sigma} f_0\|_{\dot{H}^m_x L^2_v \cap \dot{B}_2^{-\varrho,\infty} L^2_v} < \infty$. Notice that for the additional weight on the initial data we assume $\sigma > -(m+\varrho)(\gamma+2s) > 0$ for the soft potentials (1.9)*. And for the hard potentials* (1.8) *we take $\sigma = 0$.*

We point out that Proposition 3.2 is proven in Theorem 5.1 of Section 5. In the following, we observe faster decay in the hard potential case (1.8) when the initial data is microscopic, as in (1.13).

Proposition 3.3. *Suppose the initial condition f_0 in* (3.25) *with $g = 0$ satisfies:*

$$\mathbf{P} f_0 = 0. \tag{3.28}$$

Fix $m, \varrho \in \mathbb{R}$ with $m + \varrho > 0$ and $\ell \ge 0$. Then we have

$$\|w^\ell \mathbb{A}(t) f_0\|_{\dot{H}^m_x L^2_v} \lesssim (1+t)^{-\frac{\varrho+m+1}{2}} \|w^\ell f_0\|_{\dot{H}^m_x L^2_v \cap \dot{B}_2^{-\varrho,\infty} L^2_v}.$$

This faster decay is proven in the hard potential case (1.8).

Again Proposition 3.3 is proven in Theorem 5.6 of Section 5. Now we use these linear decay results, and previous developments to prove Theorem 1.3.

Proof of Theorem 1.3. Starting from (1.26), we obtain as before

$$\frac{d}{dt}\mathcal{E}_{K,\ell}(t) + \lambda\left(\mathcal{D}_K^h(t) + \mathcal{D}_{K,\ell}^m(t)\right) \leq 0. \tag{3.29}$$

where the hydrodynamic part of the dissipation, $\mathcal{D}_K^h(t)$, and the microscopic part of the dissipation, $\mathcal{D}_{K,\ell}^m(t)$, are each defined as in the following

$$\mathcal{D}_K^h(t) \overset{\text{def}}{=} \sum_{1\leq|\alpha|\leq K} \|\partial^\alpha[a,b,c]\|_{L_x^2}^2, \quad \mathcal{D}_{K,\ell}^m(t) \overset{\text{def}}{=} \sum_{|\alpha|+|\beta|\leq K} \|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f(t)\|_{L_x^2 N_{\ell-|\beta|\rho}^{s,\gamma}}^2.$$

Here we also recall (1.24) for the definition of ρ. At first we focus on the case of Theorem 1.3 when $\varrho \in (0, n/2]$ and $\|f_0\|_{\dot{B}_2^{-\varrho,\infty}L_v^2} < \infty$. In this situation, we still then have (3.19) and we can re-use exactly either the first or the second proof of Theorem 1.1. This establishes (1.6) when $\varrho \in (0, n/2]$.

In the remainder of our proof, we suppose (1.8) and we consider the case when $\|\mathbf{P}f_0\|_{\dot{B}_2^{-\varrho,\infty}L_v^2} < \infty$ and also $\|\{\mathbf{I}-\mathbf{P}\}f_0\|_{\dot{B}_2^{-\varrho+1,\infty}L_v^2} < \infty$ for $\varrho \in (n/2, (n+2)/2]$. Notice that in this very singular situation we do not have either (3.19) or even (3.5). Then we will use the time-weighted estimates, and the linear decay theory, since in this case the interpolation method seems to fail without (3.5).

Fix $s \geq 0$ to be chosen later, and $\varepsilon > 0$ small (also determined below), we now multiply (3.29) by the time weight $(1+\varepsilon t)^s$ to obtain

$$\frac{d}{dt}\left((1+\varepsilon t)^s\mathcal{E}_{K,\ell}(t)\right) + \lambda(1+\varepsilon t)^s\left(\mathcal{D}_K^h(t) + \mathcal{D}_{K,\ell}^m(t)\right) \leq s\varepsilon(1+\varepsilon t)^{s-1}\mathcal{E}_{K,\ell}(t). \tag{3.30}$$

We use (1.23), the decomposition (1.13) with (1.14), and estimates which are analogous to the one used in the proof of (3.4) to obtain

$$\mathcal{E}_{K,\ell}(t) \lesssim \|[a,b,c]\|_{L_x^2}^2(t) + \mathcal{D}_K^h(t) + \sum_{|\alpha|+|\beta|\leq K} \|w^{\ell-|\beta|\rho}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f(t)\|_{L_x^2L_v^2}^2. \tag{3.31}$$

We will handle each of the terms in the upper bound of (3.31) separately.

Initially, our focus will be on the first term in the upper bound of (3.31). As in (3.27), with g given by (3.26), we expand the solution to (1.10) as

$$f(t) = \mathbb{A}(t)\mathbf{P}f_0 + \mathbb{A}(t)\{\mathbf{I}-\mathbf{P}\}f_0 + I_1(t), \tag{3.32}$$

where we additionally use (1.15) to observe that

$$I_1(t) = \int_0^t \mathbb{A}(t-s)\{\mathbf{I}-\mathbf{P}\}\Gamma(f,f)(s)ds.$$

We now apply Propositions 3.2 and 3.3 to $\mathbb{A}(t)\mathbf{P}f_0$ and $\mathbb{A}(t)\{\mathbf{I}-\mathbf{P}\}f_0$ respectively, to obtain

$$\begin{aligned}
\|\mathbb{A}(t)\mathbf{P}f_0\|_{L_v^2L_x^2} &\lesssim (1+t)^{-\frac{\varrho}{2}}\|\mathbf{P}f_0\|_{L_v^2L_x^2\cap\dot{B}_2^{-\varrho,\infty}L_v^2}, \\
\|\mathbb{A}(t)\{\mathbf{I}-\mathbf{P}\}f_0\|_{L_v^2L_x^2} &\lesssim (1+t)^{-\frac{\varrho}{2}}\|\{\mathbf{I}-\mathbf{P}\}f_0\|_{L_v^2L_x^2\cap\dot{B}_2^{-\varrho+1,\infty}L_v^2},
\end{aligned}$$

where we recall that here $\varrho \in (n/2, (n+2)/2]$. For $I_1(t)$, we have the estimate

$$
\begin{aligned}
\|I_1(t)\|_{L^2_v L^2_x} &\le \int_0^t \|\mathbb{A}(t-s)\{\mathbf{I}-\mathbf{P}\}\Gamma(f,f)(s)\|_{L^2_v L^2_x} ds \\
&\lesssim \int_0^t (1+t-s)^{-\frac{\varrho}{2}} \|\Gamma(f,f)(s)\|_{L^2_v L^2_x \cap \dot{B}_2^{-\varrho+1,\infty} L^2_v} ds \\
&\lesssim \int_0^t (1+t-s)^{-\frac{\varrho}{2}} \|\Gamma(f,f)(s)\|_{L^2_v L^2_x \cap L^2_v L^p_x} ds.
\end{aligned}
$$

In the last inequality we used the following embedding $\dot{B}_2^{-\varrho+1,\infty} L^2_v \supset L^2_v L^p_x$ for $p = \frac{n}{\varrho - 1 + \frac{n}{2}} \in [1,2)$ when $\varrho \in (n/2, (n+2)/2]$ which follows from Lemma 4.6. Notice further that by interpolation, [25, Equation (3.22)], and $p \in [1,2)$, we have

$$
\|\Gamma(f,f)(s)\|_{L^2_v L^2_x \cap L^2_v L^p_x} \lesssim \|\Gamma(f,f)(s)\|_{L^2_v L^2_x \cap L^2_v L^1_x} \lesssim \mathcal{E}_{K,\ell'}(t). \tag{3.33}
$$

The last inequality holds for $\ell' = 2(\gamma + 2s)$.

Now since $\|\mathbf{P} f_0\|_{\dot{B}_2^{-\varrho,\infty} L^2_v} < \infty$ and $\|\{\mathbf{I}-\mathbf{P}\} f_0\|_{\dot{B}_2^{-\varrho+1,\infty} L^2_v} < \infty$ we have that $\|f_0\|_{\dot{B}_2^{-\varrho+1,\infty} L^2_v} < \infty$ as in (3.21), since $f_0 \in L^2_x L^2_v$. Then for $\varrho \in (n/2, (n+2)/2]$ from the first part of Theorem 1.3 which was already proven we have that

$$
\mathcal{E}_{K,\ell'}(t) \lesssim (1+t)^{-\varrho+1}. \tag{3.34}
$$

Here, we are implicitly using the fact that $\epsilon_{K,\ell'}$ is sufficiently small, which follows from the assumptions of Theorem 1.3 and the fact that $\ell_0 \ge \ell'$ by (1.30). We collect all of the previous estimates and use (3.32) to conclude that

$$
\begin{aligned}
\|[a,b,c]\|_{L^2_x}(t) &\lesssim \|f(t)\|_{L^2_x L^2_v} \\
&\lesssim \|\mathbb{A}(t)\mathbf{P} f_0\|_{L^2_v L^2_x} + \|\mathbb{A}(t)\{\mathbf{I}-\mathbf{P}\} f_0\|_{L^2_v L^2_x} + \|I_1(t)\|_{L^2_v L^2_x} \\
&\lesssim (1+t)^{-\frac{\varrho}{2}} + \int_0^t ds\ (1+t-s)^{-\frac{\varrho}{2}} (1+s)^{-\varrho+1}.
\end{aligned} \tag{3.35}
$$

The first inequality above used estimate (2.8). If $\varrho > 2$, then we evaluate the time integral as in [24, Proposition 4.5], to conclude that when $\varrho \in (2, (n+2)/2]$

$$
\|[a,b,c]\|_{L^2_x}(t) \lesssim (1+t)^{-\frac{\varrho}{2}}. \tag{3.36}
$$

This is the desired estimate for the first term in the upper bound of (3.31).

For the third term in the upper bound of (3.31) following the procedure exactly as in (3.11) and (3.12) we obtain the following uniform estimate

$$
\begin{aligned}
\|w^{\ell-|\beta|\rho}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f(t)\|^2_{L^2_x L^2_v} &\lesssim \|w^{\ell-|\beta|\rho}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f(t)\|^2_{L^2_x L^2_{\gamma+2s}} \\
&\lesssim (1+\varepsilon t)\|w^{\ell-|\beta|\rho}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_{L^2_x L^2_{\gamma+2s}}.
\end{aligned} \tag{3.37}
$$

Here we explicitly used the hard potentials (1.8) assumption.

Collecting (3.37) and (3.36) into (3.31) and choosing $\varepsilon > 0$ sufficiently small, we plug these estimates into (3.30) to obtain

$$
\frac{d}{dt}\left((1+\varepsilon t)^s \mathcal{E}_{K,\ell}(t)\right) + \lambda (1+\varepsilon t)^s \left(\mathcal{D}^h_K(t) + \mathcal{D}^m_{K,\ell}(t)\right) \lesssim (1+\varepsilon t)^{s-1-\varrho}. \tag{3.38}
$$

Now choose $s = \varrho + \delta$ for any small $\delta \in (0,1)$ and integrate this in time to obtain

$$\begin{aligned}(3.39) \quad \mathcal{E}_{K,\ell}(t) &\lesssim (1+\varepsilon t)^{-\varrho-\delta}\mathcal{E}_{K,\ell}(0) + (1+\varepsilon t)^{-\varrho-\delta}\int_0^t du\ (1+\varepsilon u)^{\delta-1} \\ &\lesssim (1+t)^{-\varrho-\delta}(1+t)^{\delta} \approx (1+t)^{-\varrho}.\end{aligned}$$

The constant is uniform in $t \geq 0$. We thus have Theorem 1.3 when $\varrho \in (2,(n+2)/2]$.

Alternatively suppose that we only have $\varrho \in (n/2, 2]$. Let us note this case only occurs for $n = 3$, and hence $\varrho \in (\frac{3}{2}, 2]$. Then from the estimate (3.35) and the time integral estimate [24, Proposition 4.5] instead of (3.36) we only have

$$(3.40) \qquad \|[a,b,c]\|_{L^2_x}(t) \lesssim (1+t)^{-\left(\frac{3}{2}\varrho-2\right)}A(t) \lesssim (1+t)^{-\left(\frac{3}{2}\varrho-2\right)+a}.$$

Here $A(t) = 1$ if $\varrho \neq 2$ and $A(t) = \log(1+t)$ if $\varrho = 2$ so that $a = 0$ if $\varrho \neq 2$ and $a > 0$ can be taken arbitrarily small when $\varrho = 2$. Then we can apply the all the same estimates above to observe that

$$\mathcal{E}_{K,\ell}(t) \lesssim (1+t)^{-(3\varrho-4)+2a},$$

when $\varrho \in (n/2, 2]$. In the above estimates, we are always working with $\ell \geq \ell_0$ for ℓ_0 defined as in (1.30). Now this estimate is an improvement over (3.34), and we use it in place of (3.34) to improve the decay rate by running again the full argument above. Since $3\varrho - 4 \leq \varrho$, we need to iterate this procedure again. We argue as before, and note that we can apply (3.33) with ℓ' replaced by ℓ since $\ell \geq \ell_0 \geq \ell'$ to deduce that

$$\|[a,b,c]\|_{L^2_x} \lesssim (1+t)^{-\frac{\varrho}{2}} + (1+t)^{-\left(\frac{7\varrho}{2}-5\right)+3a}$$

and so

$$\mathcal{E}_{K,\ell}(t) \lesssim (1+t)^{-\varrho} + (1+t)^{-(7\varrho-10)+6a}$$

Here, we note that $\frac{7\varrho}{2} - 5 = (3\varrho - 4) + (\frac{\varrho}{2} - 1)$, as in [24, Proposition 4.5]. In the case when $7\varrho - 10 > \varrho$, i.e. when $\varrho > \frac{5}{3}$, we are done. Otherwise, we have to iterate the procedure. We define a sequence (T_n) inductively by:

$$(3.41) \qquad T_1 := 7\varrho - 10,\ T_{k+1} := 2\big(T_k + \Big(\frac{\varrho}{2} - 1\Big)\big).$$

By induction, one can check that:

$$T_k = (2^{k+2} - 1)\varrho - (3 \cdot 2^{k+1} - 2).$$

By the previous argument, it can be seen that the claim holds for ϱ after $k-1$ more iterations if $T_k > \varrho$. By using the explicit formula for T_k, we can see that this is equivalent to:

$$\varrho > \frac{3 \cdot 2^{k+1} - 2}{2 \cdot 2^{k+1} - 2}.$$

Since the right hand side of the above inequality converges to $\frac{3}{2}$ as $k \to \infty$, it follows that Theorem 1.3 holds when $\varrho \in (3/2, 2]$, when $n = 3$. This was the final case. □

The next section is dedicated to the proof of the functional interpolation inequalities we needed to use in order to prove the differential and integral inequalities in Sections 2 and 3, as well as the linear decay estimates Proposition 3.2 and Proposition 3.3. In Section 5, we prove the linear decay estimates.

4. Functional interpolation inequalities, and auxiliary results

In this section, we develop several functional type Sobolev inequalities which we use to rigorously justify the proofs of the nonlinear energy estimates in Sections 2 and 3. The main idea will be to use analogues of the Calderón-Zygmund theory in the functional framework. In particular, we will observe that the generalizations of the Littlewood-Paley Inequality and the Hörmander-Mikhlin Multiplier Theorem hold in the functional setting. We note that some of these results can be derived from existing results in the literature [4], but we will give the main ideas of the proofs for all of the results we are using in order to make this section self-contained.

An additional subtlety of working in the functional setting is that the Besov seminorm, as given in (1.21), doesn't correspond to the convention for mixed norms given in (1.19). Namely, the use of the definition on (1.19) would require taking the Besov norm of the function $\|f(\cdot)\|_{\mathcal{H}_v}$, whereas in (1.21), we localize in the frequency variable dual to x inside the $\mathcal{H}_v$ norm. Hence, we can't automatically use any of the Sobolev embeddings in mixed norm spaces, but we have to rederive them by looking at the dyadic components separately and by using the functional Calderón-Zygmund theory.

We remark that all of the estimates in this section are proven, say, in the class of Schwartz functions. Then the general inequalities which are stated below can be justified by standard approximation arguments. Furthermore, we point out further that all of the estimates below are true for functions which vanish at infinity sufficiently fast, which is not an additional restriction for functions in $H^K(\mathbb{R}^n_x \times \mathbb{R}^n_v)$ as is the case for our assumptions.

Furthermore, we note that the Besov seminorms in the x variable, as well as the functional Besov seminorms defined in (1.21), have a nullspace given by functions which are polynomial in the x variable. Since we will be considering functions which vanish at infinity sufficiently rapidly in all variables, they will not lie in this nullpace unless they are identically zero. Hence, we can essentially treat all the Besov seminorms as norms.

In Sub-section 4.1, we summarize the main facts we want to use from vector-valued Calderón-Zygmund theory. In Sub-section 4.2, we recall the basic properties of homogeneous Besov spaces and their embedding properties in the scalar-valued setting. Sub-section 4.3 is devoted to the main properties of functional Besov spaces. In particular, we study the Sobolev-type inequalities which one can prove in these spaces. Furthermore, we study the functional Littlewood-Paley theory and Hörmander-Mikhlin Multiplier theory in Sub-section 4.4. Finally, in Sub-section 4.5, we prove the product estimates which we used in Section 2. More precisely, we prove the estimates we needed in order to deduce (2.7), (2.14) and (2.22).

4.1. **Hilbert Space-valued Calderón-Zygmund Theory.** In this sub-section, we collect some useful facts about Calderón-Zygmund operators which act on functions that take values in a Hilbert Space. These results are well-known, but we summarize them for completeness. For a more detailed discussion of vector-valued extensions of Calderón-Zygmund theory, we refer the reader to [4, Chapter 5.5].

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert Spaces [2]. Let us denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\Delta \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be the *diagonal*

[2] We suppose the spaces we are working with are separable in order to rigorously to define the vector-valued spaces $L^p\mathcal{H}_j$. For more details, we refer the reader to Chapter 5.5 of [4].

set given by $\Delta = \{(x,x), x \in \mathbb{R}^n\}$. Suppose that $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Let T be the operator whose associated kernel is K. More precisely, if $f \in L^\infty_x \mathcal{H}_1$ has compact support in x, then:

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \text{ whenever } x \notin supp(f).$$

We note that the function Tf takes values in the Hilbert space $\mathcal{H}_2$.

