# THE GROSS-PITAEVSKII HIERARCHY ON GENERAL RECTANGULAR TORI

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ABSTRACT. In this work, we study the Gross-Pitaevskii hierarchy on general –rational and irrational– rectangular tori of dimension two and three. This is a system of infinitely many linear partial differential equations which arises in the rigorous derivation of the nonlinear Schrödinger equation. We prove a conditional uniqueness result for the hierarchy. In two dimensions, this result allows us to obtain a rigorous derivation of the defocusing cubic nonlinear Schrödinger equation from the dynamics of many-body quantum systems. On irrational tori, this question was posed as an open problem in previous work of Kirkpatrick, Schlein, and Staffilani.

## 1. INTRODUCTION

1.1. **Setup of the problem.** Let $d \geq 2$ be fixed. Suppose that $\theta_1, \theta_2, \ldots, \theta_d > 0$ are given parameters. We consider the following domain:

$$\Lambda_d = \Lambda_d (\theta_1, \theta_2, \ldots, \theta_d) := (\mathbb{R} / \frac{2\pi}{\theta_1} \mathbb{Z}) \times (\mathbb{R} / \frac{2\pi}{\theta_2} \mathbb{Z}) \times \cdots \times (\mathbb{R} / \frac{2\pi}{\theta_d} \mathbb{Z}).$$

For the purpose of this paper we call $\Lambda_d$ a *general (rectangular d-dimensional) torus*.

In the context of nonlinear dispersive equations, general rectangular tori were first studied in the work of Bourgain [10], where it was noted that the number-theoretic methods employed in the case of the classical $d$-dimensional torus $\mathbb{T}^d = (\mathbb{R} / 2\pi\mathbb{Z})^d$, i.e. $\theta_1 = \theta_2 = \cdots = \theta_d = 1$, cannot be used in general. As a result, proving dispersive estimates is more challenging on general tori.

In this work, we will consider the *Gross-Pitaevskii hierarchy on the spatial domain* $\Lambda_d$. We recall that this is a system of infinitely many equations, which is given by:

$$\begin{cases} i\partial_t \gamma^{(k)} + (\Delta_{\vec{x}_k} - \Delta_{\vec{x}'_k})\gamma^{(k)} = b_0 \cdot \sum_{j=1}^k B_{j,k+1}(\gamma^{(k+1)}) \\ \gamma^{(k)}|_{t=0} = \gamma_0^{(k)}. \end{cases} \tag{1}$$

We use similar notation as in [45, 72]: For fixed $k \in \mathbb{N}$, $\gamma_0^{(k)}$ is a complex-valued function on $\Lambda_d^k \times \Lambda_d^k$. Such a function is in general called *a density matrix of order $k$ on $\Lambda_d$*. Furthermore, each $\gamma^{(k)} = \gamma^{(k)}(t)$ is a time-dependent density matrix of order $k$ on $\Lambda_d$. If we denote by $(\vec{x}_k; \vec{x}'_k) = (x_1, \ldots, x_k; x'_1, \ldots, x'_k)$ the spatial variable of $\Lambda_d^k \times \Lambda_d^k$, then $\Delta_{\vec{x}_k} := \sum_{j=1}^k \Delta_{x_j}$ is the Laplace operator in the first component of $\Lambda_d^k$ and $\Delta_{\vec{x}'_k} := \sum_{j=1}^k \Delta_{x'_j}$ is the Laplace operator in the second component of

$\Lambda_d^k$. Given $j \in \{1, 2, \ldots, k\}$, $B_{j,k+1}$ denotes the *collision operator*, which is defined as $B_{j,k+1}(\sigma^{(k+1)}) := Tr_{x_{k+1}}\big[\delta(x_j - x_{k+1}), \sigma^{(k+1)}\big]$, whenever $\sigma^{(k+1)}$ is a density matrix of order $k + 1$ on $\Lambda_d$. Here, $\delta$ denotes the Dirac delta function and $Tr_{x_{k+1}}$ denotes the trace in the $x_{k+1}$ variable, which we sometimes also denote as $Tr_{k+1}$, for simplicity of notation. A more detailed definition of the collision operator is given in (10) below. Finally, $b_0 \in \mathbb{R}$ is a non-zero coupling constant. If $b_0 > 0$, we say that the problem (1) is *defocusing*, and if $b_0 < 0$, we say that it is *focusing*.