Let us now recall the definition of boundedness for operators acting on vector-valued functions:

Definition 4.1. *Suppose that we are given $1 \leq p \leq \infty$ and an operator $S : L^p_x \mathcal{H}_1 \to L^p_x \mathcal{H}_2$ (which is not necessarily linear).*

i) *We say that S is bounded from $L^p_x \mathcal{H}_1$ to $L^p_x \mathcal{H}_2$ if there exists a constant $C > 0$ such that for all $f \in L^p_x \mathcal{H}_1$, one has:*

$$\|Sf\|_{L^p_x \mathcal{H}_2} \leq C \|f\|_{L^p_x \mathcal{H}_1}.$$

ii) *If $1 \leq p < \infty$, we say that S is weakly bounded from $L^p_x \mathcal{H}_1$ to $L^p_x \mathcal{H}_2$ if there exists a constant $C > 0$ such that for all $f \in L^p_x \mathcal{H}_1$ and $\lambda > 0$, one has:*

$$|\{x \in \mathbb{R}^n; \|(Sf)(x)\|_{\mathcal{H}_2} > \lambda\}| \leq \Big(\frac{C}{\lambda}\|f\|_{L^p_x \mathcal{H}_1}\Big)^p.$$

Above $|A|$ denotes the Lebesgue measure of the set A.

Then the following result holds:

Theorem 4.2. *Suppose that the operator T defined as above can be extended to a bounded linear operator from $L^2_x \mathcal{H}_1$ to $L^2_x \mathcal{H}_2$. Suppose furthermore that there exists a constant $C > 0$ such that the associated kernel K satisfies:*

i) $\int_{|x-y|>2|y-z|} \|K(x,y) - K(x,z)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} dx \leq C$, *for all* $y, z \in \mathbb{R}^n$.
ii) $\int_{|x-y|>2|x-w|} \|K(x,y) - K(w,y)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} dy \leq C$, *for all* $x, w \in \mathbb{R}^n$.

Then, the operator T is bounded from $L^p_x \mathcal{H}_1$ to $L^p_x \mathcal{H}_2$ for all $1 < p < \infty$, and it is weakly bounded from $L^1_x \mathcal{H}_1$ to $L^1_x \mathcal{H}_2$.

This is the analogue of the classical Calderón-Zygmund Theorem [3] for functions which take values in a Hilbert Space. We will sketch the main ideas of the proof for completeness. The proof is similar to the scalar case. As in the scalar case, one has to have an interpolation result for weakly bounded operators given by the Marcinkiewicz Inequality.

Definition 4.3. *An operator S from a vector space of measurable functions on $\mathbb{R}^n$ with values in $\mathcal{H}_1$ to a vector space of measurable functions with values in $\mathcal{H}_2$ is called sublinear if the following hold:*

i) $\|S(f_0 + f_1)(x)\|_{\mathcal{H}_2} \leq \|Sf_0(x)\|_{\mathcal{H}_2} + \|Sf_1(x)\|_{\mathcal{H}_2}$.
ii) $\|S(\lambda f)\|_{\mathcal{H}_2} = |\lambda| \|Sf\|_{\mathcal{H}_2}$, *for all* $\lambda \in \mathbb{C}$.

Proposition 4.4. *Suppose that $1 \leq p_0 < p_1 < \infty$ and suppose that T is a sublinear operator which is defined on $L^{p_0}_x \mathcal{H}_1 + L^{p_1}_x \mathcal{H}_1$. Furthermore, suppose that $T : L^{p_0}_x \mathcal{H}_1 \to L^{p_0}_x \mathcal{H}_2$ and $T : L^{p_1}_x \mathcal{H}_1 \to L^{p_1}_x \mathcal{H}_2$ are weakly bounded. Then, for all $p \in (p_0, p_1), T : L^p_x \mathcal{H}_1 \to L^p_x \mathcal{H}_2$ is bounded with operator norm $O_p(1)$.*

The proof of Proposition 4.4 is reduced to the scalar valued case, since we can use the distributional characterization of the $L^p_x\mathcal{H}_2$ norm for $1 \le p < \infty$:

$$\|f\|^p_{L^p_x\mathcal{H}_2} = p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R}^n; \|f(x)\|_{\mathcal{H}_2} > \lambda\}|d\lambda.$$

Sketch of the proof of Theorem 4.2. In order to prove Theorem 4.2, given a function $f \in L^1_{loc,x}\mathcal{H}_1$, one should form the Calderón-Zygmund decomposition of the function $F(x) \overset{\text{def}}{=} \|f(x)\|_{\mathcal{H}_2}$. We note that $F \in L^1_{loc,x}$. We can then use Proposition 4.4 and the proof of the scalar Calderón-Zygmund Theorem to deduce that $T : L^1_x\mathcal{H}_1 \to L^1_x\mathcal{H}_2$ is weakly bounded and that $T : L^p_x\mathcal{H}_1 \to L^p_x\mathcal{H}_2$ is bounded for $1 < p \le 2$. Now, we use the Riesz Representation Theorem for Hilbert Spaces to deduce, for the adjoint K^*, that for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, one has:

$$\|K^*(x, y)\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)} = \|K(x, y)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}.$$

Hence, the condition $ii)$ of the assumptions of Theorem 4.2 implies that the dual operator T^* is weakly bounded from $L^1_x\mathcal{H}_2$ to $L^1_x\mathcal{H}_1$ and that it is bounded from $L^p_x\mathcal{H}_2$ to $L^p_x\mathcal{H}_1$ for $1 < p \le 2$. To prove the claim for $2 < p < \infty$, we argue by duality. Namely, let us take $2 < p < \infty, f \in L^p_x\mathcal{H}_1,\ g \in (L^p_x\mathcal{H}_2)^*$. Since $\mathcal{H}_2$ is a Hilbert space, it follows that $(L^p_x\mathcal{H}_2)^* \approx L^{p'}_x\mathcal{H}_2$, where p' denotes the Hölder conjugate of p. Here, the duality is taken with respect to the pairing $\langle\cdot,\cdot\rangle$ given by:

$$\langle f, g\rangle_{L^2\mathcal{H}_j} \overset{\text{def}}{=} \int \langle f(x), g(x)\rangle_{\mathcal{H}_j} dx, \quad j \in \{1, 2\},$$

where $\langle\cdot,\cdot\rangle_{\mathcal{H}_j}$ is the inner product on $\mathcal{H}_j$. Since $p' \in (1, 2)$, it follows that $T^* : L^{p'}_x\mathcal{H}_2 \to L^{p'}_x\mathcal{H}_1$ is bounded. We conclude by noting that:

$$|\langle Tf, g\rangle_{L^2_x\mathcal{H}_2}| = |\langle f, T^*g\rangle_{L^2_x\mathcal{H}_1}| \le \|f\|_{L^p_x\mathcal{H}_1}\|T^*g\|_{L^{p'}_x\mathcal{H}_1} \le C\|f\|_{L^p_x\mathcal{H}_1}\|g\|_{L^{p'}_x\mathcal{H}_2}.$$

We thus conclude our sketch of the proof. □

4.2. **Homogeneous Besov spaces.** For an integrable function $g : \mathbb{R}^n_x \to \mathbb{R}$, its Fourier transform is defined by

$$\widehat{g}(\xi) = \mathcal{F}g(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} e^{-2\pi \mathrm{i}x\cdot\xi}g(x)dx, \quad x\cdot\xi \overset{\text{def}}{=} \sum_{j=1}^n x_j\xi_j, \quad \xi \in \mathbb{R}^n,$$

where $\mathrm{i} = \sqrt{-1}$. We define Λ^k, the Riesz potential of order $k \in \mathbb{R}$, by:

$$\mathcal{F}(\Lambda^k f)(\xi) \overset{\text{def}}{=} |\xi|^k\widehat{f}(\xi).$$

We now describe a standard Littlewood-Paley decomposition on $\mathbb{R}^n_x$ as follows. Let $\phi \in C^\infty_c(\mathbb{R}^n_\xi)$ be such that $\phi(\xi) = 1$ when $|\xi| \le 1$ and $\phi(\xi) = 0$ when $|\xi| \ge 2$. Let $\varphi(\xi) = \phi(\xi) - \phi(2\xi)$ and $\varphi_j(\xi) = \varphi(\frac{\xi}{2^j})$ for $j \in \mathbb{Z}$. Then of course,

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \quad \xi \ne 0.$$

Further let $\mathcal{F}(\psi)(\xi) = \varphi(\xi)$ and then $\psi_j(x) = 2^{nj}\psi(2^jx)$ satisfies $\mathcal{F}(\psi_j) = \varphi_j$. We define

$$\Delta_j f \overset{\text{def}}{=} (\psi_j * f)(x).$$

And if, say, $f \in L^p(\mathbb{R}^n_x)$, for $1 < p < \infty$, then $f = \sum_{j\in\mathbb{Z}} \Delta_j(f)$, with convergence in L^p. Now we define the homogeneous Besov seminorm for $1 \le q < \infty$ by

$$\|f\|_{\dot{B}^{\varrho,q}_p} \overset{\text{def}}{=} \left(\sum_{j\in\mathbb{Z}} \left(2^{\varrho j}\|\Delta_j f\|_{L^p_x}\right)^q\right)^{1/q}, \quad \|f\|_{\dot{B}^{\varrho,\infty}_p} \overset{\text{def}}{=} \sup_{j\in\mathbb{Z}} \left(2^{\varrho j}\|\Delta_j f\|_{L^p_x}\right).$$

The following embeddings are known:

$$\dot{B}^{\varrho,q'}_p \subset \dot{B}^{\varrho,q}_p, \quad \text{for } q \ge q'.$$

Of course also $\|\cdot\|_{\dot{B}^{\varrho,2}_2} \approx \|\cdot\|_{\dot{H}^{\varrho}(\mathbb{R}^n_x)}$. We note that we always use the Besov spaces acting only on the spatial, x, variables.

Let us prove a useful interpolation estimate in Besov spaces:

Lemma 4.5. *Suppose $k \ge 0$ and $m, \varrho > 0$. Then we have the estimate*

$$\|g\|_{\dot{B}^{k,1}_2} \lesssim \|g\|^{\theta}_{\dot{B}^{k+m,\infty}_2}\|g\|^{1-\theta}_{\dot{B}^{-\varrho,\infty}_2} \lesssim \|g\|^{\theta}_{\dot{B}^{k+m,2}_2}\|g\|^{1-\theta}_{\dot{B}^{-\varrho,\infty}_2},$$

where $\theta = \frac{\varrho+k}{\varrho+k+m}$.

We notice that Lemma 3.1 is implied by Lemma 4.5, Lemma 4.13, Lemma 4.14, and the fact that $\dot{B}^{k,1}_2 \subset \dot{B}^{k,2}_2$.

Proof of Lemma 4.5. For $R \in \mathbb{R}$ to be chosen later, we expand out

$$\|g\|_{\dot{B}^{k,1}_2} = \sum_{j\in\mathbb{Z}} 2^{kj}\|\Delta_j g\|_{L^2_x} = \sum_{j\ge R} + \sum_{j<R}.$$

Now $\sum_{j\ge R} 2^{kj}\|\Delta_j g\|_{L^2_x} \lesssim 2^{-mR}\|g\|_{\dot{B}^{k+m,\infty}_2}$. On the other side

$$\sum_{j<R} 2^{kj}\|\Delta_j g\|_{L^2_x} \lesssim 2^{(k+\varrho)R}\|g\|_{\dot{B}^{-\varrho,\infty}_2}.$$

Choosing $R = \log_2\left(\frac{\|g\|_{\dot{B}^{k+m,\infty}_2}}{\|g\|_{\dot{B}^{-\varrho,\infty}_2}}\right)^{\theta/(k+\varrho)}$ yields the result. □

As was noted at the beginning of this section, the Besov interpolation estimates in the scalar-valued setting are difficult to apply directly in the functional setting due to our definition of the functional Besov seminorm (1.21). However, we will still make use of the bounds of the preceding section.

4.3. **Functional Sobolev-type inequalities in Besov spaces.** The main tool which is going to allow us to develop the functional Besov theory is the following Minkowski-type inequality for $f = f(x,v)$ and $g = g(x)$:

$$\|f * g\|_{\mathcal{H}_v} \le \|f\|_{\mathcal{H}_v} * |g|. \tag{4.1}$$

Above the convolution acts only on the variables $x \in \mathbb{R}^n_x$. In order to prove (4.1), we note that, for f, g as above, and for all $x \in \mathbb{R}^n$:

$$\begin{aligned}\|f * g\|^2_{\mathcal{H}_v}(x) &= \langle (f*g)(x), (f*g)(x)\rangle_{\mathcal{H}_v} \\ &= \int_{\mathbb{R}^n} dy_1\, g(x-y_1)\int_{\mathbb{R}^n} dy_2\, g(x-y_2)\langle f(y_1), f(y_2)\rangle_{\mathcal{H}_v},\end{aligned}$$

which by the Cauchy-Schwarz inequality in $\mathcal{H}_v$ is:

$$\leq \int_{\mathbb{R}^n} dy_1 \ |g(x-y_1)| \int_{\mathbb{R}^n} dy_2 \ |g(x-y_2)| \, \|f(y_1)\|_{\mathcal{H}_v} \|f(y_2)\|_{\mathcal{H}_v} = (\|f\|_{\mathcal{H}_v} * |g|)^2(x).$$

The bound (4.1) now follows.

We can use (4.1) to prove the following Bernstein-type inequalities:

$$\|\Delta_j f\|_{L^p_x \mathcal{H}_v} \lesssim 2^{\left(\frac{n}{q}-\frac{n}{p}\right)j} \|\Delta_j f\|_{L^q_x \mathcal{H}_v}, \qquad \forall 1 \leq q \leq p \leq \infty, \tag{4.2}$$

$$\|\Delta_j \Lambda^s f\|_{L^p \mathcal{H}_v} \approx 2^{js} \|\Delta_j f\|_{L^p \mathcal{H}_v}, \qquad \forall 1 \leq p \leq \infty, \ s \in \mathbb{R}. \tag{4.3}$$

These inequalities are useful since they give us estimates on the pieces we are considering in the Besov seminorm (1.21).

Let us first prove (4.2). We find $\tilde{\varphi} \in C^\infty_c(\mathbb{R}^n_\xi)$ such that $\tilde{\varphi} = 1$ near the support of the function φ defined in the previous sub-section and we let $\tilde{\varphi}_j(\xi) \stackrel{\text{def}}{=} \tilde{\varphi}(\frac{\xi}{2^j})$. We let $\mathcal{F}(\tilde{\psi})(\xi) = \tilde{\varphi}(\xi)$ and $\tilde{\psi}_j(x) \stackrel{\text{def}}{=} 2^{nj} \tilde{\psi}(2^j x)$. Then

$$\tilde{\Delta}_j f \stackrel{\text{def}}{=} (\tilde{\psi}_j * f)(x). \tag{4.4}$$

We use these calculations and (4.1) to obtain

$$\begin{aligned} \|\Delta_j f\|_{L^p_x \mathcal{H}_v} &= \|\tilde{\Delta}_j \Delta_j f\|_{L^p_x \mathcal{H}_v} = \| \|\tilde{\Delta}_j \Delta_j f\|_{\mathcal{H}_v} \|_{L^p_x} \\ &= \| \|\tilde{\psi}_j * \Delta_j f\|_{\mathcal{H}_v} \|_{L^p_x} \leq \| \|\Delta_j f\|_{\mathcal{H}_v} * |\tilde{\psi}_j| \|_{L^p_x}. \end{aligned} \tag{4.5}$$

Let us recall Young's inequality (for $1 \leq p, q, r \leq \infty$):

$$\|f * g\|_{L^p(\mathbb{R}^n_x)} \leq \|f\|_{L^q(\mathbb{R}^n_x)} \|g\|_{L^r(\mathbb{R}^n_x)}, \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{p} + 1. \tag{4.6}$$

We then apply (4.6) in the x variable and use $\|\tilde{\psi}_j\|_{L^r} \approx 2^{(n-\frac{n}{r})j}$ to deduce (4.2).

We can also deduce the $L^p_x \mathcal{H}_v$ embedding:

Lemma 4.6. *Suppose that $\varrho > 0$ and $1 \leq p \leq 2$. We have the embedding $L^p \mathcal{H}_v \subset \dot{B}^{-\varrho,\infty}_q \mathcal{H}_v$ where $\frac{1}{p} - \frac{1}{q} = \frac{\varrho}{n}$. In particular we have the estimate*

$$\|f\|_{\dot{B}^{-\varrho,\infty}_q \mathcal{H}_v} \lesssim \|f\|_{L^p_x \mathcal{H}_v}. \tag{4.7}$$

This holds for example with $\varrho = \frac{n}{2}$, $q = 2$ and $p = 1$.

Proof. Let p, q, ϱ be as in the statement. We use (4.2) to deduce that for all j:

$$2^{-\varrho j} \|\Delta_j f\|_{L^q \mathcal{H}_v} \lesssim \|\Delta_j f\|_{L^p_x \mathcal{H}_v} \lesssim \|f\|_{L^p_x \mathcal{H}_v}.$$

The second inequality above uses the definition of Δ_j and Young's inequality in the x variable. The claim follows by taking suprema in j. $\square$

In order to prove (4.3), with notation as above, we note that:

$$\Delta_j \Lambda^s f = \tilde{\Delta}_j \Delta_j \Lambda^s f = 2^{js} (2^{jn} \Psi(2^j \cdot)) * \Delta_j f,$$

where $\Psi \stackrel{\text{def}}{=} \Lambda^s \tilde{\psi}$. As in the proof of (4.2), we use (4.1) and Young's inequality (4.6) in the x variable. Since $\|2^{jn} \Psi(2^j \cdot)\|_{L^1} = \|\Psi\|_{L^1} = O(1)$, it follows that

$$\|\Delta_j \Lambda^s f\|_{L^p_x \mathcal{H}_v} \lesssim 2^{js} \|\Delta_j f\|_{L^p_x \mathcal{H}_v}.$$

We replace s by $-s$ to obtain:

$$\|\Delta_j f\|_{L^p_x \mathcal{H}_v} = \|\Lambda^{-s} \Lambda^s \Delta_j f\|_{L^p_x \mathcal{H}_v} \lesssim 2^{-js} \|\Lambda^s \Delta_j f\|_{L^p_x \mathcal{H}_v}.$$

We used that Λ^s and Δ_j commute. Then (4.3) follows from the last two inequalities.

As a consequence of (4.3), we obtain, for all $s_1, s_2 \in \mathbb{R}$ and for all $1 \le p \le \infty$:

$$\|\Lambda^{s_1} f\|_{\dot{B}^{s_2,p}_q \mathcal{H}_v} \approx \|f\|_{\dot{B}^{s_1+s_2,p}_q \mathcal{H}_v}. \tag{4.8}$$

In other words, differentiation by using the Riesz potential Λ and by using the index s in the definition of the Besov norm give rise to equivalent norms. We will use (4.8) throughout this section without explicit reference.

We will use the previous observations to prove several interpolation estimates in Besov spaces. In particular, we have

Lemma 4.7. *Suppose that $m \neq \varrho$. We have the following interpolation estimate:*

$$\|f\|_{\dot{B}^{k,1}_p \mathcal{H}_v} \lesssim \|f\|^{1-\theta}_{\dot{B}^{m,\infty}_r \mathcal{H}_v} \|f\|^{\theta}_{\dot{B}^{\varrho,\infty}_r \mathcal{H}_v}$$

where $0 < \theta < 1$ and $1 \le r \le p \le \infty$. We also require:

$$k + \frac{n}{r} - \frac{n}{p} = m(1-\theta) + \varrho\theta. \tag{4.9}$$

Proof of Lemma 4.7. Without loss of generality suppose that $m < \varrho$. For $R \in \mathbb{R}$ to be chosen later, we expand out

$$\|f\|_{\dot{B}^{k,1}_p \mathcal{H}_v} = \sum_{j \in \mathbb{Z}} 2^{kj} \|\Delta_j f\|_{L^p_x \mathcal{H}_v} = \sum_{j \ge R} + \sum_{j < R}.$$

Now using (4.2) we obtain

$$\sum_{j \ge R} 2^{kj} \|\Delta_j f\|_{L^p_x \mathcal{H}_v} \lesssim \sum_{j \ge R} 2^{kj + \left(\frac{n}{r} - \frac{n}{p}\right)j} \|\Delta_j f\|_{L^r_x \mathcal{H}_v} \lesssim 2^{\left(k + \left(\frac{n}{r} - \frac{n}{p}\right) - \varrho\right)R} \|f\|_{\dot{B}^{\varrho,\infty}_r \mathcal{H}_v}.$$

For the other term

$$\sum_{j < R} 2^{kj} \|\Delta_j f\|_{L^p_x \mathcal{H}_v} \lesssim \sum_{j < R} 2^{kj + \left(\frac{n}{r} - \frac{n}{p}\right)j} \|\Delta_j f\|_{L^r_x \mathcal{H}_v} \lesssim 2^{\left(k + \left(\frac{n}{r} - \frac{n}{p}\right) - m\right)R} \|f\|_{\dot{B}^{m,\infty}_r \mathcal{H}_v}.$$

Choosing $R = \log_2 \left(\frac{\|f\|_{\dot{B}^{\varrho,\infty}_r \mathcal{H}_v}}{\|f\|_{\dot{B}^{m,\infty}_r \mathcal{H}_v}} \right)^{1/(\varrho - m)}$ yields the result. $\square$

Notice that Lemma 4.7 directly implies an optimized functional Sobolev inequality of Gagliardo-Nirenberg-Sobolev-type. We obtain directly that for $m \neq \varrho$:

$$\|\Lambda^k g\|_{L^p_x \mathcal{H}_v} \lesssim \|\Lambda^m g\|^{1-\theta}_{L^2_x \mathcal{H}_v} \|\Lambda^\varrho g\|^{\theta}_{L^2_x \mathcal{H}_v} \tag{4.10}$$

where $0 < \theta < 1$, $2 \le p \le \infty$, and again θ satisfies (4.9). To see this, we set $r = 2$ in Lemma 4.7 and note that

$$\|\Lambda^k g\|_{L^p_x \mathcal{H}_v} = \left\| \sum_j \Delta_j \Lambda^k g \right\|_{L^p_x \mathcal{H}_v} \le \sum_j \|\Delta_j \Lambda^k g\|_{L^p_x \mathcal{H}_v} \approx \|g\|_{\dot{B}^{k,1}_p \mathcal{H}_v},$$

and furthermore

$$\|g\|_{\dot{B}^{m,\infty}_2 \mathcal{H}_v} \approx \sup_{j \in \mathbb{Z}} \|\Delta_j \Lambda^m g\|_{L^2_x \mathcal{H}_v} \le \left(\sum_j \|\Delta_j \Lambda^m g\|^2_{L^2_x \mathcal{H}_v} \right)^{\frac{1}{2}} = \|\Lambda^m g\|_{L^2_x \mathcal{H}_v},$$

where we have used (4.3) in order to deduce both bounds.

We will frequently use the following functional Sobolev type inequalities

$$\|g\|_{L^q_x \mathcal{H}_v} \lesssim \|\Lambda^\varrho g\|_{L^p_x \mathcal{H}_v}, \quad \frac{1}{p} - \frac{1}{q} = \frac{\varrho}{n}, \quad 1 < p < q < \infty, \tag{4.11}$$

which implies that $p = \frac{nq}{n+q\varrho}$. The functional inequality (4.11) follows directly from (4.1) combined with the standard fractional integration proof of (4.11) when there is not an additional function space $\mathcal{H}_v$. In other words, the inequality (4.1) allows us to reduce the proof of the vector-valued case to the scalar-valued case. For the details of the scalar-valued case, we refer the reader to [30, Proposition A.3].

We observe the following Besov space variant of (4.11); namely:

$$\|g\|_{\dot{B}^{0,2}_q \mathcal{H}_v} \lesssim \|\Lambda^\varrho g\|_{\dot{B}^{0,2}_p \mathcal{H}_v}, \quad \frac{1}{p} - \frac{1}{q} = \frac{\varrho}{n}, \quad 1 < p < q < \infty. \tag{4.12}$$

Notice that (4.12) immediately follows from applying (4.11) to the individual functions $\Delta_j g$ and then taking the ℓ^2_j norms.

We will also use the functional Sobolev embedding:

$$\|g\|_{L^q_x \mathcal{H}_v} \le C_{q,n} \|g\|_{H^k_x \mathcal{H}_v}, \quad \text{whenever} \quad k + \frac{n}{q} \ge \frac{n}{2}, \quad 2 \le q < \infty, \tag{4.13}$$

and $\|g\|_{L^\infty_x \mathcal{H}_v} \le C_n \|g\|_{H^k_x \mathcal{H}_v}$, $\forall k > \frac{n}{2}$. In the endpoint case $k + \frac{n}{q} = \frac{n}{2}$, (4.13) follows directly from (4.11). In all the other non-endpoint cases, (4.13) follows directly from (4.10). We can deduce the Besov version of (4.13):

$$\|g\|_{\dot{B}^{0,2}_q \mathcal{H}_v} \le C_{q,n} \|g\|_{H^k_x \mathcal{H}_v}, \quad \text{whenever} \quad k + \frac{n}{q} \ge \frac{n}{2}, \quad 2 \le q < \infty, \tag{4.14}$$

and $\|g\|_{\dot{B}^{0,2}_\infty \mathcal{H}_v} \le C_n \|g\|_{H^k_x \mathcal{H}_v}$, $\forall k > \frac{n}{2}$. In order to prove (4.14), we use (4.13) and (4.3) to note that for k, q as in the assumptions:

$$\begin{aligned} \|g\|^2_{\dot{B}^{0,2}_q \mathcal{H}_v} &= \sum_{j \ge 0} \|\Delta_j g\|^2_{L^q_x \mathcal{H}_v} + \sum_{j<0} \|\Delta_j g\|^2_{L^q_x \mathcal{H}_v} \lesssim \sum_{j \ge 0} \|\Delta_j g\|^2_{H^k_x \mathcal{H}_v} + \sum_{j<0} \|\Delta_j g\|^2_{H^k_x \mathcal{H}_v} \\ &\lesssim \sum_{j \ge 0} 2^{2jk} \|\Delta_j g\|^2_{L^2_x \mathcal{H}_v} + \sum_{j<0} \|\Delta_j g\|^2_{L^2_x \mathcal{H}_v} \lesssim \|g\|^2_{H^k_x \mathcal{H}_v}. \end{aligned}$$

We will use these inequalities to prove our main product estimates in Section 4.5.

First let us give one more Besov space interpolation estimate:

Lemma 4.8. *Fix $m > \ell \ge k$, and $1 \le p \le q \le r \le \infty$. We have*

$$\|g\|_{\dot{B}^{\ell,q'}_q \mathcal{H}_v} \le \|g\|^\theta_{\dot{B}^{k,r'}_r \mathcal{H}_v} \|g\|^{1-\theta}_{\dot{B}^{m,p'}_p \mathcal{H}_v}. \tag{4.15}$$

These parameters satisfy the following restrictions

$$\ell = k\theta + m(1-\theta), \quad \frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{p}, \quad \frac{1}{q'} = \frac{\theta}{r'} + \frac{1-\theta}{p'}.$$

Also $1 \le p' \le q' \le r' \le \infty$ and solving we have $\theta = \frac{m-\ell}{m-k} \in (0,1]$.

The most common case of this inequality that we will use is

$$\|g\|_{\dot{B}^{\ell,2}_q \mathcal{H}_v} \lesssim \|g\|^\theta_{\dot{B}^{0,2}_r \mathcal{H}_v} \|g\|^{1-\theta}_{\dot{B}^{\ell+m,2}_p \mathcal{H}_v}, \tag{4.16}$$

with $\ell, m > 0$, for $\theta = \frac{\ell}{\ell+m} \in (0,1)$.

Proof of Lemma 4.8. Recall $\|\Delta_j g\|_{L^q_x \mathcal{H}_v} = \| \|\Delta_j g(x,\cdot)\|_{\mathcal{H}_v} \|_{L^q_x}$. We use Hölder's inequality in x to obtain

$$\|\Delta_j g\|_{L^q \mathcal{H}_v} \le \| \|\Delta_j g(x,\cdot)\|_{\mathcal{H}_v} \|^\theta_{L^r_x} \| \|\Delta_j g(x,\cdot)\|_{\mathcal{H}_v} \|^{1-\theta}_{L^p_x} = \|\Delta_j g\|^\theta_{L^r \mathcal{H}_v} \|\Delta_j g\|^{1-\theta}_{L^p \mathcal{H}_v}.$$

Consequently, for all $j \in \mathbb{Z}$, one has:

$$2^{\ell j} \|\Delta_j g\|_{L^q \mathcal{H}_v} \le (2^{kj} \|\Delta_j g\|_{L^r \mathcal{H}_v})^\theta (2^{mj} \|\Delta_j g\|_{L^p \mathcal{H}_v})^{(1-\theta)}. \tag{4.17}$$

Using (4.17) and applying Hölder's inequality in j, it follows that

$$\left\|(2^{\ell j}\|\Delta_j g\|_{L^q\mathcal{H}_v})_j\right\|_{\ell_j^{q'}} \leq \left\|(2^{\theta kj}\|\Delta_j g\|^{\theta}_{L^r\mathcal{H}_v})_j\right\|_{\ell_j^{\frac{r'}{\theta}}} \left\|(2^{(1-\theta)mj}\|\Delta_j g\|^{1-\theta}_{L^p\mathcal{H}_v})_j\right\|_{\ell_j^{\frac{p'}{1-\theta}}}.$$

This is exactly (4.15). □

Remark 4.9. *Note that some other physical-space proofs of analogous interpolations don't easily generalize to the functional setting due to the definition* (1.21).

4.4. **The Littlewood-Paley Inequality for Hilbert Space-valued functions.** In this sub-section, we use the tools from Sub-section 4.1 to obtain additional functional Sobolev inequalities in Besov spaces. We will use the Hilbert Space-valued Littlewood-Paley inequality (mentioned for example in [19]).

In the following, we will always further suppose that $\mathcal{H}_v$ is some seperable Hilbert space acting only on the variables $v \in \mathbb{R}^n_v$ as in Section 1.2. Then $\langle \cdot, \cdot \rangle_{\mathcal{H}_v}$ is the inner product in $\mathcal{H}_v$ and $\|\cdot\|_{\mathcal{H}_v}$ will denote the norm. We sometimes also use $\mathcal{H}'_v$ as a second Hilbert space which satisfies the same assumptions.

We will need the following vector-valued Littlewood-Paley inequality:

Theorem 4.10. *Suppose that $\|f\|_{L^p_x\mathcal{H}_v} < \infty$ for some $1 < p < \infty$. Then*

$$\left\|\left(\sum_{j\in\mathbb{Z}} \|\Delta_j f(x)\|^2_{\mathcal{H}_v}\right)^{1/2}\right\|_{L^p_x} \approx \|f\|_{L^p_x\mathcal{H}_v}.$$

In order to prove Theorem 4.10, we need to apply Theorem 4.2.

Sketch of the proof of Theorem 4.10. Given a function $f \in L^2_x\mathcal{H}_v$, we define:

$$Tf \stackrel{\text{def}}{=} (\Delta_j f)_j.$$

By the Plancherel theorem, we note that the mapping $T : L^2_x\mathcal{H}_v \to L^2_x\ell^2_j\mathcal{H}_v$ is bounded and linear. Hence, we want to apply Theorem 4.2 with $\mathcal{H}_1 = \mathcal{H}_v$ and $\mathcal{H}_2 = \ell^2_j\mathcal{H}_v$. Moreover, T is realized as convolution with the kernel:

$$K(x,y) \stackrel{\text{def}}{=} (\psi_j(x-y))_j \in \mathcal{L}(\mathcal{H}_v, \ell^2_j\mathcal{H}_v).$$

Consequently:

$$\|K(x,y)\|_{\mathcal{L}(\mathcal{H}_v,\ell^2_j\mathcal{H}_v)} \approx \|(\psi_j(x-y))_j\|_{\ell^2_j}.$$

The required conditions in order to apply Theorem 4.2 are now checked directly as in the scalar-valued setting, and we can then obtain desired the upper bound for $\left\|\left\|\left(\|\Delta_j f\|_{\mathcal{H}_v}\right)_j\right\|_{\ell^2_j}\right\|_{L^p_x}$. In order to obtain the lower bound, we argue as in [32] and we use the upper bound with Δ_j replaced by $\tilde{\Delta}_j$ (as in (4.4)) to deduce that for $\{f_j\} \in L^p_x\ell^2_j\mathcal{H}_v$, one has:

$$\left\|\sum_{j\in\mathbb{Z}} \tilde{\Delta}_j f_j\right\|_{L^p_x\mathcal{H}_v} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}} \|f_j\|^2_{\mathcal{H}_v}\right)^{\frac{1}{2}}\right\|_{L^p_x}. \tag{4.18}$$

Establishing (4.18) uses a duality argument; then, in the duality step, it is crucial to use the fact that $(L^p_x\mathcal{H}_v)^* \approx L^{p'}_x\mathcal{H}_v$, where p' denotes the Hölder conjugate of p. As before, we know that the above duality holds since $\mathcal{H}_v$ is a Hilbert space. More precisely, we let $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2_x \mathcal{H}_v$, and we take $g \in L^{p'} \mathcal{H}_v$. One then obtains:

$$|\langle \sum_j \tilde{\Delta}_j f_j, g \rangle| = |\sum_j \langle f_j, \tilde{\Delta}_j^* g \rangle|,$$

where $\tilde{\Delta}_j^*$ satisfies the same properties as $\tilde{\Delta}_j$. We then use the Cauchy-Schwarz inequality in $\mathcal{H}_v$, the Cauchy-Schwarz inequality in j, Hölder's inequality in x and the upper bound with $\tilde{\Delta}_j$ replaced by $\tilde{\Delta}_j^*$ to deduce that the above expression is:

$$\leq \int \sum_j \|f_j\|_{\mathcal{H}_v} \|\tilde{\Delta}_j^* g\|_{\mathcal{H}_v} dx$$
$$\leq \|(\sum_j \|f_j\|^2_{\mathcal{H}_v})^{\frac{1}{2}}\|_{L^p_x} \|(\sum_j \|\tilde{\Delta}_j^* g\|^2_{\mathcal{H}_v})^{\frac{1}{2}}\|_{L^{p'}_x} \leq \|(\sum_j \|f_j\|^2_{\mathcal{H}_v})^{\frac{1}{2}}\|_{L^p_x} \|g\|_{L^{p'}_x \mathcal{H}_v}.$$

Now (4.18) follows by duality. □

From the Littlewood-Paley Theorem, we can deduce a Sobolev embedding bound:

Lemma 4.11. *Fix $2 \leq p < \infty$, $s \in \mathbb{R}$. Then we have:*

$$\|\Lambda^s f\|_{L^p_x \mathcal{H}_v} \lesssim \|f\|_{\dot{B}^{s,2}_p \mathcal{H}_v} \approx \|\Lambda^s f\|_{\dot{B}^{0,2}_p \mathcal{H}_v}. \tag{4.19}$$

Proof. By Theorem 4.10 we have

$$\|f\|_{L^p_x \mathcal{H}_v} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \|\Delta_j f\|^2_{\mathcal{H}_v} \right)^{\frac{1}{2}} \right\|_{L^p_x} \lesssim \left(\sum_{j \in \mathbb{Z}} \|\Delta_j f\|^2_{L^p_x \mathcal{H}_v} \right)^{\frac{1}{2}} \approx \|f\|_{\dot{B}^{0,2}_p \mathcal{H}_v},$$

where we used $p \geq 2$ to obtain the second inequality above.

In the last step, we are using the following fact about mixed-norm spaces. For measure spaces (Y_1, μ_1) and (Y_2, μ_2), if $f = f(y_1, y_2)$ is a measurable function on $Y_1 \times Y_2$, and $1 \leq q_1 \leq q_2$, then the following inequality holds:

$$\|f\|_{L^{q_2}_{y_2} L^{q_1}_{y_1}} \leq \|f\|_{L^{q_1}_{y_1} L^{q_2}_{y_2}}. \tag{4.20}$$

In particular, above we are taking $Y_1 = \mathbb{Z}$, $Y_2 = \mathbb{R}^n$, $q_1 = 2$, $q_2 = p$.

The inequality (4.20) holds by definition when $q_1 = q_2$. When $q_1 = 1$, it holds by Minkowski's inequality. The full (4.20) then follows by interpolation in mixed-norm spaces. For a more detailed discussion of this we refer the reader to [32].