The problem (1) is closely linked to the *cubic nonlinear Schrödinger equation (NLS)* on $\Lambda_d$:

$$\begin{cases} i\partial_t \phi_t + \Delta\phi_t = b_0 \cdot |\phi_t|^2 \phi_t \\ \phi_t\big|_{t=0} = \phi \end{cases} \tag{2}$$

The coupling constant $b_0$ is the same as in (1). From a solution $\phi_t$ to (2) we can construct a solution of (1) with initial data $\gamma_0^{(k)} = |\phi\rangle\langle\phi|^{\otimes k}$. This is the solution given by:

$$\gamma^{(k)}(t) := |\phi_t\rangle\langle\phi_t|^{\otimes k}.$$

Here, $|\cdot\rangle\langle\cdot|$ denotes the Dirac bracket, which is given by $|h\rangle\langle g|(x; x') := h(x) \cdot \overline{g(x')}$. The sequence $(\gamma^{(k)}(t))_k$ is then called a *factorized solution of* (1).

The factorized solutions of (1) play a key role in the rigorous derivation of the NLS from the dynamics of many-body quantum systems. More precisely, let us first start from a real-valued potential $V$, which is defined on $\Lambda_d$. Given $N \in \mathbb{N}$, we build the corresponding $N$-body Hamiltonian $H_N$ on a dense subspace of $L^2_{sym}(\Lambda_d^N)$, which is the space of all permutation-symmetric elements of $L^2(\Lambda^N)$. The operator $H_N$ is given by:

$$H_N := -\sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N}\sum_{\ell<j}^{N} V_N(x_\ell - x_j).$$

Here $V_N(x) := N^{3\beta}V(N^\beta x)$, for $\beta > 0$ a parameter. Given initial data $\Psi_{N,0}$, we can solve the $N$-body Schrödinger equation associated to $H_N$:

$$\begin{cases} i\partial_t \Psi_{N,t} = H_N \Psi_{N,t} \\ \Psi_{N,t}\big|_{t=0} = \Psi_{N,0}. \end{cases}$$

From the solution, we can define:

$$\gamma_{N,t}^{(k)} := Tr_{k+1,\ldots,N}\big|\Psi_{N,t}\rangle\langle\Psi_{N,t}\big|. \tag{3}$$

Here, $Tr_{k+1,\ldots,N}$ denotes the partial trace in the $x_{k+1},\ldots,x_N$ variables. By definition, for $k > N$, one takes $\gamma_{N,t}^{(k)} := 0$.

The constructed sequence $(\gamma_{N,t}^{(k)})_k$ solves the *Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy* on $\Lambda_d$:

$$i\partial_t \gamma_{N,t}^{(k)} + \big(\Delta_{\vec{x}_k} - \Delta_{\vec{x}_k'}\big)\gamma_{N,t}^{(k)}$$
$$= \frac{1}{N}\sum_{\ell<j}^{k}\big[V_N(x_\ell - x_j), \gamma_{N,t}^{(k)}\big] + \frac{N-k}{N}\sum_{j=1}^{k} Tr_{k+1}\big[V_N(x_j - x_{k+1}), \gamma_{N,t}^{(k+1)}\big].$$

Formally speaking, the BBGKY hierarchy converges to the GP hierarchy with $b_0 = \int_{\Lambda_d} V(x)\,dx$ as $N \to \infty$. In order to make this formal argument rigorous, one wants to choose $(\Psi_N)_N$ in an appropriate way in terms of $\phi$ and show that

there exists a sequence $N_j \to \infty$, which does not depend on $k \in \mathbb{N}$ and $t$ with the property that:

$$Tr \left| \gamma_{N_j,t}^{(k)} - |\phi_t\rangle\langle\phi_t|^{\otimes k} \right| \to 0 \tag{4}$$

as $j \to \infty$, for $t$ belonging to a finite time interval. Here, $Tr| \cdot |$ denotes the trace norm. We refer to (4) as *a rigorous derivation of the cubic NLS on $\Lambda_d$ from the dynamics of many-body quantum systems.*

1.2. **Statement of the results.** In our paper, we will primarily study the case $d = 2$ and $d = 3$, i.e. the setting of two- and three-dimensional general rectangular tori. The following class of time-dependent density matrices is defined on $\Lambda_d$:

**Definition 1.1.** *Given $\alpha \in \mathbb{R}$, let $\widetilde{\mathcal{A}} = \widetilde{A}(\alpha)$ denote the class of all time-dependent sequences $\widetilde{\Gamma}(t) = (\widetilde{\gamma}^{(k)}(t))$, where each $\widetilde{\gamma}^{(k)} : \mathbb{R}_t \times \Lambda_d^k \times \Lambda_d^k \to \mathbb{C}$ satisfies:*

i) *For all $t \in \mathbb{R}, x_1, \ldots, x_k, x_1', \ldots, x_k' \in \Lambda_d$, and for all $\sigma \in S^k$:*

$$\widetilde{\gamma}^{(k)}(t, x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x_{\sigma(1)}', \ldots, x_{\sigma(k)}') = \widetilde{\gamma}^{(k)}(t, x_1, \ldots, x_k; x_1', \ldots, x_k').$$

ii) *There exist positive and continuous functions $\widetilde{f}, \widetilde{g} : \mathbb{R} \to \mathbb{R}$, which are independent of $k$, such that for all $t \in \mathbb{R}$ and for all $j \in \{1, 2, \ldots, k\}$:*

$$\int_{t-\widetilde{g}(t)}^{t+\widetilde{g}(t)} \|S^{(k,\alpha)} B_{j,k+1}(\widetilde{\gamma}^{(k+1)})(s)\|_{L^2(\Lambda_d^k \times \Lambda_d^k)} ds \leq \widetilde{f}^{k+1}(t).$$

Here $S^{(k,\alpha)}$ denotes the operator of fractional differentiation of order $\alpha$ on density matrices of order $k$, as is defined in (13) below. The class $\widetilde{\mathcal{A}}$ corresponds to the a priori bound needed in the analysis of [56]. The time dependence of the parameters $\widetilde{f}(t)$ and $\widetilde{g}(t)$ was subsequently introduced in [45].

In the two-dimensional case, we will prove the following conditional uniqueness result:

**Theorem 1.** *Let $\alpha > \frac{1}{2}$ be given. Then, solutions to the Gross-Pitaevskii hierarchy on $\Lambda_2$ are unique in the class $\widetilde{\mathcal{A}}(\alpha)$.*

Theorem 1 resolves an open problem for the irrational torus stated in [55]. In particular, we can implement the result obtained in Theorem 1 into the derivation strategy and combine the analysis of [55] adapted to the case of general two-dimensional tori to deduce:

**Theorem 2.** *The convergence in (4) holds on $\Lambda_2$ for $V \in W^{2,\infty}(\Lambda_2)$ with $V \geq 0$, $\int_{\Lambda_2} V(x)\,dx = b_0 > 0$, for $\beta \in (0, \frac{3}{4})$, and for $(\psi_N)_N \in \bigoplus_N L^2(\Lambda_2^N)$ satisfying the assumptions of:*

i) *Bounded energy per particle:*

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle < \infty. \tag{5}$$

ii) *Asymptotic factorization, i.e. there exists $\phi \in H^1(\Lambda_2)$ with $\|\phi\|_{L^2(\Lambda_2)} = 1$ such that:*

$$Tr|\gamma_N^{(1)} - |\phi\rangle\langle\phi|^{\otimes k}| \to 0 \ \ as \ N \to \infty. \tag{6}$$

*The function $\phi$ is taken as the initial data in (2).*

Theorem 1 is stated as Theorem 3.11 and Theorem 2 is stated as Theorem 3.12 below.

In the three-dimensional setting, we can prove the following conditional uniqueness result:

**Theorem 3.** *Let $\alpha > 1$ be given. Then, solutions to the Gross-Pitaevskii hierarchy on $\Lambda_3$ are unique in the class $\widetilde{\mathcal{A}}(\alpha)$. Moreover, whenever $\alpha \geq 1$, the class $\widetilde{\mathcal{A}}(\alpha)$ is non-empty and it contains the factorized solutions corresponding to initial data in $H^\alpha(\Lambda_3)$.*

As in [45], the uniqueness in Theorem 3 is above the regularity of the energy space and hence is not enough to obtain a rigorous derivation result. However, it is possible to use the multilinear estimates on the irrational torus [75] –see also the recent preprint [54]– and adapt the arguments in [72] to $\Lambda_3$ to obtain the following unconditional uniqueness result:

**Theorem 4.** *Suppose that $\Gamma(t) = (\gamma^{(k)}(t))_k \in L^\infty_{t\in[0,T]}\mathfrak{H}^1$ is a mild solution to the Gross-Pitaevskii hierarchy on $\Lambda_3$ such that each component $\gamma^{(k)}(t)$ can be written as a limit in the weak-$*$ topology of the trace class on $L^2_{sym}(\Lambda_3^k \times \Lambda_3^k)$ of $Tr_{k+1,\dots,N}\Gamma_{N,t}$. Here, each $\Gamma_{N,t} \in L^2_{sum}(\Lambda_3^N \times \Lambda_3^N)$ is non-negative as an operator and it has trace equal to 1. Then $\Gamma(t)$ is uniquely determined by the initial data $\Gamma(0)$.*

Theorem 4 can hence be used to deduce the following derivation result:

**Theorem 5.** *Let $V : \mathbb{R}^3 \to \mathbb{R}$ be non-negative, smooth, compactly supported and suppose that $b_0 = \int_{\mathbb{R}^3} V(x)\, dx = b_0 > 0$. Moreover, let $\beta \in (0, \frac{3}{5})$, and let $(\psi_N)_N \in \bigoplus_N L^2(\Lambda_3^N)$ satisfy the assumptions (5) and (6).*

Then, the convergence (4) holds. Theorem 3, Theorem 4, and Theorem 5 are stated respectively as Theorem 4.4, Theorem 4.7, and Theorem 4.8 below. A key idea in the proof of Theorem 1 and Theorem 3 is to prove certain spacetime estimates. The spacetime estimate in two dimensions is given in Proposition 3.1 and sharpness is shown in Proposition 3.2. In three dimensions, the spacetime estimate is given in Proposition 4.1 and sharpness is shown in Proposition 4.2. The sharp spacetime estimate in the general case of $d$ dimensions is noted in Remark 4.3. Once we have a good spacetime estimate, it is possible to apply the analysis in the work of T. Chen and Pavlović [21] to obtain local-in-time solutions to the hierarchy. The details of this construction are recalled in Section 5.

1.3. **Physical interpretation.** Finally, we briefly recall the physical interpretation, similarly to [45, Subsection 1.3.2]: The Gross-Pitaevskii hierarchy and the nonlinear Schrödinger equation occur in the framework of Bose-Einstein condensation. This is a state of matter which is made up of bosonic particles which are cooled to a temperature close to absolute zero. In such an environment, they tend to occupy the lowest quantum state. This state corresponds to a ground state of an energy functional associated to an NLS-type equation. In such a context, the NLS equation is sometimes refered to as the *Gross-Pitaevskii equation*, after the work of Gross [46] and Pitaevskii [68]. The phenomenon of Bose-Einstein condensation was theoretically predicted in 1924-1925 in the work of Bose [8] and Einstein [33]. The existence of this state of matter was experimentally verified independently by the research teams of Cornell and Wieman [3] and Ketterle [31] in 1995. Both experimental groups were awarded the Nobel Prize in Physics in 2001 for their achievement.

1.4. **Previous results.** In this subsection, we will briefly discuss further related results and provide references, which in part is similar to the exposition in [45, 72].

Properties of the ground state for the $N$-body Hamiltonian $H_N$ have been studied by Lieb and Seiringer [63], Lieb, Seiringer and Yngvason [66, 67], and Lieb, Seiringer, Yngvason, and Solovej [64]. In these works, the assumption of *asymptotic factorization* as in (6) of the initial data was rigorously verified for a sequence consisting of appropriate ground states. These results are summarized in the expository work [65].

A method for proving (4) by means of the BBGKY hierarchy was first developed by Spohn [74] when the spatial domain is $\mathbb{R}^d$. Using this approach, the result obtained in [74] is a rigorous derivation of the Hartree equation $iu_t + \Delta u = (V * |u|^2)u$ on $\mathbb{R}^d$ with $V \in L^\infty(\mathbb{R}^d)$. Due to the regularity assumptions on the convolution potential, in this case it is possible to explicitly compute the infinite Duhamel expansions $\gamma_{\infty,t}^{(k)}$ and show that it is the appropriate limit of $(\gamma_{N,t}^{(k)})_N$. This result could be extended to the situation of more singular $V$ in [4, 5, 40] by adapting the following two-step derivation strategy. In the first step, it is shown that the sequence $(\gamma_{N,t}^{(k)})_N$ satisfies certain compactness properties, and that the obtained subsequential limits solve the GP hierarchy. In the second step, it is shown that solutions to the GP hierarchy are unique in a class of objects containing this limit. Due to the complexity of the system, the uniqueness step is significantly non-trivial.