The inequality (4.19) follows in the special case that $s = 0$. Furthermore, we note that the general case follows by using (4.3). □

We will also need to use a Hilbert space-valued version of the Hörmander-Mikhlin Multiplier Theorem. First note that given a function $m : \mathbb{R}^n_\xi \to \mathbb{C}$, we define the Fourier multiplier operator $m(D)$ on $L^2_x \mathcal{H}_v$ by:

$$\mathcal{F}(m(D)f)(\xi) \stackrel{\text{def}}{=} m(\xi) \widehat{f}(\xi).$$

Then we have

Proposition 4.12. *Suppose that $m : \mathbb{R}^n_\xi \to \mathbb{C}$ is a bounded function such that*

$$|\nabla^k_\xi m(\xi)| \lesssim \frac{1}{|\xi|^k}, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad 0 \leq k \leq n+2.$$

Then $m(D) : L^p_x \mathcal{H}_v \to L^p_x \mathcal{H}_v$ is a bounded operator for all $1 < p < \infty$.

The proof of Proposition 4.12 is similar to the proof of the analogous statement in the scalar-valued setting. For a detailed discussion in the scalar-valued setting, we refer the reader to [31]. We now sketch a proof in the vector-valued setting.

Sketch of the proof of Proposition 4.12. We write: $m(D)f = \sum_j (m(D)\tilde{\Delta}_j)\Delta_j f$ using (4.4), and we use (4.18) as well as the fact that the operator $m(D)\tilde{\Delta}_j$ satisfies the same bounds as the Littlewood-Paley projection (as in Theorem 4.10). More precisely, with Δ_j and $\tilde{\Delta}_j$ as in (4.4), we note that, for $1 < p < \infty$

$$\|m(D)f\|_{L^p_x\mathcal{H}_v} = \|\sum_j (m(D)\tilde{\Delta}_j)\Delta_j f\|_{L^p_x\mathcal{H}_v}.$$

We use the Leibniz rule and the assumption on m to deduce that for all j:

$$|\nabla^k_\xi (m(\xi)\tilde{\psi}_j(\xi))| \lesssim \frac{1}{|\xi|^k}, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad 0 \leq k \leq n+2.$$

We note that these were the assumptions that guaranteed that we could substitute $m(D)\tilde{\Delta}_j$ for the Littlewood-Paley projections in Theorem 4.10. Hence, by the version of (4.18) we obtain, we know:

$$\|\sum_j (m(D)\tilde{\Delta}_j)\Delta_j f\|_{L^p_x\mathcal{H}_v} \lesssim \|(\sum_j \|\Delta_j f\|^2_{\mathcal{H}_v})^{\frac{1}{2}}\|_{L^p_x}$$

By applying the upper bound in Theorem 4.10, this expression is $\lesssim \|f\|_{L^p_x\mathcal{H}_v}$. $\square$

By using Proposition 4.12, we can deduce:

Lemma 4.13. *Let $\mathcal{H}_v$ be a Hilbert space and let $1 < p < \infty$. For all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = k$, the following bound holds:*

$$\|\partial^\alpha f\|_{L^p_x\mathcal{H}_v} \lesssim \|\Lambda^k f\|_{L^p_x\mathcal{H}_v}.$$

Proof. The proof follows from Proposition 4.12 when we take $m(\xi) \overset{\text{def}}{=} \frac{\xi^\alpha}{|\xi|^k}$. $\square$

Moreover, when $p = 2$, we can use Plancherel's Theorem to deduce:

Lemma 4.14. *Suppose that k is a non-negative integer. One then obtains:*

$$\|\Lambda^k f\|_{L^2_x\mathcal{H}_v} \lesssim \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^2_x\mathcal{H}_v}.$$

We note that Lemma 4.13 and Lemma 4.14 are important because the product estimates (2.7), (2.14) and (2.22) in Section 2 are given in terms of $\|\partial^\alpha f\|_{L^p_x\mathcal{H}_v}$, whereas the estimates in this section are given in terms of $\|\Lambda^k f\|_{L^p_x\mathcal{H}_v}$. Furthermore, we note that the right-hand sides of the above mentioned estimates in Section 2 are given in terms of $\|\partial^\alpha f\|_{L^2_x\mathcal{H}_v}$, so in Lemma 4.14, we only need to consider the case when $p = 2$.

4.5. **The main product estimates.** In this sub-section, we prove product estimates which allow us to deduce (2.7), (2.14) and (2.22). The first bound we prove is Lemma 4.15 which holds in the framework of general Hilbert spaces $\mathcal{H}_v$ and $\mathcal{H}'_v$ as in Section 1.2. The second result is Lemma 4.16 in which the Hilbert spaces are L^2_v and $N^{s,\gamma}$. The proofs of both results are based on a case-by-case analysis in which one uses the functional Sobolev type inequalities from Sub-section 4.3. We have not attempted to optimize the choice of p and q, nor of the weight ℓ' used in Lemma 4.16. The first product estimate we prove is:

Lemma 4.15. *For any $k \in \{0, 1, \dots, K\}$ and $i \in \{0, 1, \dots, k\}$ there exists $p, q \geq 2$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ such that we have*

$$\|\Lambda^{k-i} f\|_{L^q_x \mathcal{H}_v} \|\Lambda^i g\|_{L^p_x \mathcal{H}'_v} \lesssim \|g\|_{H^{K^*_n}_x \mathcal{H}'_v} \|\Lambda^{j_*} f\|_{L^2_x \mathcal{H}_v} + \|f\|_{H^{K^*_n}_x \mathcal{H}_v} \|\Lambda^{j_*} g\|_{L^2_x \mathcal{H}'_v},$$

where $j_* \overset{def}{=} \min\{k+1, K\}$. Also $\mathcal{H}_v$ and $\mathcal{H}'_v$ are Hilbert spaces as in Section 1.2

Note clearly that by using Lemma 4.15, together with Lemma 4.13 and Lemma 4.14, we can directly deduce (2.7) and (2.22).

Proof of Lemma 4.15. We prove this lemma in a series of several special cases. Suppose first that $i = 0$. Then we choose $q = 2^* = \frac{2n}{n-2}$, as in (4.11), so that

$$\|\Lambda^k f\|_{L^{2^*}_x \mathcal{H}_v} \lesssim \|\Lambda^{k+1} f\|_{L^2_x \mathcal{H}_v}. \tag{4.21}$$

In this case $p = n$ and we use (4.13) to obtain

$$\|g\|_{L^n_x \mathcal{H}'_v} \lesssim \|g\|_{H^{K^*_n - 1}_x \mathcal{H}'_v}, \tag{4.22}$$

since $K^*_n - 1 \geq \frac{n}{2} - 1$. This then establishes Lemma 4.15 if $i = 0$ and $j_* = k + 1$. If $i = 0$ and $j_* = k = K$ we choose $q = 2$ so that $p = \infty$ and we use the embedding $L^\infty_x \mathcal{H}'_v \supset H^{K^*_n}_x \mathcal{H}'_v$ as in (4.13). Then we obtain Lemma 4.15 in this case as well. The cases $i = k$, $k = 0$ and $k = 1$ can both be handled similarly.

Next we consider the case $k \geq 2$, $j_* = k = K$ and $i \in \{1, \dots, k-1\}$. In this situation we choose $q = 2\frac{k}{k-i}$ and $p = 2\frac{k}{i}$. Since $q, p \geq 2$, we can use (4.19) and (4.16) with $\theta = \frac{i}{k}$ twice to obtain:

$$\begin{aligned}
\|\Lambda^{k-i} f\|_{L^{2k/(k-i)}_x \mathcal{H}_v} \|\Lambda^i g\|_{L^{2k/i}_x \mathcal{H}'_v} &\lesssim \|\Lambda^{k-i} f\|_{\dot{B}^{0,2}_{\frac{2k}{k-i}} \mathcal{H}_v} \|\Lambda^i g\|_{\dot{B}^{0,2}_{\frac{2k}{i}} \mathcal{H}'_v} \\
&\lesssim \|f\|^{\theta}_{\dot{B}^{0,2}_\infty \mathcal{H}_v} \|\Lambda^{j_*} f\|^{1-\theta}_{\dot{B}^{0,2}_2 \mathcal{H}_v} \|g\|^{1-\theta}_{\dot{B}^{0,2}_\infty \mathcal{H}'_v} \|\Lambda^{j_*} g\|^{\theta}_{\dot{B}^{0,2}_2 \mathcal{H}'_v} \\
&\lesssim \|g\|_{H^{K^*_n}_x \mathcal{H}'_v} \|\Lambda^{j_*} f\|_{L^2_x \mathcal{H}_v} + \|f\|_{H^{K^*_n}_x \mathcal{H}_v} \|\Lambda^{j_*} g\|_{L^2_x \mathcal{H}'_v}.
\end{aligned}$$

Here we used Young's inequality, (4.14) and the fact that $\|f\|_{\dot{B}^{0,2}_2 \mathcal{H}_v} \approx \|f\|_{L^2_x \mathcal{H}_v}$. Then, in this case, we also obtain the result.

We suppose then in the rest of our argument that we have $k \in \{2, \dots, K-1\}$ and $i \in \{1, \dots, k-1\}$ so that always $j_* = k+1$. This is the last case to consider in our proof. We first use (4.11) with $2^* = \frac{2n}{n-2} \leq q < \infty$ to obtain

$$\|\Lambda^{k-i} f\|_{L^q_x \mathcal{H}_v} \lesssim \|\Lambda^{k+1-i} f\|_{L^{q'}_x \mathcal{H}_v}, \quad q' = \frac{nq}{n+q} \geq 2. \tag{4.23}$$

Then we apply (4.19) and (4.16) with $\theta = \frac{i}{k+1}$ to obtain:

$$\|\Lambda^{k+1-i} f\|_{L^{q'}_x \mathcal{H}_v} \lesssim \|\Lambda^{k+1-i} f\|_{\dot{B}^{0,2}_{q'} \mathcal{H}_v} \lesssim \|f\|^{\theta}_{\dot{B}^{0,2}_{r'} \mathcal{H}_v} \|\Lambda^{k+1} f\|^{1-\theta}_{\dot{B}^{0,2}_2 \mathcal{H}_v}, \tag{4.24}$$

where r' is obtained from the restriction $\frac{1}{q'} = \frac{\theta}{r'} + \frac{1-\theta}{2}$.

We use (4.19) and (4.16) again (switching the θ and $1-\theta$) to obtain:

$$\|\Lambda^i g\|_{L^p_x \mathcal{H}'_v} \lesssim \|\Lambda^i g\|_{\dot{B}^{0,2}_p \mathcal{H}'_v} \lesssim \|g\|^{1-\theta}_{\dot{B}^{0,2}_r \mathcal{H}'_v} \|\Lambda^{k+1} g\|^{\theta}_{\dot{B}^{0,2}_2 \mathcal{H}'_v}, \quad \theta = \frac{i}{k+1},$$

where r is obtained from $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{2}$.

The value of θ is the same in both inequalities by design. For these inequalities to hold, from (4.16) and $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, we require that q satisfies:

$$\frac{2n}{n-2}\le q<\infty,\qquad \frac{2q}{q-2}=p\le\frac{2(k+1)}{i},\qquad \frac{nq}{n+q}=q'\le\frac{2(k+1)}{k+1-i}. \tag{4.25}$$

We note that the first condition is equivalent to $q'\ge 2$, which we use in order to apply (4.16). The second condition is equivalent to $\frac{1}{p}\ge\frac{\theta}{2}$, and the third condition is $\frac{1}{q'}\ge\frac{1-\theta}{2}$. These allow us to solve for r and r', which will then be at least 2.

We claim that we can find $q\ge 2^*$ satisfying (4.25). Supposing for the moment that this is the case, we obtain:

$$\begin{aligned}\|\Lambda^{k-i}f\|_{L^q_x\mathcal{H}_v}\|\Lambda^i g\|_{L^p_x\mathcal{H}'_v}&\lesssim\|\Lambda^{k+1-i}f\|_{\dot B^{0,2}_{q'}\mathcal{H}_v}\|\Lambda^i g\|_{\dot B^{0,2}_{p}\mathcal{H}'_v}\\&\lesssim\|f\|^{\theta}_{\dot B^{0,2}_{r'}\mathcal{H}_v}\|\Lambda^{j_*}f\|^{1-\theta}_{\dot B^{0,2}_{2}\mathcal{H}_v}\|g\|^{1-\theta}_{\dot B^{0,2}_{r}\mathcal{H}'_v}\|\Lambda^{j_*}g\|^{\theta}_{\dot B^{0,2}_{2}\mathcal{H}'_v}\\&\lesssim\|g\|_{H^{K^*_n}_x\mathcal{H}'_v}\|\Lambda^{j_*}f\|_{L^2_x\mathcal{H}_v}+\|f\|_{H^{K^*_n}_x\mathcal{H}_v}\|\Lambda^{j_*}g\|_{L^2_x\mathcal{H}'_v}.\end{aligned}$$

This follows by collecting the estimates in this paragraph, using the embedding (4.14) and applying Young's inequality.

Thus we will have proven Lemma 4.15 as soon as we establish (4.25). The second condition in (4.25) is equivalent to

$$q\ge\frac{2(k+1)}{k+1-i}.$$

The third condition in (4.25) is equivalent to:

$$\frac{1}{q}\ge\frac{(n-2)(k+1)-ni}{2n(k+1)}. \tag{4.26}$$

If we assume that $i<\frac{n-2}{n}(k+1)$, then (4.26) is equivalent to:

$$q\le\frac{2n(k+1)}{(n-2)(k+1-i)-ni}.$$

We can check that in this case $\left(\max\left\{\frac{2(k+1)}{k+1-i},\frac{2n}{n-2}\right\},\frac{2n(k+1)}{(n-2)(k+1-i)-ni}\right)$ is a non-empty open interval, hence we observe that we can always satisfy (4.25) when $i<\frac{n-2}{n}(k+1)$. Alternatively if $i\ge\frac{n-2}{n}(k+1)$ then (4.26) is immediately satisfied since the right-hand side is non-positive. $\square$

An additional product estimate which we will use is as follows.

Lemma 4.16. *For any $k\in\{0,1,\dots,K\}$ and $i\in\{0,1,\dots,k\}$ there exists $p,q\ge 2$ satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ such that we have*

$$\|\Lambda^{k-i}f\|_{L^q_xL^2_v}\|\Lambda^i f\|_{L^p_xN^{s,\gamma}}\lesssim\left(\|f\|_{H^{K^*_n}_xN^{s,\gamma}_{\ell'}}+\|w^{\ell'}f\|_{H^K_xL^2_v}\right)\sum_{j_*\le m\le K}\|\Lambda^m f\|_{L^2_xN^{s,\gamma}},$$

where $j_*=\min\{k+1,K\}$ and we suppose that $\ell'\ge\ell'_0$, where ℓ'_0 is defined in (1.31).

By using Lemma 4.16, together with the results of Lemma 4.13 and Lemma 4.14, we can directly deduce (2.14).

Proof of Lemma 4.16. We first prove this inequality for the hard potentials (1.8), when $\gamma + 2s \geq 0$. In this case we have: $|f|_{L^2_v} \leq |f|_{L^2_{\gamma+2s}} \leq |f|_{N^{s,\gamma}}$. Hence:

$$\|\Lambda^{k-i} f\|_{L^q_x L^2_{\gamma+2s}} \|\Lambda^i f\|_{L^p_x N^{s,\gamma}} \leq \|\Lambda^{k-i} f\|_{L^q_x N^{s,\gamma}} \|\Lambda^i f\|_{L^p_x N^{s,\gamma}}$$

We then use Lemma 4.15 to deduce this expression is:

$$\lesssim \|f\|_{H^{K^*_n}_x N^{s,\gamma}} \|\Lambda^{j_*} f\|_{L^2_x N^{s,\gamma}}$$

Hence, the claim follows in the case of hard potentials if we take $\ell' \geq 0$.

Thus the rest of this proof is focused on the soft potential case of (1.9), when $\gamma + 2s < 0$. We again prove it in a series of special cases. Consider when $i = k$ and $j_* = k = K$. In this situation choose $p = 2$ and $q = \infty$, and use the functional Sobolev embedding $L^\infty_x L^2_v \supset H^{K^*_n}_x L^2_v$ as in (4.13). In this case the lemma holds for any $\ell' \geq 0$. If alternatively $i = k$ and $j_* = k+1$ we choose $p = 2^* = \frac{2n}{n-2}$ and $q = n$ and use the same estimates as in (4.21) and (4.22) (we again only need $\ell' \geq 0$). Of course the case $k = 0$ is also covered by this aforementioned analysis.

Next we turn to the case when $i = 0$ and $j_* = k = K$. In this situation we must choose $q = 2$ and $p = \infty$. By interpolation

$$\|\Lambda^k f\|_{L^2_x L^2_v} \lesssim \|\Lambda^k f\|^{\tilde{\theta}}_{L^2_x L^2_{\gamma+2s}} \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde{\theta}}{1-\tilde{\theta}}} \Lambda^k f\|^{1-\tilde{\theta}}_{L^2_x L^2_v}, \tag{4.27}$$

which holds for any $\tilde{\theta} \in (0,1)$. In order to prove (4.27), we note that:

$$|\Lambda^k f|_{L^2_v} = ||w^{\frac{\alpha}{\tilde{\theta}}} \Lambda^k f|^{\tilde{\theta}} |w^{\frac{-\alpha}{1-\tilde{\theta}}} \Lambda^k f|^{1-\tilde{\theta}}|_{L^2_v} \leq |w^{\frac{\alpha}{\tilde{\theta}}} \Lambda^k f|^{\tilde{\theta}}_{L^2_v} |w^{\frac{-\alpha}{1-\tilde{\theta}}} \Lambda^k f|^{1-\tilde{\theta}}_{L^2_v},$$

by Hölder's inequality. We then apply Hölder's inequality in x and we further choose $\alpha \overset{\text{def}}{=} \frac{(\gamma+2s)\tilde{\theta}}{2}$ to obtain (4.27).

On the other hand, we use (4.10) (with θ and $1-\theta$ reversed) to obtain:

$$\|f\|_{L^\infty_x N^{s,\gamma}} \lesssim \|\Lambda^{m'} f\|^{\theta'}_{L^2_x N^{s,\gamma}} \|\Lambda^{j_*} f\|^{1-\theta'}_{L^2_x N^{s,\gamma}}.$$

for appropriate m' and θ'. Now from (4.10) since $i = 0$, $j_* = k = K$ and $p = \infty$ we have $\theta' = \frac{K - \frac{n}{2}}{K - m'}$. Furthermore always $K > \frac{n}{2}$ so that we choose $m' = 0$. Hence, the condition $m' \neq j_*$, which is an assumption in order to use (4.10) is satisfied. Moreover, we note that $\theta' = \frac{2K-n}{2K} \in (0,1)$. Thus, we just choose $\tilde{\theta} \overset{\text{def}}{=} \theta' \in (0,1)$.

By using the bounds in the previous two paragraphs, and Young's inequality, it follow that the contribution from this case is:

$$\begin{aligned}
\|\Lambda^k f\|_{L^2_x L^2_v} \|f\|_{L^\infty_x N^{s,\gamma}} &\lesssim \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde{\theta}}{1-\tilde{\theta}}} \Lambda^k f\|_{L^2_x L^2_v} \|\Lambda^{j_*} f\|_{L^2_x N^{s,\gamma}} \\
&\quad + \|\Lambda^{j_*} f\|_{L^2_x L^2_{\gamma+2s}} \|f\|_{L^2_x N^{s,\gamma}} \\
&\lesssim \left(\|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde{\theta}}{1-\tilde{\theta}}} f\|_{H^K_x L^2_v} + \|f\|_{H^{K^*_n}_x N^{s,\gamma}_{\ell'}} \right) \|\Lambda^{j_*} f\|_{L^2_x N^{s,\gamma}},
\end{aligned}$$

provided that $\ell' \geq 0$. Finally, we note that we have to take:

$$\ell' \geq \frac{(-\gamma - 2s)}{2} \frac{\tilde{\theta}}{1-\tilde{\theta}} = \frac{|\gamma + 2s|}{2} \left(\frac{2K-n}{n} \right). \tag{4.28}$$

This condition contributes to the size of the weight in (1.31).

Suppose next that $i=0$ and $j_*=k+1$. Then we choose $q=2^*=\frac{2n}{n-2}$, use (4.21) and interpolation (as in (4.27) with k replaced by $k+1$) to achieve that:

$$\|\Lambda^k f\|_{L^{2^*}_x L^2_v} \lesssim \|\Lambda^{k+1} f\|_{L^2_x L^2_v} \lesssim \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde\theta}{1-\tilde\theta}}\Lambda^{k+1} f\|^{(1-\tilde\theta)}_{L^2_x L^2_v}\|\Lambda^{k+1} f\|^{\tilde\theta}_{L^2_x L^2_{\gamma+2s}}.$$

For the other term we use (4.10) (since $K\geq 2K^*_n$) to deduce:

$$\|f\|_{L^n_x N^{s,\gamma}} \lesssim \|f\|^{\theta'}_{L^2_x N^{s,\gamma}}\|\Lambda^K f\|^{1-\theta'}_{L^2_x N^{s,\gamma}}.$$

Now from (4.10) since $i=0$ and $p=n$ we have $\theta'=\frac{K-\frac{n}{2}+1}{K}\in(0,1)$. We now choose $\tilde\theta=\theta'$. By using the obtained bounds, Young's inequality, the fact that $j_*=k+1\leq K$, and arguing as earlier, we obtain that, since $\ell'\geq 0$

$$\begin{aligned}\|\Lambda^k f\|_{L^{2^*}_x L^2_v}\|f\|_{L^n_x N^{s,\gamma}} &\lesssim \|f\|_{L^2_x N^{s,\gamma}}\|\Lambda^{j_*} f\|_{L^2_x N^{s,\gamma}} \\ &\quad + \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde\theta}{1-\tilde\theta}}\Lambda^{k+1} f\|_{L^2_x L^2_v}\|\Lambda^K f\|_{L^2_x N^{s,\gamma}} \\ &\lesssim \left(\|f\|_{H^{K^*_n}_x N^{s,\gamma}_{\ell'}} + \|w^{\frac{-\gamma-2s}{2}\frac{\tilde\theta}{1-\tilde\theta}} f\|_{H^K_x L^2_v}\right)\sum_{j_*\leq m\leq K}\|\Lambda^m f\|_{L^2_x N^{s,\gamma}}.\end{aligned}$$

The term coming from this contribution satisfies the required bound provided that:

$$\ell' \geq \frac{-\gamma-2s}{2}\frac{\tilde\theta}{1-\tilde\theta} = \frac{-\gamma-2s}{2}\frac{2K-n+2}{n-2} = \frac{-\gamma-2s}{2}\left(\frac{2K}{n-2}-1\right). \tag{4.29}$$

We note that the case $k=1$ is covered by the previous arguments.

Now we consider the case when $k\geq 2$, $i\in\{1,\dots,k-1\}$ with $j_*=k=K$. Take $q=\frac{2K}{K-i}$ and $\theta=\frac{K-i}{K}$. We use (4.19), (4.16) and then (4.27) to obtain

$$\begin{aligned}\|\Lambda^{K-i} f\|_{L^{2K/(K-i)}_x L^2_v} &\lesssim \|\Lambda^{K-i} f\|_{\dot B^{0,2}_{2K/(K-i)} L^2_v} \lesssim \|f\|^{1-\theta}_{\dot B^{0,2}_\infty L^2_v}\|\Lambda^K f\|^{\theta}_{\dot B^{0,2}_2 L^2_v} \\ &= \|f\|^{1-\theta}_{\dot B^{0,2}_\infty L^2_v}\|\Lambda^K f\|^{\theta}_{L^2_x L^2_v} \lesssim \|f\|^{1-\theta}_{\dot B^{0,2}_\infty L^2_v}\|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde\theta}{1-\tilde\theta}}\Lambda^K f\|^{\theta(1-\tilde\theta)}_{L^2_x L^2_v}\|\Lambda^K f\|^{\theta\tilde\theta}_{L^2_x L^2_{\gamma+2s}}.\end{aligned}$$

Here again $\tilde\theta\in(0,1)$ is arbitrary.

Furthermore, $p=\frac{2K}{i}$ and, for $\theta'=\frac{K-\frac{n}{2}\left(\frac{K-i}{K}\right)-i}{K}$, we use (4.10) to obtain

$$\|\Lambda^i f\|_{L^{2K/i}_x N^{s,\gamma}} \lesssim \|f\|^{\theta'}_{L^2_x N^{s,\gamma}}\|\Lambda^K f\|^{1-\theta'}_{L^2_x N^{s,\gamma}}.$$

Combining the previous estimates, it follows that:

$$\begin{aligned}&\|\Lambda^{K-i} f\|_{L^{2K/(K-i)}_x L^2_v}\|\Lambda^i f\|_{L^{2K/i}_x N^{s,\gamma}} \\ &\lesssim \|f\|^{1-\theta}_{\dot B^{0,2}_\infty L^2_v}\|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde\theta}{1-\tilde\theta}}\Lambda^K f\|^{\theta(1-\tilde\theta)}_{L^2_x L^2_v}\|\Lambda^K f\|^{\theta\tilde\theta}_{L^2_x L^2_{\gamma+2s}}\|f\|^{\theta'}_{L^2_x N^{s,\gamma}}\|\Lambda^K f\|^{1-\theta'}_{L^2_x N^{s,\gamma}},\end{aligned}$$

Since $\theta>\theta'$, we can choose $\tilde\theta=\frac{\theta'}{\theta}=1-\frac{n}{2}\frac{1}{K}\in(0,1)$. By the condition that $(1-\theta')+\theta\tilde\theta=1$, it follows that the above product is:

$$\lesssim \|\Lambda^K f\|_{L^2_x N^{s,\gamma}}(\|f\|_{H^{K^*_n}_x L^2_v} + \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde\theta}{1-\tilde\theta}}\Lambda^K f\|_{L^2_x L^2_v} + \|f\|_{L^2_x N^{s,\gamma}}).$$

Here, we used Young's inequality and (4.14). Note that $K>\frac{n}{2}\left(\frac{K-i}{K}\right)+i$ since $\frac{n}{2}\frac{1}{K}<1$. As before, we deduce that ℓ' has to satisfy the bound:

$$\ell' \geq \frac{-\gamma-2s}{2}\frac{\tilde\theta}{1-\tilde\theta} = \frac{-\gamma-2s}{2}\left(\frac{2}{n}K-1\right). \tag{4.30}$$

This completes the our estimate in this case.

Notice the only case remaining is when $i \in \{1, \dots, k-1\}$ with $k \in \{2, \dots, K-1\}$. Let us first consider the subcase when:

$$k + 1 \geq \frac{n}{2}. \tag{4.31}$$

Notice that (4.31) covers all the remaining cases when $n \leq 6$.

The first step now is to use (4.23) and (4.24) with $\mathcal{H}_v = L^2_v$ to deduce:

$$\|\Lambda^{k-i} f\|_{L^q_x L^2_v} \lesssim \|\Lambda^{k+1-i} f\|_{L^{q'}_x L^2_v} \lesssim \|f\|^{\theta}_{\dot{B}^{0,2}_{r'} L^2_v} \|\Lambda^{k+1} f\|^{1-\theta}_{\dot{B}^{0,2}_2 L^2_v},$$

whenever $q \geq \frac{2n}{n-2} = 2^*$, $q' = \frac{nq}{n+q} \geq 2$, $r' \geq 2$ are such that for $\theta = \frac{i}{k+1}$, one has:

$$\frac{1}{q'} = \frac{\theta}{r'} + \frac{1-\theta}{2} = \frac{i}{r'(k+1)} + \frac{k+1-i}{2(k+1)}. \tag{4.32}$$

Let us suppose for now that we can find such a q and r'. Then, by doing an interpolation similar to (4.27) we obtain that the previous upper bound is

$$\lesssim \|f\|^{\theta}_{\dot{B}^{0,2}_{r'} L^2_v} \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde{\theta}}{1-\tilde{\theta}}} \Lambda^{k+1} f\|^{(1-\theta)(1-\tilde{\theta})}_{L^2_x L^2_v} \|\Lambda^{k+1} f\|^{(1-\theta)\tilde{\theta}}_{L^2_x L^2_{\gamma+2s}},$$

for any $\tilde{\theta} \in (0,1)$. At the same time we use (4.10) to deduce that for $p \geq 2$:

$$\|\Lambda^i f\|_{L^p_x N^{s,\gamma}} \lesssim \|f\|^{1-\theta'}_{L^2_x N^{s,\gamma}} \|\Lambda^{k+1} f\|^{\theta'}_{L^2_x N^{s,\gamma}},$$

for

$$\theta' = \frac{\frac{n}{2} - \frac{n}{p} + i}{k+1}, \tag{4.33}$$

provided that $\theta' \in (0,1)$. We will choose $\tilde{\theta} \in (0,1)$, such that $1 - \theta' = (1-\theta)\tilde{\theta}$. This is possible whenever $1 - \theta' < 1 - \theta$ or equivalently $\theta' > \theta$ which is automatic, provided $p > 2$.

We suppose that we can choose $q \in [2^*, \infty), r' \geq 2$ satisfying (4.32). Then, since $q \in (2, \infty)$, we can find $p > 2$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Furthermore, we suppose that θ' from (4.33) belongs to $(0,1)$. Under these assumptions, we can use Young's inequality and argue as before to deduce that the given contribution is:

$$\begin{aligned}\|\Lambda^{k-i} f\|_{L^q_x L^2_v} \|\Lambda^i f\|_{L^p_x N^{s,\gamma}} &\lesssim \|f\|_{H^{K^*_n}_x L^2_v} \|\Lambda^{k+1} f\|_{L^2_x N^{s,\gamma}} \\ &+ \left(\|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde{\theta}}{1-\tilde{\theta}}} \Lambda^{k+1} f\|_{L^2_x L^2_v} + \|f\|_{L^2_x N^{s,\gamma}} \right) \|\Lambda^{k+1} f\|_{L^2_x N^{s,\gamma}},\end{aligned}$$

where we used (4.14) for the first term. We must choose

$$\ell' \geq \frac{-\gamma - 2s}{2} \frac{\tilde{\theta}}{1-\tilde{\theta}}. \tag{4.34}$$

Then this term satisfies the desired bound.

We now choose r' for which all of the assumptions will be satisfied. If we take $r' \stackrel{\text{def}}{=} n$, from (4.32), we deduce that: $q' \in (2, n)$. Consequently, $q = \frac{nq'}{n-q'} > 2^* > 2$ and q is finite. Thus, we can choose $p \in (2, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. We now explicitly compute $\frac{1}{q}$ from (4.32) and the fact that $\frac{1}{q} = \frac{1}{q'} - \frac{1}{n}$:

$$\frac{1}{q} = \frac{\theta}{n} + \frac{1-\theta}{2} - \frac{1}{n} = \frac{i}{n(k+1)} + \frac{k+1-i}{2(k+1)} - \frac{1}{n} = \left(\frac{1}{2} - \frac{1}{n}\right)\left(1 - \frac{i}{k+1}\right). \tag{4.35}$$

We substitute the middle expression into (4.33) and use $\frac{1}{2} - \frac{1}{p} = \frac{1}{q}$ to deduce:

$$\theta' = \frac{\frac{i}{k+1} + \frac{n}{2}\left(\frac{k+1-i}{k+1}\right) - 1 + i}{k+1}.$$

Since $i \geq 1$, it follows that $\theta' > 0$. To check that $\theta' < 1$, it suffices to show

$$\frac{i}{k+1} + \frac{n}{2}\left(\frac{k+1-i}{k+1}\right) - 1 + i < k+1.$$

which is equivalent to:

$$\frac{n}{2} + i\left(\frac{1}{k+1} - \frac{n}{2(k+1)} + 1\right) - 1 < k+1. \tag{4.36}$$

By assumption (4.31), it follows that $\frac{1}{k+1} - \frac{n}{2(k+1)} + 1 > 0$. Hence, we need to verify (4.36) when $i = k-1$. In this case, the condition is equivalent to $2k+4 > n$ which holds by (4.31).

In order to find the precise value for the lower bound for ℓ' in (4.34), we need to compute $\tilde{\theta}$ explicitly. In order to do this, using (4.35), we have

$$\tilde{\theta} = \frac{1-\theta'}{1-\theta} = \frac{k+1-i-\frac{k+1-i}{k+1}\left(\frac{n}{2}-1\right)}{k+1-i} = 1 - \frac{n}{k+1}\left(\frac{1}{2} - \frac{1}{n}\right).$$

From here, it follows that:

$$\frac{\tilde{\theta}}{1-\tilde{\theta}} = \frac{1 - \frac{n}{k+1}\left(\frac{1}{2} - \frac{1}{n}\right)}{\frac{n}{k+1}\left(\frac{1}{2} - \frac{1}{n}\right)} = \frac{2(k+1)}{n-2} - 1 \leq \frac{2K}{n-2} - 1.$$

Consequently, in (4.34), we can take:

$$\ell' \geq \frac{-\gamma - 2s}{2}\left(\frac{2K}{n-2} - 1\right). \tag{4.37}$$

This grants the desired bound under (4.38).

We now consider the second subcase (and the last case) when

$$k < \frac{n}{2} - 1. \tag{4.38}$$

Notice that this case is only needed when $n > 6$. Also recall that $i \in \{1, \dots, k-1\}$ with $k \in \{2, \dots, K-1\}$. In the following we recall that $\tilde{j}$ from (1.29) is the largest integer which is strictly less than $\frac{n}{2}$. Let us take $q \geq \frac{2n}{n-2\tilde{j}}$. We note that for even n, $\tilde{j} = \frac{n+1}{2} - 1 = \frac{n-1}{2}$ and, for odd n, $\tilde{j} = \frac{n}{2} - 1 = \frac{n-2}{2}$. Consequently, the above condition on q becomes:

$$q \geq \begin{cases} 2n & \text{for } n \text{ odd,} \\ n & \text{for } n \text{ even.} \end{cases} \tag{4.39}$$

We use (4.19), (4.12) and (4.16) with $q' = \frac{nq}{n+q\tilde{j}}$ and $\theta = \frac{i}{k+\tilde{j}}$ to deduce that:

$$\|\Lambda^{k-i} f\|_{L^q_x L^2_v} \lesssim \|\Lambda^{k-i} f\|_{\dot{B}^{0,2}_q L^2_v} \lesssim \|\Lambda^{k+\tilde{j}-i} f\|_{\dot{B}^{0,2}_{q'} L^2_v} \lesssim \|f\|^{\theta}_{\dot{B}^{0,2}_r L^2_v} \|\Lambda^{k+\tilde{j}} f\|^{1-\theta}_{L^2_x L^2_v}.$$

We note that $q' \geq 2$ since $q \geq \frac{2n}{n-2\tilde{j}}$, and $r \geq 2$ is obtained from the relation $\frac{1}{q'} = \frac{\theta}{r} + \frac{1-\theta}{2}$.

By using (4.14), and an interpolation similar to (4.27), we obtain that the above expression is

$$\lesssim \|f\|^{\theta}_{H^{K^*_n}_x L^2_v} \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde{\theta}}{1-\tilde{\theta}}} \Lambda^{k+\tilde{j}} f\|^{(1-\theta)(1-\tilde{\theta})}_{L^2_x L^2_v} \|\Lambda^{k+\tilde{j}} f\|^{(1-\theta)\tilde{\theta}}_{L^2_x L^2_{\gamma+2s}},$$

for any $\tilde{\theta} \in (0,1)$. Simultaneously we use (4.10) to obtain that for $p \geq 2$ we have

$$\|\Lambda^i f\|_{L^p_x N^{s,\gamma}} \lesssim \|f\|^{1-\theta'}_{L^2_x N^{s,\gamma}} \|\Lambda^{k+\tilde{j}} f\|^{\theta'}_{L^2_x N^{s,\gamma}}, \quad \theta' = \frac{\frac{n}{2} - \frac{n}{p} + i}{k+\tilde{j}}. \tag{4.40}$$

We want to choose $1-\theta' = \tilde{\theta}(1-\theta)$ which requires that $1-\theta' < 1-\theta$ or equivalently $\theta' > \theta$, which follows if we take $p > 2$. Finally, given $q \in [\frac{2n}{n-2\tilde{j}}, \infty)$ as before, we can find $p > 2$ such that $\frac{1}{p} + \frac{1}{q} = 2$.