A breakthrough in that direction was the rigorous derivation of the defocusing cubic NLS on $\mathbb{R}^3$ in the work of Erdős, Schlein, and Yau [36, 37, 38, 39]. Here, the uniqueness step was obtained by the use of Feynman graph expansions. A combinatorial reformulation of this step under an additional a priori assumption on the solution was subsequently given by Klainerman and Machedon [56].

An alternative rigorous derivation of NLS-type equations based on Fock space techniques was developed by Hepp [50] and Ginibre and Velo [43, 44].

The first derivation result in the periodic problem was given in the case of $\mathbb{T}^2$ in the work of Kirkpatrick, Schlein, and Staffilani [55]. We note that, in the case of $\mathbb{T}^3$, the first step in the derivation strategy was carried out in the work of Elgart, Erdős, Schlein, and Yau [34], which builds on the work previously done in [35]. A conditional uniqueness result for the GP hierarchy on $\mathbb{T}^3$ was shown in a class of density matrices of regularity $\alpha > 1$ by Gressman, the second author, and Staffilani in [45]. This was done by the use of a spacetime estimate as in [55, 56]. In [45], it was shown that the obtained range of regularity exponents in this estimate was sharp. Since the regularity is above the natural energy space, which corresponds to the regularity $\alpha = 1$, it is not immediately possible to apply this result in the second step of the derivation.

In a recent paper, T. Chen, Hainzl, Pavlović, and Seiringer [16] give an alternative proof of the uniqueness result on $\mathbb{R}^3$ by use of the *(Weak) Quantum de Finetti Theorem*, formulated in [1, 2, 59]. This approach was subsequently adapted to the setting of $\mathbb{T}^3$ by the second author in [72], in which the open question of uniqueness from [34] was resolved. As a result, one could obtain a rigorous derivation of the defocusing cubic NLS on $\mathbb{T}^3$. We note that the uniqueness result of [72] does not directly extend the uniqueness result of [45] since the papers deal with different classes of density matrices. Methods based on the Quantum de Finetti Theorem were applied to related problems in [17, 30, 52].

Moreover, once one has a rigorous derivation result, based on either method, it is possible to study the rate of convergence in (4). This problem was first addressed by Rodnianski and Schlein [69], and subsequently reformulated in [58]. The Cauchy problem associated to the GP hierarchy was studied in its own right in the work of T. Chen and Pavlović [18], with later work in [20, 21, 22, 24, 25, 61, 62]. Randomization techniques were studied in the context of the Cauchy problem associated to the GP hierarchy in [71, 73]. In a recent preprint [60], a derivation of the nonlinear Gibbs measure from many-body dynamics is given. A related result in a discrete setting had also been proved in [57]. The Klainerman-Machedon a priori bound was studied in further detail in [23, 26, 28, 29]. Generalizations of the spacetime estimate for the GP hierarchy were studied in a different context in [6, 7], with related work in [27]. The case of singular convolution potentials was revisited in [41, 42]. For a more detailed discussion on all of these results and further references, we refer the reader to [45, Subsection 1.3.2] and [72, Section 1.1 and Section 1.3], as well as to the expository works [19, 70].

As was noted above, the study of the NLS on general rectangular tori was first started in the work of Bourgain [10]. Here, it was shown that certain Strichartz estimates with a loss of derivative hold. The estimates were much weaker than those proved in the setting of the classical torus [9], due to number-theoretical difficulties. The NLS on irrational tori has been studied further in [15, 32, 47, 75]. A different approach to the problem, which has led to stronger results, including the stronger Strichartz estimates conjectured in [10], was recently taken in the work of Bourgain and Demeter [11, 12, 13], with subsequent work by Killip and Vişan [54].

1.5. **Main ideas of the proof.** We choose the setup of [10, 47, 75]. To this end, we will rescale the domain $\Lambda_d$ to the classical torus $\Lambda$. Consequently, we will be able to work with Fourier series defined on the standard lattice $\mathbb{Z}^d$. Due to the scaling transformation, we will have to work with the modified Laplacian $\Delta_Q$, as well as with the associated quadratic form $Q$. In the rescaled setting, we will consider a Gross-Pitaevskii hierarchy with modified Laplacian on $\Lambda$ as in (22).