For such a pair (p,q) and for θ' as defined in (4.40), we can see that $\theta' \in (0,1)$. Indeed since $p > 2$, it follows that $\theta' > 0$. On the other hand, in order to check that $\theta' = \frac{\frac{n}{q}+i}{k+\tilde{j}} < 1$, since $i \leq k-1$, we must observe that

$$\frac{n}{q} + (k-1) < k + \tilde{j}. \tag{4.41}$$

From (1.29), then (4.41) holds if $q > n/\lceil n/2 \rceil$, which is always the case by (4.39).

By using the previous estimates, and arguing similarly as before, we obtain

$$\begin{aligned}&\|\Lambda^{k-i} f\|_{L^q_x L^2_v} \|\Lambda^i f\|_{L^p_x N^{s,\gamma}} \\ &\quad\lesssim \left(\|f\|_{H^{K^*_n}_x L^2_v} + \|w^{\left(\frac{-\gamma-2s}{2}\right)\frac{\tilde{\theta}}{1-\tilde{\theta}}} \Lambda^{k+\tilde{j}} f\|_{L^2_x L^2_v} + \|f\|_{L^2_x N^{s,\gamma}} \right) \|\Lambda^{k+\tilde{j}} f\|_{L^2_x N^{s,\gamma}}.\end{aligned}$$

Because of (4.38), (1.29), and $K \geq 2K^*_n = 2\lfloor \frac{n}{2} + 1 \rfloor$, it follows that $k + \tilde{j} \leq K$. Hence, we are done if we have (4.34) with $\tilde{\theta}$ as given in this part.

As before, we compute $\tilde{\theta}$ explicitly:

$$\tilde{\theta} = \frac{1-\theta'}{1-\theta} = \frac{1 - \frac{\frac{n}{q}+i}{k+\tilde{j}}}{1 - \frac{i}{k+\tilde{j}}} = 1 - \frac{n}{q(k+\tilde{j}-i)}.$$

Therefore, using (4.38), (4.39) and (1.29), we can choose $q = n$ for odd n and we deduce:

$$\frac{\tilde{\theta}}{1-\tilde{\theta}} = \frac{q}{n}\left(k + \tilde{j} - i\right) - 1 = (k + \lceil n/2 \rceil - i - 1) - 1 \leq \frac{n}{2} + \lceil n/2 \rceil - 3.$$

For even n, we take $q = 2n$ and we deduce that:

$$\frac{\tilde{\theta}}{1-\tilde{\theta}} = 2\left(k + \lceil n/2 \rceil - i - 1\right) - 1 \leq n + 2\lceil n/2 \rceil - 7.$$

Recall that in this case, $n > 6$. Hence, we recall the definition of ℓ^d_0 in (1.27) and deduce that in this case, we need to take:

$$\ell' \geq \frac{-\gamma - 2s}{2} \ell^d_0. \tag{4.42}$$

By using (4.28), (4.29), (4.30), (4.37), (4.42), and the definition of ℓ'_0 in (1.31), the lemma now follows. $\square$

5. Linear decay in Besov spaces

For the linearized Boltzmann equation (1.10), by dropping the non-linear term we obtain the Cauchy problem for the linear Boltzmann equation (3.25) when $g = 0$:

$$\partial_t f + \mathbf{B} f = 0, \quad f(0,x,v) = f_0(x,v), \quad \mathbf{B} \stackrel{\text{def}}{=} v \cdot \nabla_x + L. \tag{5.1}$$

Then as in (3.27) we can represent solutions to (5.1) with the solution operator:

$$f(t) = \mathbb{A}(t) f_0, \quad \mathbb{A}(t) \stackrel{\text{def}}{=} e^{-t\mathbf{B}}. \tag{5.2}$$

In this section we will establish the large time decay rates for the linear Boltzmann equation (5.1). In the first part of this section, we obtain Besov space decay estimates for general initial data belonging to an appropriate Besov space.

In the second part of this section, we study the hard potential case (1.8) and we obtain improved decay estimates for initial data which belongs to an appropriate Besov space, but which is also microscopic. We will see that, in this case, we obtain an additional decay factor of $t^{-\frac{1}{2}}$. The key to obtaining the additional decay will be a detailed understanding the spectral properties of the Fourier transform of the linearized Boltzmann operator for small frequencies. Our analysis of these spectral properties is motivated by the work of Ellis and Pinsky [11].

5.1. **Linear decay rates in Besov spaces.**

We are interested in obtaining decay estimates for the general linear problem (5.1).

Theorem 5.1. *Suppose that $m, \varrho \in \mathbb{R}$ with $m + \varrho > 0$, $1 \le p \le \infty$ and $\ell \in \mathbb{R}$. Smooth solutions to (5.1) satisfy, uniform in $t \ge 0$, the large time decay estimate*

$$\| w^\ell \mathbb{A}(t) f_0 \|_{\dot{B}_2^{m,p} L_v^2} \lesssim \| w^{\ell+\sigma} f_0 \|_{\dot{B}_2^{m,p} L_v^2 \cap \dot{B}_2^{-\varrho,\infty} L_v^2} (1+t)^{-(m+\varrho)/2}.$$

This holds if $\| w^{\ell+\sigma} f_0 \|_{\dot{B}_2^{m,p} L_v^2 \cap \dot{B}_2^{-\varrho,\infty} L_v^2} < \infty$. Now for the soft potentials (1.9) we need $\sigma > -(m+\varrho)(\gamma + 2s)$, and for the hard potentials (1.8) we can take $\sigma = 0$.

To prove Theorem 5.1, we will use the following Lyapunov functional constructed in [25, Theorem 2.3]:

Theorem 5.2. *Fix $\ell \in \mathbb{R}$. Let $f(t,x,v)$ be the solution to the Cauchy problem (5.1). Then there is a weighted time-frequency functional $\mathcal{E}_\ell(t,\xi)$ such that*

$$\mathcal{E}_\ell(t,\xi) \approx | w^\ell \hat{f}(t,\xi) |_{L_v^2}^2, \tag{5.3}$$

where for any $t \ge 0$ and $\xi \in \mathbb{R}^n$ we have

$$\partial_t \mathcal{E}_\ell(t,\xi) + \lambda \left(1 \wedge |\xi|^2\right) | w^\ell \hat{f}(t,\xi) |_{L^2_{\gamma+2s}}^2 \le 0. \tag{5.4}$$

We use the notation $1 \wedge |\xi|^2 \stackrel{def}{=} \min\{1, |\xi|^2\}$.

Proof of Theorem 5.1. Using Theorem 5.2, as in [25, Eqn (2.21)], we have

$$\mathcal{E}_\ell(t,\xi) \lesssim e^{-\lambda\left(1\wedge|\xi|^2\right)t} \mathcal{E}_\ell(0,\xi). \tag{5.5}$$

The bound (5.5) only holds for the hard-potentials (1.8). For the soft-potentials (1.9), it follows from [25, just below eqn (2.21)] that we have the estimate

$$\mathcal{E}_\ell(t,\xi) \lesssim \left(\frac{t\left(1 \wedge |\xi|^2\right)}{\sigma} + 1 \right)^{-\sigma} \mathcal{E}_{\ell+\sigma'}(0,\xi). \tag{5.6}$$

This estimate (5.6) holds for any $\sigma > 0$ with $\sigma' = -\sigma(\gamma + 2s) > 0$. Now because of the degeneracy of the soft potentials in (1.9), the upper bound (5.6) loses a weight of order σ' on the initial data. Alternatively, the hard potential case (1.8) does not lose a weight on the initial data in the time-frequency estimate (5.5).[3]

In the following we will prove Theorem 5.1 for the soft potential (1.9) case of (5.6). However notice that the hard potential (1.8) estimate (5.5) is better than (5.6). Thus the proof below will clearly apply in both situations.

We now recall the smooth function $\varphi_j(\xi)$ which is supported on $|\xi| \approx 2^j$ from Section 4.2. Then multiplying (5.6) by $\varphi_j^2(\xi)$ and integrating over $\xi \in \mathbb{R}^n_\xi$ we obtain from (5.6), (5.3) and the Plancherel theorem that

$$\|w^\ell(\Delta_j f)(t)\|_{L^2_x L^2_v} \lesssim \left(\frac{t\left(1 \wedge 2^{2j}\right)}{\sigma} + 1\right)^{-\sigma/2} \|w^{\ell+\sigma'} \Delta_j f_0\|_{L^2_x L^2_v}. \tag{5.7}$$

If $j \geq 0$, we conclude from (5.7), for any $\sigma > 0$, that

$$\left(2^{mj}\|w^\ell(\Delta_j f)(t)\|_{L^2_x L^2_v}\right) \lesssim (t+1)^{-\sigma/2} \left(2^{mj}\|w^{\ell+\sigma'}\Delta_j f_0\|_{L^2_x L^2_v}\right).$$

If $j < 0$, we alternatively conclude from (5.7) that

$$\begin{aligned}&\left(2^{mj}\|w^\ell(\Delta_j f)(t)\|_{L^2_x L^2_v}\right)\\ &\qquad\lesssim t^{-(m+\varrho)/2}\left(\sqrt{t}2^j\right)^{(m+\varrho)}\left(\left(\sqrt{t}2^j\right)^2+1\right)^{-\sigma/2}\|w^{\ell+\sigma'} f_0\|_{\dot{B}^{-\varrho,\infty}_2 L^2_v}.\end{aligned}$$

We complete our proof of Theorem 5.1 by taking the ℓ^p_j norm of both sides of the last two inequalities. In particular we notice that for $\sigma > (m+\varrho)$ then

$$\left\|\left(\sqrt{t}2^j\right)^{(m+\varrho)}\left(\left(\sqrt{t}2^j\right)^2+1\right)^{-\sigma/2}\right\|_{\ell^p_j} \lesssim 1, \tag{5.8}$$

for any $p \in [1,\infty]$ and the upper bound does not depend upon $t \geq 0$. $\square$

5.2. **Faster Linear Decay rates.** If we assume that the initial data is microscopic as in (1.13) and (3.28), we will see that it is possible to obtain a better decay bound in the linear problem. From the analysis, we will see that the key to using the additional information is to understand the behavior of the degenerate macroscopic part of the solution in the low spatial frequency regime. As a result of a precise analysis of the spectral properties of this operator, given by Proposition 5.3, we can obtain an additional factor of the frequency (c.f. the factor κ in (5.11)), which will result in a better estimate. The arguments in this subsection are restricted to the hard potential case (1.8) due to the degeneracy of the operators needed in order to obtain the spectral decomposition (c.f. (5.16)).

Given $\xi \in \mathbb{R}^n$, let us look at the following operator:

$$\widehat{\mathbf{B}}(\xi) \overset{\mathrm{def}}{=} 2\pi \mathrm{i} v\cdot\xi + L. \tag{5.9}$$

We define $\kappa \overset{\mathrm{def}}{=} |\xi|$, and, assuming $\xi \neq 0$, we let $\omega \overset{\mathrm{def}}{=} \frac{\xi}{|\xi|}$. We will prove the following fact about the eigenvalues of $\widehat{\mathbf{B}}(\xi)$ for ξ sufficiently close to zero.

Proposition 5.3. *There exists $\kappa_0 > 0$ and smooth radial functions $\lambda_j = \lambda_j(\xi)$ such that $\lambda_j(\xi) \in C^\infty(\{\xi \in \mathbb{R}^n, 0 < |\xi| \leq \kappa_0\})$ for $j = 1, \ldots, n+2$ and*

[3]Notice the changing of the definition of the weight w in (1.22) from [25].

i) *Every* $\lambda_j(\xi)$ *is an eigenvalue of* $\widehat{\mathbf{B}}(\xi)$.

ii) *The* $\lambda_j(\xi)$ *have the asymptotic expansion:*

$$\lambda_j(\xi) = i\lambda_j^{(1)}\kappa + \lambda_j^{(2)}\kappa^2 + \mathcal{O}(\kappa^3), \text{ as } \kappa \to 0. \tag{5.10}$$

Here $\lambda_j^{(1)} \in \mathbb{R}$ *and* $\lambda_j^{(2)} > 0$.

iii) *Let us denote by* $\mathbf{P}_j(\xi)$ *the eigenprojection corresponding to the eigenvalue* $\lambda_j(\xi)$. *Then, assuming that* $\kappa_0 > |\xi| > 0$, *one has:*

$$\mathbf{P}_j(\xi) = \mathbf{P}_j^{(0)}(\omega) + \kappa \mathbf{P}_j^{(1)}(\xi) \tag{5.11}$$

and the eigenvalues $\lambda_j(\xi)$ *are semisimple, in the sense that they don't give rise to a generalized eigenspace.*

Moreover, $\mathbf{P}_j^{(0)}(\omega)$ *are orthogonal projections on* L^2_v *which satisfy:*

$$\mathbf{P} = \sum_{j=1}^{n+2} \mathbf{P}_j^{(0)}(\omega). \tag{5.12}$$

Finally, the $\mathbf{P}_j^{(1)}(\xi)$ *are uniformly bounded on* L^2_v *for* $|\xi| \leq \kappa_0$.

Let us point out that this type of result was first established by Ellis and Pinsky [11] in the setting of cut-off hard spheres. As we will see, their result also holds in the case of the non cut-off hard potential Boltzmann equation.

Let us first give an outline of the argument to obtain the existence of the eigenvalues. We would like to solve the eigenvalue equation

$$\left(\widehat{\mathbf{B}}(\xi) - \lambda\right)\phi = (2\pi \mathrm{i} v \cdot \xi + L - \lambda)\phi = 0. \tag{5.13}$$

We add to (5.13) the macroscopic projection $\mathbf{P}$ as

$$\left(\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda\right)\phi = \mathbf{P}\phi. \tag{5.14}$$

We notice that the operator $L + \mathbf{P}$ is coercive. Then for sufficiently small λ and $|\xi|$ the operator $\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda$ is invertible with bounded inverse. In order to obtain the invertibility, let us note that by (1.17), and the fact that $\gamma + 2s \geq 0$:

$$\begin{aligned} Re\,\langle(\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda)f, f\rangle &= Re\,\langle(L + \mathbf{P} - \lambda)f, f\rangle \\ &\geq C|\{\mathbf{I} - \mathbf{P}\}f|^2_{N^{s,\gamma}} + |\mathbf{P}f|^2_{L^2_v} - Re\,\lambda|f|^2_{L^2_v} \\ \geq C|\{\mathbf{I} - \mathbf{P}\}f|^2_{L^2_v} &+ |\mathbf{P}f|^2_{L^2_v} - Re\,\lambda|f|^2_{L^2_v} \geq (C - Re\,\lambda)|f|^2_{L^2_v} \geq C_1|f|^2_{L^2_v}, \end{aligned}$$

for some $C_1 > 0$ if $Re\,\lambda < \delta$ for $\delta > 0$ sufficiently small. In particular, the latter condition holds if λ is sufficiently close to zero. Let us henceforth fix λ satisfying this condition in order to obtain invertibility of the operator $\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda$. Let us suppose that $(\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda)f = g$. By taking L^2_v inner products with f and then taking real parts, it follows from the above noted coercivity property that

$$|f|_{L^2_v} \lesssim |(\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda)f|_{L^2_v}. \tag{5.15}$$

We can hence rewrite (5.14) as

$$\phi = \left(\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda\right)^{-1}\mathbf{P}\phi. \tag{5.16}$$

This equation (5.16) says that if we know the $n+2$ coefficients of $\mathbf{P}\phi$ then we can calculate ϕ itself. Thus we take $\mathbf{P}$ of both sides of (5.16) to obtain

$$\mathbf{P}\phi = \mathbf{P}\left(\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda\right)^{-1} \mathbf{P}\phi. \tag{5.17}$$

This grants a system of $n+2$ equations with $n+2$ unknowns, the macroscopic components (1.14), with parameter λ. One now expands out this system of equations and does a comparison of coefficients of the velocity basis vectors to obtain the exact form of this system for the macroscopic components from (1.14).

Note that when $\lambda = |\xi| = 0$ the system (5.17) has an exact solution for any element of $N(L)$. These are all of the solutions. One therefore studies the expanded system (5.17) for λ and $|\xi|$ close to zero, and tries to prove the existence of a branch of $\lambda(|\xi|)$ for $|\xi|$ near zero such that the system (5.17) has non-trivial solutions.

More precisely, we can find $\psi \in N(L)$ satisfying (5.17). Then, with $\mathbf{P}^1 = \mathbf{I} - \mathbf{P}$, we define ψ^1 to be

$$\psi^1 = \mathbf{P}^1\left(\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda\right)^{-1} \psi. \tag{5.18}$$

And we take $\phi \overset{\text{def}}{=} \psi + \psi^1$. From (5.17) and (5.18), and the fact that $\mathbf{P}\phi = \psi$, and $\mathbf{P}^1\phi = \psi^1$, it will follow that ϕ solves (5.16)

Proof of Proposition 5.3. We organize the proof in two parts, both parts generally follow [11]. The first part is devoted to the existence of the eigenvalues their smoothness in κ. The second part looks at their additional properties.

Part 1: Existence of the eigenvalues and their smoothness properties. We start by solving (5.17). We write

$$\mathbf{P}\phi = \sum_{j=0}^{n+1} C_j \chi_j, \tag{5.19}$$

for some $(C_0, C, C_{n+1}) \in \mathbb{R}^{n+2}$ with $C = (C_1, \dots, C_n)$. We use the notation

$$\chi_0 = \sqrt{\mu}, \quad \chi_i = v_j\sqrt{\mu} \quad (i = 1, \dots, n), \quad \text{and} \quad \chi_{n+1} = \frac{1}{\sqrt{2n}}(|v|^2 - n)\sqrt{\mu},$$

similar to (1.13). We then substitute (5.19) into (5.17). We also define

$$\Phi(\lambda, \xi) \overset{\text{def}}{=} \mathbf{P}(\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda)^{-1}.$$

We take L^2_v inner products of the result with the χ_j to obtain an equivalent system for (C_0, C, C_{n+1}):

$$C_j = \sum_{k=0}^{n+1} C_k \langle \Phi(\lambda, \xi)\chi_k, \chi_j\rangle, \quad (j = 0, \dots, n+1).$$

In the v integration of the inner product above we apply the rotation $\mathcal{O}$ for which $\mathcal{O}^t\xi = \kappa e_1$ for $e_1 = (1, 0, \dots, 0)$, and use the symmetries of this system to deduce that it is equivalent to the following:

$$C_0 = C_0\langle\bar{\Phi}(\lambda,\kappa)\chi_0, \chi_0\rangle + (C\cdot\omega)\langle\bar{\Phi}(\lambda,\kappa)\chi_1, \chi_0\rangle + C_{n+1}\langle\bar{\Phi}(\lambda,\kappa)\chi_{n+1}, \chi_0\rangle. \tag{5.20}$$

Also, for $1 \le j \le n$, we have

$$\begin{aligned} C_j = {} & \omega_j C_0\langle\bar{\Phi}(\lambda,\kappa)\chi_0, \chi_1\rangle + [C_j - (C\cdot\omega)\omega_j]\langle\bar{\Phi}(\lambda,\kappa)\chi_2, \chi_2\rangle \\ & + \omega_j(C\cdot\omega)\langle\bar{\Phi}(\lambda,\kappa)\chi_1, \chi_1\rangle + \omega_j C_{n+1}\langle\bar{\Phi}(\lambda,\kappa)\chi_{n+1}, \chi_1\rangle. \end{aligned} \tag{5.21}$$

And lastly

$$C_{n+1} = C_0\langle \bar{\Phi}(\lambda,\kappa)\chi_0, \chi_{n+1}\rangle + (C\cdot\omega)\langle \bar{\Phi}(\lambda,\kappa)\chi_1, \chi_{n+1}\rangle + C_{n+1}\langle \bar{\Phi}(\lambda,\kappa)\chi_{n+1}, \chi_{n+1}\rangle, \tag{5.22}$$

where $\bar{\Phi}(\lambda,\kappa) \stackrel{\text{def}}{=} \Phi(\lambda,\kappa e_1)$.

Let us now multiply the j-th equation in (5.21) by ω_j, sum in j, use the fact that $\omega\in\mathbb{S}^{n-1}$, and $\langle v_j\sqrt{\mu}, v_j\sqrt{\mu}\rangle = \langle v_1\sqrt{\mu}, v_1\sqrt{\mu}\rangle = 1$ to deduce that

$$(C\cdot\omega) = C_0\langle \bar{\Phi}(\lambda,\kappa)\chi_0, \chi_1\rangle + (C\cdot\omega)\langle \bar{\Phi}(\lambda,\kappa)\chi_1, \chi_1\rangle + C_{n+1}\langle \bar{\Phi}(\lambda,\kappa)\chi_{n+1}, \chi_1\rangle. \tag{5.23}$$

For $j=1,\dots,n$, we can now multiply (5.23) by ω_j and subtract from the corresponding equation in (5.21) to obtain:

$$(C_j - (C\cdot\omega)\omega_j)\langle(\bar{\Phi}(\lambda,\kappa) - I)\chi_2, \chi_2\rangle = 0. \tag{5.24}$$

Here, we also used that for all j, $\langle v_j\sqrt{\mu}, v_j\sqrt{\mu}\rangle = \langle v_2\sqrt{\mu}, v_2\sqrt{\mu}\rangle = 1$. We now observe that (5.21) holds if and only if (5.23) and (5.24) both hold.

Our goal now is to simultaneously solve (5.20), (5.22), (5.23), and (5.24). Hence, we have reduced a system of $n+2$ equations to a system of three equations and an additional scalar equation. Let us observe that we can view (5.20), (5.22) and (5.23) as a system for the unknowns $C_0, (C\cdot\omega), C_{n+1}\in\mathbb{R}$. We consider separately the cases when this system has a non-trivial solution or not.

Case 1: The system for $C_0, (C\cdot\omega), C_{n+1}$ has a nontrivial solution: We observe that there exists a non-trivial solution $(C_0, (C\cdot\omega), C_{n+1})\in\mathbb{R}^3$ to this system when:

$$D(\lambda,\kappa) \stackrel{\text{def}}{=} \left| \int_{\mathbb{R}^n} dv\,\left(\bar{\Phi}(\lambda,\kappa) - I\right) \begin{pmatrix} \chi_0\chi_0 & \chi_1\chi_0 & \chi_{n+1}\chi_0 \\ \chi_0\chi_1 & \chi_1\chi_1 & \chi_{n+1}\chi_1 \\ \chi_0\chi_{n+1} & \chi_1\chi_{n+1} & \chi_{n+1}\chi_{n+1} \end{pmatrix} \right| = 0.$$

In the expression above the operator $\int_{\mathbb{R}^n} dv\,\left(\bar{\Phi}(\lambda,\kappa) - I\right)$ acts on each element of the 3×3 matrix individually. Let us observe that $D(0,0)=0$, and furthermore all of the entries of the matrix for $(\lambda,\kappa)=(0,0)$ are equal to zero, so we obtain a zero of order three at the origin. We then define the function:

$$G(z,\kappa) \stackrel{\text{def}}{=} \frac{1}{\kappa^3} D(z\kappa,\kappa).$$

We want to see what is $\lim_{\kappa\to 0} G(z,\kappa)$.

We can extend $G(z,\kappa)$ to a smooth function at $\kappa=0$ by defining:[4]

$$G(z,0) = \left| \int_{\mathbb{R}^n} dv\, \frac{d}{d\kappa}|_{\kappa=0} \bar{\Phi}(z\kappa,\kappa) \begin{pmatrix} \chi_0\chi_0 & \chi_1\chi_0 & \chi_{n+1}\chi_0 \\ \chi_0\chi_1 & \chi_1\chi_1 & \chi_{n+1}\chi_1 \\ \chi_0\chi_{n+1} & \chi_1\chi_{n+1} & \chi_{n+1}\chi_{n+1} \end{pmatrix} \right|.$$

[4]Here, we are implicitly using that $\bar{\Phi}(0,0) = (L+\mathbf{P})^{-1}$ is the identity when restricted to $N(L)$.

Furthermore we have: $\frac{d}{d\kappa}|_{\kappa=0}\bar{\Phi}(z\kappa,\kappa) = -\mathbf{P}\big[(L+\mathbf{P})^{-1}(2\pi \mathrm{i} v_1 - z)(L+\mathbf{P})^{-1}\big]$. Then by a standard calculation we can compute that

$$G(z,0) = \begin{vmatrix} z & -2\pi\mathrm{i} & 0 \\ -2\pi\mathrm{i} & z & -2\pi\mathrm{i}\langle\chi_{n+1}, v_1\chi_1\rangle \\ 0 & -2\pi\mathrm{i}\langle\chi_{n+1}, v_1\chi_1\rangle & z \end{vmatrix} = \left(z^3 - z(\eta_1^2+\eta_2^2)\right),$$

where $\eta_1 \stackrel{\text{def}}{=} -2\pi\mathrm{i}$ and $\eta_2 \stackrel{\text{def}}{=} -2\pi\mathrm{i}\langle\chi_{n+1}, v_1\chi_1\rangle$. In particular, we obtain that $G(z,0)=0$ if and only if $z = 0, \pm\sqrt{\eta_1^2+\eta_2^2}$. Moreover, for $z=0$, we note that $G'(z,0) = -(\eta_1^2+\eta_2^2) \neq 0$ and for $z = \pm\sqrt{\eta_1^2+\eta_2^2}$, it follows that $G'(z,0) = 2(\eta_1^2+\eta_2^2) \neq 0$. Let $z_1 \stackrel{\text{def}}{=} -\sqrt{\eta_1^2+\eta_2^2}$, $z_2 \stackrel{\text{def}}{=} 0$, $z_3 \stackrel{\text{def}}{=} \sqrt{\eta_1^2+\eta_2^2}$. Therefore, we can use the Implicit Function Theorem to deduce that, for all $1 \le j \le 3$, there exists an open neighborhood V_j of 0 and a unique smooth function ζ_j on V_j such that $G(\zeta_j(\kappa),\kappa) = 0$ for all $\kappa \in V_j$ and such that $\zeta_j(0) = z_j$. For $1 \le j \le 3$, we then take $\lambda_j(\xi) \stackrel{\text{def}}{=} \kappa\zeta_j(\kappa)$. Hence, we have obtained three zeros of $D(\cdot,\kappa)$, for κ sufficiently close to the origin. By construction, these zeros are smooth radial functions. We now observe that there are no other zeros.

By another rescaling argument, we note that: $D(\lambda,0) = \lambda^3 H(\lambda)$. We use that $\frac{d}{dt}|_{t=0}(L+\mathbf{P}-t)^{-1} = (L+\mathbf{P})^{-2}$ and argue as before to deduce that:

$$H(0) = \left| \int_{\mathbb{R}^n} dv \begin{pmatrix} \chi_0\chi_0 & \chi_1\chi_0 & \chi_{n+1}\chi_0 \\ \chi_0\chi_1 & \chi_1\chi_1 & \chi_{n+1}\chi_1 \\ \chi_0\chi_{n+1} & \chi_1\chi_{n+1} & \chi_{n+1}\chi_{n+1} \end{pmatrix} \right| = 1 \neq 0.$$

Now, for fixed κ sufficiently close to zero, we look at the holomorphic map $\lambda \mapsto D(\lambda,\kappa)$. We note that when λ traverses a small circle $\mathcal{C}$ centered at the origin, then $D(\lambda,0)$ goes around the origin three times. For κ sufficiently small, the same property will hold for $D(\lambda,\kappa)$. Hence, by applying the Argument Principle, we deduce that $D(\lambda,\kappa)$ has exactly three roots for κ sufficiently small. It follows, as in [11], that the eigenvalues $\lambda_j(\kappa)$ for $j=1,2,3$ are semisimple, i.e. they don't give rise to a generalized eigenspace.

We now construct the appropriate eigenfunctions $e^j(\kappa)$; $j=1,2,3$. The first fact to note is that we can choose $C_1(\kappa)$, $(C\cdot\omega)(\kappa)$, $C_{n+1}(\kappa)$ to depend smoothly on κ by the process of Gaussian Elimination. More precisely, the entries of the reduced row echelon form of the matrix of the system for $C_0, C\cdot\omega, C_{n+1}$ will depend smoothly on κ, hence we can choose elements of the nullspace in a smooth way. We find $C_j = C_j(\kappa)$ for $j=1,\dots,n$ by: $C_j \stackrel{\text{def}}{=} (C\cdot\omega)\omega_j$. In this way, (5.24) is also satisfied. Finally, we substitute C_0, $(C\cdot\omega)$, C_{n+1} into (5.19) and we then obtain ϕ from (5.16). The dependence on κ will then be smooth.

Case 2: The system for $C_0, (C\cdot\omega), C_{n+1}$ has no nontrivial solution: In this case, we can only take $C_0 = (C\cdot\omega) = C_{n+1} = 0$. Although we require that $C = (C_1,\dots,C_n) \neq 0$. Consequently, we obtain $n-1$ degrees of freedom for choosing C. The only constraint we now have to satisfy is (5.24).

We satisfy (5.24) this by choosing $\lambda = \lambda(\kappa)$ such that:

$$D_0(\lambda,\kappa) \stackrel{\text{def}}{=} \langle(\bar{\Phi}(\lambda,\kappa) - I)\chi_2, \chi_2\rangle = 0. \tag{5.25}$$

As in the previous case, we rescale to define:

$$G_0(z,\kappa) \overset{\text{def}}{=} \frac{1}{\kappa} D_0(z\kappa,\kappa),$$

and we compute:

$$\frac{d}{d\kappa}|_{\kappa=0} D_0(z,\kappa) = \langle -(L+\mathbf{P})^{-1}(2\pi \mathrm{i} v_1 - z)(L+\mathbf{P})^{-1}\chi_2, \chi_2\rangle$$
$$= \langle -(L+\mathbf{P})^{-1}(2\pi v_1 - z)\chi_2, \chi_2\rangle = z\langle \chi_2,\chi_2\rangle = z.$$

Hence, similarly as before, we take: $G_0(z,0) \overset{\text{def}}{=} z$. We note that $G_0(0,0) = 0$ and $\frac{d}{dz}G_0(\cdot,0) = 1 \neq 0$. Hence, by the Implicit Function Theorem, it follows that we can find a neighborhood $U \subseteq \mathbb{R}$ of the origin and $\zeta \in C^\infty(U)$ such that $\zeta(0) = 0$ and $G_0(\zeta(\kappa),\kappa) = 0$ for all $\kappa \in U$. We can then take $\lambda(\kappa) \overset{\text{def}}{=} \kappa\zeta(\kappa)$. This will be the only solution of (5.25) for κ sufficiently close to zero. Let us now choose $C^{(j)} \in \mathbb{R}^n; j = 4,\dots,n+2$, linearly independent such that $C^{(j)}\cdot\omega = 0$. (The $C^{(j)}$ don't depend on κ). From (5.16), we find eigenvectors $e^{(j)}(\xi)$ $(j = 4,\dots,n+2)$ by:

$$e^{(j)}(\xi) = (\widehat{\mathbf{B}}(\xi) + \mathbf{P} - \lambda(|\xi|))^{-1}(C^{(j)}\cdot v\sqrt{\mu}).$$

The $e^{(j)}$ are then smooth radial functions as well, so we can write $e^{(j)}(\kappa)$ instead of $e^{(j)}(\xi)$. In this way, we obtain an $n-1$ dimensional eigenspace. We need to check that there is no generalized eigenspace, i.e. that the eigenvalue $\lambda(\kappa) = \kappa\zeta(\kappa)$ is semisimple. It suffices to show that we can normalize the $e^{(j)}$ such that for all $j,k = 4,\dots,n+2$, one has:

$$\langle e^{(j)}(-\kappa), e^{(k)}(\kappa)\rangle = \delta_{j,k}. \tag{5.26}$$

In order to do this, we argue similarly as in the previous case to deduce that:

$$\overline{D_0(\lambda,\kappa)} = D_0(\bar{\lambda},-\kappa) = D_0(\bar{\lambda},\kappa).$$

In particular, we observe that $\lambda(\kappa) = \lambda(-\kappa) = \overline{\lambda(\kappa)}$, so that λ is a real and even function. We use this observation to compute:

$$\langle e^{(j)}(-\kappa), e^{(k)}(\kappa)\rangle$$
$$= \langle [\widehat{\mathbf{B}}(-\kappa e_1) + \mathbf{P} - \lambda(-\kappa)]^{-1}(C^{(j)}\cdot v\sqrt{\mu}), [\widehat{\mathbf{B}}(\kappa e_1) + \mathbf{P} - \lambda(\kappa)]^{-1}(C^{(k)}\cdot v\sqrt{\mu})\rangle$$
$$= \langle C^{(j)}\cdot v\sqrt{\mu}, [\widehat{\mathbf{B}}(\kappa e_1) + \mathbf{P} - \lambda(\kappa)]^{-2}(C^{(k)}\cdot v\sqrt{\mu})\rangle,$$

which by symmetry equals:

$$\frac{1}{n}\sum_{\ell=1}^{n} \langle C^{(j)}\cdot v\sqrt{\mu}, [\widehat{\mathbf{B}}(\kappa e_\ell) + \mathbf{P} - \lambda(\kappa)]^{-2}(C^{(k)}\cdot v\sqrt{\mu})\rangle,$$

where $e_\ell \in \mathbb{R}^n$ is the ℓ-th canonical unit vector. By symmetry arguments again:

$$\frac{1}{n}\langle \chi_1, \Big(\widehat{\mathbf{B}}(\kappa e_1) + \mathbf{P} - \lambda(\kappa)\Big)^{-2}\chi_1\rangle \sum_{\ell=1}^{n} C_\ell^{(j)}C_\ell^{(k)}.$$
$$+ (n-1)\langle \chi_2, \Big(\widehat{\mathbf{B}}(\kappa e_1) + \mathbf{P} - \lambda(\kappa)\Big)^{-2}\chi_2\rangle \sum_{\ell=1}^{n} C_\ell^{(j)}C_\ell^{(k)}.$$

By orthogonality of the $C^{(m)}$, it follows that the above expression is zero for $j \neq k$. Furthermore, since the $(e^{(j)}(\kappa))$ and $(e^{(j)}(-\kappa))$ span the same space, it follows that we can indeed normalize the vectors $e^{(j)}$ to satisfy the condition (5.26).

Part 2: Additional properties of the eigenvalues: We have established the existence and smoothness of the eigenvalues and of the eigenvectors above. Now once we have the invertibility of L away from $N(L)$–which follows from (1.17)–one can use [11, Section 4] to establish the additional properties in Proposition 5.3.

We note that $\mathbf{P}_j^{(0)}(\omega)$ from Proposition 5.3 is in $N(L)$. We furthermore notice that we can split $\mathbf{P}_j^{(1)}(\xi)$ from Proposition 5.3 as $\mathbf{P}_j^{(1)}(\xi) = \mathbf{P}_j^{(1,1)}(\xi) + \mathbf{P}_j^{(1,2)}(\xi)$ where $\mathbf{P}_j^{(1,1)}(\xi)$ maps into $N(L)$ and $\mathbf{P}_j^{(1,2)}(\xi)$ is orthogonal to $N(L)$. Again $\mathbf{P}_j(\xi) \overset{\text{def}}{=} \mathbf{P}_j^{(0)}(\omega) + \kappa\mathbf{P}_j^{(1)}(\xi)$ is the eigenprojection corresponding to the eigenvalue $\lambda_j(\xi)$ of $\widehat{\mathbf{B}}(\xi)$, where $\mathbf{P}_j^{(0)}(\xi)$, $\mathbf{P}_j^{(1)}(\xi)$, $\mathbf{P}_j^{(1,1)}(\xi)$ and $\mathbf{P}_j^{(1,2)}(\xi)$ are bounded on L^2_v, with uniformly bounded operator norm when $|\xi|$ is sufficiently small. □

The following lemma will be useful in proving the additional decay.

Lemma 5.4. *If we choose κ sufficiently small, then there exists $C > 0$ such that*

$$\sum_{j=1}^{n+2} |\mathbf{P}_j(\xi) f|^2_{L^2_v} \geq C|\mathbf{P} f|^2_{L^2_v}.$$

Proof. We notice that by construction, as in [11], we have

$$\mathbf{P}_j^{(1,1)}(\xi)\mathbf{P} f = \mathbf{P}_j^{(1,1)}(\xi) f, \tag{5.27}$$

and additionally $\mathbf{P}^1\mathbf{P}_j^{(1,1)}(\xi) = 0$ and $\mathbf{P}\mathbf{P}_j^{(1,2)}(\xi) = 0$. Hence for $C_1, C_2 > 0$:

$$\begin{aligned}
|\mathbf{P}_j(\xi) f|^2_{L^2_v} &= |\mathbf{P}^0_j(\omega) f + \kappa \mathbf{P}_j^{(1,1)}(\xi) f|^2_{L^2_v} + \kappa^2 |\mathbf{P}_j^{(1,2)}(\xi) f|^2_{L^2_v} \\
&\geq C_1 |\mathbf{P}^0_j(\omega) f|^2_{L^2_v} - C_2\kappa^2 |\mathbf{P}_j^{(1,1)}(\xi) f|^2_{L^2_v} \\
&= C_1 |\mathbf{P}^0_j(\omega) f|^2_{L^2_v} - C_2\kappa^2 |\mathbf{P}_j^{(1,1)}(\xi)\mathbf{P} f|^2_{L^2_v}.
\end{aligned}$$

The final equality follows from (5.27).