A crucial ingredient in proving conditional uniqueness for this hierarchy on $\Lambda$ is a spacetime estimate associated to $\mathcal{U}_Q^{(k)}(t)$, the free Schrödinger evolution on density matrices of order $k$ on $\Lambda$ corresponding to $\Delta_Q$. The operator $\mathcal{U}_Q^{(k)}(t)$ is precisely defined in (16). This estimate is stated in two dimensions in Proposition 3.1 and in three dimensions in Proposition 4.1. By standard techniques, the spacetime bound is reduced to a pointwise estimate on a corresponding multiplier $I(\tau, p)$ as in [45].

A challenge in proving the pointwise estimate lies in the fact that the sum in the multiplier is taken over a larger set than in the classical setting. In particular, the sum does not contain a $\delta$-function, but it contains a characteristic function of the interval $[0, 1]$. Geometrically speaking, we are no longer summing over lattice points that lie on a curve, but over lattice points that lie near a curve. This is a general phenomenon which occurs, due to the fact that $e^{it\Delta_Q}$ is in general no longer periodic in time. To this sum, we apply the dyadic decomposition arguments from [45]. Due to the presence of the quadratic form $Q$, associated to $\Delta_Q$, which can take irrational values, the arguments based on the determinant of a lattice from [45] do not apply in this setting. We will remedy this difficulty by using a Fourier analytic fact used in [10], which allows us to estimate the number of points in a set in terms of an integral (28). We then rewrite the integral in the upper bound in a factorized way

(35), keeping in mind all of the dyadic localization of the frequencies. In certain cases, this allows us to apply the oscillatory sum estimates given in Subsection 2.2 below. The remaining cases are then obtained from this one by applying the geometric arguments from [45].

In proving the sharpness of the Sobolev exponents in the spacetime estimates, we again encounter the difficulty that $e^{it\Delta_Q}$ is no longer periodic in time. More precisely, we need to estimate the integral in $\tau$, the frequency variable corresponding to $t$. In other words, we need to estimate the behavior of the spacetime Fourier transform of the expression left-hand side of the estimate for $\tau$ taking values in an interval, rather than at a single point. In order to prove this fact, we observe that the spacetime Fourier transform can bounded from below by an integral of a constant function. In the two-dimensional problem, this is seen in (41) below. This idea generalizes to all dimensions $d \geq 2$.

Once one proves the spacetime estimate, it is possible to use standard arguments to deduce a conditional uniqueness result for the Gross-Pitaevskii hierarchy with a modified Laplacian on $\Lambda$. An additional scaling argument then allows us to show conditional uniqueness for the Gross-Pitaevskii hierarchy on $\Lambda_d$. We note that, by this method, it is not immediately possible to deduce conditional uniqueness for the Gross-Pitaevskii hierarchy on $\Lambda_d$ from the spacetime estimate on $\Lambda$. This point is explained in more detail in Remark 3.5 below.

1.6. **Organization of the paper.** In Section 2, we will define the notation and we will recall some useful preliminary facts from Fourier analysis. In Subsection 2.1, we will define more precisely the notions related to density matrices. In Subsection 2.2, we will recall several estimates for oscillatory sums.

Section 3 is devoted to the study of the two-dimensional problem. In Subsection 3.1, we will prove a sharp spacetime estimate for the free evolution operator $\mathcal{U}_Q^{(k)}(t)$ in two dimensions. A conditional uniqueness result for the GP hierarchy on $\Lambda_2$ is given in Subsection 3.2. This result will be used to obtain a rigorous derivation of the defocusing cubic NLS on $\Lambda_2$ in Subsection 3.3.

The three-dimensional problem is studied in Section 4. In Subsection 4.1, we prove a sharp spacetime estimate for $\mathcal{U}_Q^{(k)}(t)$. This result is used in order to deduce a conditional uniqueness result for the GP hierarchy on $\Lambda_3$ in Subsection 4.2. The sharp spacetime estimate for general dimensions is given in Remark 4.3. An unconditional uniqueness result, which is used to obtain a rigorous derivation of the defocusing cubic NLS on $\Lambda_3$ is given in Subsection 4.3.

In Section 5, we comment on the existence of local-in-time solutions to the GP hierarchy on general $\Lambda_d$.