By Proposition 5.3, it follows that the $\mathbf{P}_j^{(1,1)}(\xi)$ are uniformly bounded on L^2_v for $|\xi|$ sufficiently small, hence it follows that, for some $C_3 > 0$

$$|\mathbf{P}_j(\xi) f|^2_{L^2_v} \geq C_1 |\mathbf{P}^0_j(\omega) f|^2_{L^2_v} - C_3\kappa^2 |\mathbf{P} f|^2_{L^2_v}. \tag{5.28}$$

We now sum (5.28) in $j = 1, \ldots, n+2$ to deduce that

$$\begin{aligned}
\sum_{j=1}^{n+2} |\mathbf{P}_j(\xi) f|^2_{L^2_v} &\geq C_1 \sum_{j=1}^{n+2} |\mathbf{P}^0_j(\omega) f|^2_{L^2_v} - C_3(n+1)\kappa^2 |\mathbf{P} f|^2_{L^2_v} \\
&\geq C_1 |\mathbf{P} f|^2_{L^2_v} - C_3(n+1)\kappa^2 |\mathbf{P} f|^2_{L^2_v} \geq C |\mathbf{P} f|^2_{L^2_v},
\end{aligned}$$

where we have used Proposition 5.3 and chosen $\kappa > 0$ sufficiently small. □

The next result establishes a relevant orthogonality property of $\widehat{\mathbf{B}}(\xi)$

Lemma 5.5. *Given $j = 1, \ldots, n+2$ and $\mathbf{P}_j(\xi)$, $\lambda_j(\xi)$ as in Proposition 5.3, there exists $R_j(\xi)$ such that $\widehat{\mathbf{B}}(\xi) = \lambda_j(\xi)\mathbf{P}_j(\xi) + R_j(\xi)$ and $\mathbf{P}_j(\xi) R_j(\xi)^* = 0$.*

Proof. We define $R_j(\xi) \overset{\text{def}}{=} \widehat{\mathbf{B}}(\xi) - \lambda_j(\xi)\mathbf{P}_j(\xi)$. Then since $\widehat{\mathbf{B}}(\xi)\mathbf{P}_j(\xi) = \lambda_j(\xi)\mathbf{P}_j(\xi)$ we have that $R_j(\xi)\mathbf{P}_j(\xi) = 0$ or taking adjoints $\mathbf{P}_j(\xi) R_j(\xi)^* = 0$. □

As we will see, the above result will be useful when we want to separately study the evolution of each eigencomponent in time. In the end, we will obtain decay estimates coming from the precise asymptotics of the eigenvalues given by (5.10) in Proposition 5.3. Putting all of these components together will allow us to obtain good decay estimates by using Lemma 5.4. We have

Theorem 5.6. *Suppose that the initial condition f_0 in* (5.1) *satisfies* (3.28) *and that we are in the hard potential case* (1.8)*. Let m, $\varrho \in \mathbb{R}$ with $m+\varrho > 0$, $2 \le p \le \infty$ and $\ell \ge 0$. Then smooth solutions to* (5.1) *satisfy the large time decay estimate*

$$\|w^\ell \mathbb{A}(t) f_0\|_{\dot{B}^{m,p}_2 L^2_v} \lesssim \|w^\ell f_0\|_{\dot{B}^{m,p}_2 L^2_v \cap \dot{B}^{-\varrho,\infty}_2 L^2_v} (1+t)^{-(m+\varrho+1)/2}.$$

This holds uniformly in $t \ge 0$ if $\|w^\ell f_0\|_{\dot{B}^{m,p}_2 L^2_v \cap \dot{B}^{-\varrho,\infty}_2 L^2_v} < \infty$.

Notice that Theorem 5.6 is more general than and directly implies Proposition 3.3 from Section 3. This follows from the definitions in Section 4 if we take $p = 2$.

Proof. We will prove Theorem 5.6 in three steps. In the first step, we study the case when $|\xi| \ge \kappa_0$ for any small $\kappa_0 > 0$. In the second step we use the eigenvalue decomposition of $\widehat{\mathbf{B}}$ from (5.1) on $|\xi| \le \kappa_0$ for a small $\kappa_0 > 0$ as in Proposition 5.3 to obtain the decay of the macroscopic part of the solution. Then in the last step we use an estimate from [25] to prove the decay of the microscopic part.

We recall the method used in the proof of Theorem 5.2. To begin, we notice that from (5.5) we have that

$$\|w^\ell (\Delta_k f)(t)\|_{L^2_x L^2_v} \lesssim e^{-\lambda(1\wedge 2^k)t} \|w^\ell \Delta_k f_0\|_{L^2_x L^2_v}. \tag{5.29}$$

Given $\kappa_0 > 0$ small choose $M > 0$ such that $2^{2k} \ge \kappa_0$ whenever $k \ge -M$. Define

$$c_k \stackrel{\text{def}}{=} \|w^\ell (\Delta_k f)(t)\|_{L^2_x L^2_v}, \quad \|c_k\|_{\ell^p_{k\ge -M}} \stackrel{\text{def}}{=} \left(\sum_{k \ge -M} |c_k|^p \right)^{1/p}, \quad 1 \le p < \infty,$$

with an obvious modification when $p = \infty$. Then from (5.29) and the above we deduce that

$$\|2^{mk} c_k\|_{\ell^p_{k\ge -M}} \lesssim (1+t)^{-j/2} \|w^\ell f_0\|_{\dot{B}^{m,p}_2 L^2_v}. \tag{5.30}$$

This will hold for any $j > 0$.

For the second step, let us consider the Fourier transform of the problem (5.1):

$$\partial_t \widehat{f} + \widehat{\mathbf{B}} \widehat{f} = 0, \quad \widehat{f}(0,x,v) = \widehat{f}_0(x,v). \tag{5.31}$$

Here we recall (5.9). We apply the Littlewood-Paley Projection Δ_k to obtain:

$$\partial_t \widehat{\Delta_k f} + \widehat{\mathbf{B}} \widehat{\Delta_k f} = 0, \quad \widehat{\Delta_k f}(0,x,v) = \widehat{\Delta_k f_0}(x,v). \tag{5.32}$$

We will further take the complex conjugate of (5.32), use that

$$\overline{\widehat{\mathbf{B}}} = \widehat{\mathbf{B}}^* = \overline{\lambda_j(\xi)} \mathbf{P}_j(\xi) + R_j(\xi)^*,$$

and use Lemma 5.5 when we apply $\mathbf{P}_j(\xi)$ to the result to deduce

$$\partial_t \mathbf{P}_j(\xi) \overline{\widehat{\Delta_k f}(\xi)} + \overline{\lambda_j(\xi)} \mathbf{P}_j(\xi) \overline{\widehat{\Delta_k f}(\xi)} = 0.$$

In particular, it follows that

$$\mathbf{P}_j(\xi) \overline{\widehat{\Delta_k f}(t,\xi,v)} = e^{-\overline{\lambda_j(\xi)} t} \mathbf{P}_j(\xi) \overline{\widehat{\Delta_k f_0}(\xi,v)}.$$

Notice that the assumption (3.28) and its analogoue, $\mathbf{P}\overline{\widehat{f_0}} = 0$, both hold when we take the Fourier transform in x and apply the Littlewood-Paley projection Δ_k. Moreover, we use (5.11) from Proposition 5.3 to deduce that for $|\xi|$ sufficiently small

$$\mathbf{P}_j(\xi)\overline{\widehat{\Delta_k f}(t,\xi,v)} = |\xi| e^{-\overline{\lambda_j(\xi)}t}\mathbf{P}_j^{(1)}(\xi)\overline{\widehat{\Delta_k f_0}(\xi,v)}.$$

We note that, in this way, we have gained an extra factor of $|\xi|$. Consequently, since by Proposition 5.3, $\mathbf{P}_j^{(1)}(\xi)$ is bounded on L^2_v, it follows that

$$\int dv\ |\mathbf{P}_j(\xi)\overline{\widehat{\Delta_k f}(t,\xi,v)}|^2 \lesssim e^{-2\mathrm{Re}(\lambda_j(\xi))t}|\xi|^2 \int dv\ |\widehat{\Delta_k f_0}(\xi,v)|^2.$$

Notice that $\mathrm{Re}\,(\lambda_j(\xi)) = \mathrm{Re}\left(\overline{\lambda_j(\xi)}\right)$. By (5.10) of Proposition 5.3 the above is

$$\lesssim e^{-\lambda_j^{(2)}|\xi|^2 t}|\xi|^2 \int dv\ |\widehat{\Delta_k f_0}(\xi,v)|^2.$$

Here, the $\lambda_j^{(2)} > 0$ no longer depend on ξ. We also used that $|\xi|$ is sufficiently small. Then for all $j = 1,\dots,n+2$ and $\sigma > 0$, and for all $|\xi|$ sufficiently small

$$|\mathbf{P}_j(\xi)\overline{\widehat{\Delta_k f}(t,\xi)}|^2_{L^2_v} \leq \frac{C}{t}(1+|\xi|^2 t)^{-\sigma}|\widehat{\Delta_k f_0}(\xi)|^2_{L^2_v}.$$

We sum the above inequality in $j = 1,\dots,n+2$, we note that the projection $\mathbf{P}$ is real, and we use Lemma 5.4 to deduce that, for $|\xi|$ sufficiently small

$$|\mathbf{P}\widehat{\Delta_k f}(t,\xi)|^2_{L^2_v} \leq \frac{C}{t}(1+|\xi|^2 t)^{-\sigma}|\widehat{\Delta_k f_0}(\xi,v)|^2_{L^2_v}.$$

Consequently, by Plancherel's Theorem, it follows that for k sufficiently negative, i.e. for $k < -M$ for some $M > 0$ large

$$\|\Delta_k \mathbf{P} f(t)\|_{L^2_x L^2_v} \lesssim \frac{1}{\sqrt{t}}(1+t2^{2k})^{-\frac{\sigma}{2}}\|\Delta_k f_0\|_{L^2_x L^2_v}. \tag{5.33}$$

Now, analogously to the proof of Theorem 5.1, using the estimate (5.30) when $k \geq -M$, and $\ell = 0$ we deduce for $1 \leq p \leq \infty$ the following uniform in $t \geq 0$ inequality

$$\|\mathbf{P} f(t)\|_{\dot{B}^{m,p}_2 L^2_v} \lesssim (1+t)^{-\frac{\varrho+m+1}{2}}\|f_0\|_{\dot{B}^{m,p}_2 L^2_v \cap \dot{B}^{-\varrho,\infty}_2 L^2_v}. \tag{5.34}$$

This gives us the bound on the macroscopic term.

We now estimate the microscopic term. Let us recall that we are working in the hard potential case (1.8), and hence by the discussion after [25, Equation (2.25)]

$$\begin{aligned} |\{\mathbf{I}-\mathbf{P}\}\widehat{f}(t,\xi)|^2_{L^2_v} \lesssim e^{-\lambda t}&|\{\mathbf{I}-\mathbf{P}\}\widehat{f}_0(\xi)|^2_{L^2_v} \\ &+ \int_0^t ds\, e^{-\lambda(t-s)}|\xi|^2|\mathbf{P}\widehat{f}(s,\xi)|^2_{L^2_v}, \end{aligned} \tag{5.35}$$

Let us now consider the second term in the upper bound of (5.35). We multiply this term by the Littlewood-Paley projection $\varphi_k^2(\xi)$ (from Section 4.2), also multiply it by $|\xi|^{2m}$, integrate over $\xi \in \mathbb{R}^n_\xi$ and then sum the $\ell^p_{k\leq -M}$ norm for $2 \leq p \leq \infty$. We use the $\ell^p_{k\leq -M}$ to denote the standard ℓ^p_k norm, as defined in Section 1.2, except that $\ell^p_{k\leq -M}$ is only summed over frequencies $k \leq -M$ for some large $M > 1$. In the first step of this procedure, similar to (5.33), we have, for all $\varrho > 0$

$$\left(2^{2(m+1)k}\|\Delta_k \mathbf{P} f(s)\|^2_{L^2_x L^2_v}\right) \lesssim s^{-1-(m+1)-\varrho}\left(s2^{2k}\right)^{m+1+\varrho}(1+s2^{2k})^{-\sigma}\|f_0\|^2_{\dot{B}^{-\varrho,\infty}_2 L^2_v}.$$

Then similarly to (5.8) we have that

$$\left\| \left(2^{(m+1)k} \|\Delta_k \mathbf{P} f(s)\|_{L^2_x L^2_v} \right) \right\|^2_{\ell^p_{k\leq -M}} \lesssim (1+s)^{-1-(m+1)-\varrho} \|f_0\|^2_{\dot{B}^{-\varrho,\infty}_2 L^2_v} \tag{5.36}$$

Here we use (WLOG) that $s \geq 1$.

Now we plug the estimate (5.36) into (5.35) to obtain

$$\begin{aligned} \left\| \left(2^{mk} \|\{\mathbf{I}-\mathbf{P}\}\Delta_k \widehat{f}(t)\|_{L^2_v L^2_x} \right) \right\|^2_{\ell^p_{k\leq -M}} &\lesssim e^{-\lambda t} \|\{\mathbf{I}-\mathbf{P}\} f_0\|^2_{\dot{B}^{m,p}_2 L^2_v} \\ &\quad + \|f_0\|^2_{\dot{B}^{-\varrho,\infty}_2 L^2_v} \int_0^t ds\, e^{-\lambda(t-s)} (1+s)^{-1-(m+1)-\varrho} \end{aligned}$$

Here we have supposed that $p \geq 2$ and we initially took the $\ell^{p/2}_{k\leq -M}$ norm of the dyadic estimates and used Minkowski's inequality. Finally, we use [29, Lemma A.1] to estimate the time integral and deduce that

$$\begin{aligned} \left\| \left(2^{mk} \|\{\mathbf{I}-\mathbf{P}\}\Delta_k \widehat{f}(t)\|_{L^2_v L^2_x} \right) \right\|^2_{\ell^p_{k\leq -M}} &\lesssim e^{-\lambda t} \|\{\mathbf{I}-\mathbf{P}\} f_0\|^2_{\dot{B}^{m,p}_2 L^2_v} \\ &\quad + \|f_0\|^2_{\dot{B}^{-\varrho,\infty}_2 L^2_v} (1+t)^{-m-\varrho-2}. \end{aligned} \tag{5.37}$$

Collecting (5.37) with (5.34) and (5.30) yields Theorem 5.6 when $\ell = 0$

Lastly, when $\ell > 0$, we recall the estimate [25, Equation (2.9)]:

$$\begin{aligned} \frac{d}{dt} |w^\ell \{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2} + \lambda |w^\ell \{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_{\gamma+2s}} \\ \leq C_\lambda |\xi|^2 |w^{-\sigma} \hat{f}(t,\xi)|^2_{L^2_v} + C |\{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2(B_C)}. \end{aligned} \tag{5.38}$$

This estimate will hold for any large $\sigma > 0$. Here $L^2(B_C)$ denotes the L^2_v norm on B_C, the ball of radius $C > 0$ centered at the origin. This estimate (5.38) is actually slightly stronger that what is stated in [25, Equation (2.9)], however following the proof of [25, Equation (2.9)] then (5.38) can be directly deduced.

From [25, Equation (2.11)] we also have

$$\frac{d}{dt} |\{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_v} + \lambda |\{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_{\gamma+2s}} \leq C_\lambda |\xi|^2 |\mathbf{P} \hat{f}(t,\xi)|^2_{L^2_v}. \tag{5.39}$$

Taking a suitable linear combination of (5.38) and (5.39) we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\delta |w^\ell \{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_v} + |\{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_v} \right) \\ &+ \lambda \left(\delta |w^\ell \{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_{\gamma+2s}} + |\{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_{\gamma+2s}} \right) \leq C |\xi|^2 |w^{-\sigma} \hat{f}(t,\xi)|^2_{L^2_v}, \end{aligned}$$

for some suitably small $\delta > 0$, since $|\mathbf{P} \hat{f}(t,\xi)|_{L^2_v} \lesssim |w^{-\sigma} \hat{f}(t,\xi)|_{L^2_v}$. We use the fact that $\gamma + 2s \geq 0$ to deduce

$$\begin{aligned} &\frac{d}{dt} \left(\delta |w^\ell \{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_v} + |\{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_v} \right) \\ &+ \lambda \left(\delta |w^\ell \{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_v} + |\{\mathbf{I}-\mathbf{P}\} \hat{f}(t,\xi)|^2_{L^2_v} \right) \leq C |\xi|^2 |w^{-\sigma} \hat{f}(t,\xi)|^2_{L^2_v}, \end{aligned} \tag{5.40}$$

Now we use the integrating factor $e^{-\lambda t}$. Then, as in (5.35), it follows from (5.40), when $\ell \geq 0$, that

$$|w^\ell \{\mathbf{I}-\mathbf{P}\}\widehat{f}(t,\xi)|^2_{L^2_v} \lesssim e^{-\lambda t}|w^\ell \{\mathbf{I}-\mathbf{P}\}\widehat{f}_0(\xi)|^2_{L^2_v} + \int_0^t ds\, e^{-\lambda(t-s)}|\xi|^2 |w^{-\sigma}\widehat{f}(s,\xi)|^2_{L^2_v},$$

which holds for any $\sigma > 0$. In order to obtain the above inequality, we also used the fact that $\delta|w^\ell\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,\xi)|^2_{L^2_v} + |\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,\xi)|^2_{L^2_v} \approx |w^\ell\{\mathbf{I}-\mathbf{P}\}\hat{f}(t,\xi)|^2_{L^2_v}$. From here we follow the argument from (5.35) to (5.37) and the explanations below it to obtain Theorem 5.6 when $\ell \geq 0$. □

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