## 2. NOTATION AND SOME PRELIMINARY FACTS

2.1. **Density matrices and their properties.** Given $f \in L^2(\Lambda_d)$, its Fourier transform is defined as follows:

$$\widehat{f}(\xi) := \int_0^{\frac{2\pi}{\theta_1}} \int_0^{\frac{2\pi}{\theta_2}} \cdots \int_0^{\frac{2\pi}{\theta_d}} f(x^1, x^2, \ldots, x^d) \cdot e^{-ix^1 \cdot \xi^1 - ix^2 \cdot \xi^2 - \cdots - ix^d \cdot \xi^d} \, dx^d \cdots dx^2 \, dx^1.$$

$$(7)$$

Here, $\xi = (\xi^1, \xi^2, \ldots, \xi^d) \in \theta_1 \cdot \mathbb{Z} \times \theta_2 \cdot \mathbb{Z} \times \cdots \times \theta_d \cdot \mathbb{Z}$. On $\Lambda_d$, we consider the standard Laplace operator given by:

$$\Delta = \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \cdots + \frac{\partial^2}{\partial(x^d)^2}.$$

Here, $x^j$ denotes the $\mathbb{R} \big/ \frac{2\pi}{\theta_j} \mathbb{Z}$ variable. In particular, given a function $f \in L^2(\Lambda_d)$ such that $\Delta f \in L^2(\Lambda_d)$, and $\xi = (\xi^1, \xi^2, \ldots, \xi^d) \in \theta_1 \cdot \mathbb{Z} \times \theta_2 \cdot \mathbb{Z} \times \cdots \times \theta_d \cdot \mathbb{Z}$ as above, it is the case that:

$$(\Delta f)\widehat{\ }(\xi) = (-(\xi^1)^2 - (\xi^2)^2 - \cdots - (\xi^d)^2) \cdot \widehat{f}(\xi),$$

When working with density matrices of order $k$, we will use the shorthand notation $\vec{x}_k := (x_1, x_2, \ldots, x_k)$, $\vec{x}'_k := (x'_1, x'_2, \ldots, x'_k)$, where $x_j = ((x_j)^1, (x_j)^2, \ldots, (x_j)^d) \in \Lambda_d$, $x'_j = ((x'_j)^1, (x'_j)^2, \ldots, (x'_j)^d) \in \Lambda_d$.

Given $\gamma^{(k)}$, a density matrix of order $k$ on $\Lambda_d$, its Fourier transform is defined as:

$$(\gamma^{(k)})\widehat{\ }(\vec{\xi}_1; \vec{\xi}'_k) := \int_{\Lambda_d^k \times \Lambda_d^k} \gamma^{(k)}(\vec{x}_k; \vec{x}'_k) e^{-i \cdot \sum_{j=1}^k x_j \cdot \xi_j + i \cdot \sum_{j=1}^k x'_j \cdot \xi'_j} \, d\vec{x}_k \, d\vec{x}'_k \qquad (8)$$

for $\vec{\xi}_k = (\xi_1, \ldots, \xi_k), \vec{\xi}'_k := (\xi'_1, \ldots, \xi'_k) \in (\theta_1 \cdot \mathbb{Z} \times \theta_2 \cdot \mathbb{Z} \times \cdots \times \theta_d \cdot \mathbb{Z})^k$. Furthermore, if $\gamma^{(k)} = \gamma^{(k)}(t)$ depends on time, we can define its spacetime Fourier transform as:

$$(\gamma^{(k)})\widetilde{\ }(\tau, \vec{\xi}_1; \vec{\xi}'_k) := \int_{\mathbb{R}} \int_{\Lambda_d^k \times \Lambda_d^k} \gamma^{(k)}(t, \vec{x}_k; \vec{x}'_k) e^{-it\tau - i \cdot \sum_{j=1}^k x_j \cdot \xi_j + i \cdot \sum_{j=1}^k x'_j \cdot \xi'_j} \, d\vec{x}_k \, d\vec{x}'_k \, dt,$$
$$(9)$$

for $\tau \in \mathbb{R}$ and $\vec{\xi}_k, \vec{\xi}'_k \in (\theta_1 \cdot \mathbb{Z} \times \theta_2 \cdot \mathbb{Z} \times \cdots \times \theta_d \cdot \mathbb{Z})^k$.

The *collision operator* $B_{j,k+1}$, for $j \in \{1, 2, \ldots, k\}$ is defined on density matrices of order $k+1$ by:

$$B_{j,k+1}\gamma^{(k+1)}(\vec{x}_k; \vec{x}'_k) := Tr_{x_{k+1}} \left[ \delta(x_j - x_{k+1}), \gamma^{(k+1)} \right](\vec{x}_k; \vec{x}'_k) \qquad (10)$$

$$= \int_{\Lambda_d} dx_{k+1} \Big( \delta(x_j - x_{k+1}) - \delta(x'_j - x_{k+1}) \Big) \gamma^{(k+1)}(\vec{x}_k, x_{k+1}; \vec{x}'_k, x_{k+1}).$$

In particular, $B_{j,k+1}\gamma^{(k+1)}$ is a density matrix of order $k$. We can write the above difference as $B^+_{j,k+1}\gamma^{(k+1)} - B^-_{j,k+1}\gamma^{(k+1)}$. In this way, we define the operators $B^+_{j,k+1}$ and $B^-_{j,k+1}$. More precisely,

$$B^+_{j,k+1}\gamma^{(k+1)}(\vec{x}_k; \vec{x}'_k) := \int_{\Lambda_d} dx_{k+1} \, \delta(x_j - x_{k+1}) \gamma^{(k+1)}(\vec{x}_k, x_{k+1}; \vec{x}'_k, x_{k+1}) \qquad (11)$$

and

$$B^-_{j,k+1}\gamma^{(k+1)}(\vec{x}_k; \vec{x}'_k) := \int_{\Lambda_d} dx_{k+1} \, \delta(x'_j - x_{k+1}) \gamma^{(k+1)}(\vec{x}_k, x_{k+1}; \vec{x}'_k, x_{k+1}). \qquad (12)$$

Given $\alpha \in \mathbb{R}$, we define the *differentiation operator* $S^{(k,\alpha)}$ on density matrices of order $k$ on $\Lambda_d$ by:

$$(S^{(k,\alpha)}\gamma^{(k)})\widehat{\ }(\vec{\xi}_k; \vec{\xi}'_k) := \prod_{j=1}^k \langle \xi_j \rangle^\alpha \cdot \prod_{j=1}^k \langle \xi'_j \rangle^\alpha \cdot \widehat{\gamma}^{(k)}(\vec{\xi}_k; \vec{\xi}'_k), \qquad (13)$$

whenever $\vec{\xi}_k, \vec{\xi}_k' \in (\theta_1 \cdot \mathbb{Z} \times \theta_2 \cdot \mathbb{Z} \times \cdots \times \theta_d \cdot \mathbb{Z})^k$. Here, we use the convention for the Japanese bracket given by:

$$\langle x \rangle := \sqrt{1 + |x|^2}.$$

Using $\Delta$, we define $\mathcal{U}^{(k)}(t)$ by:

$$\mathcal{U}^{(k)}(t) \gamma^{(k)} := e^{it \sum_{j=1}^k \Delta_{x_j}} \gamma^{(k)} e^{-it \sum_{j=1}^k \Delta_{x_j'}}.$$

This operator corresponds to the *free Schrödinger evolution* on density matrices.

In the remainder of this section, we will rescale the torus $\Lambda_d$ to the classical torus $\Lambda = \mathbb{T}^d = (\mathbb{R} / 2\pi\mathbb{Z})^d$. We will henceforth work on the simpler domain $\Lambda$ at the expense of working with a modified Laplacian operator as it was done in the context of the NLS in [10]. More precisely, we will consider the domain $\Lambda = \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ with the modified Laplacian operator:

$$\Delta_Q = \theta_1^2 \cdot \frac{\partial^2}{\partial(x^1)^2} + \theta_2^2 \cdot \frac{\partial^2}{\partial(x^2)^2} + \cdots + \theta_d^2 \cdot \frac{\partial^2}{\partial(x^d)^2}.$$

Here, we write the spatial variable as $x = (x^1, x^2, \ldots, x^d) \in \mathbb{T}^d$.

The action of $\Delta_Q$ on the Fourier side is given by:

$$(\Delta_Q g)\hat{\phantom{x}}(\xi) := (-\theta_1^2 \cdot (\xi^1)^2 - \theta_2^2 \cdot (\xi^2)^2 - \cdots - \theta_d^2 \cdot (\xi^d)^2) \cdot \widehat{g}(\xi) \qquad (14)$$

for $\xi = (\xi^1, \xi^2, \ldots, \xi^d) \in \mathbb{Z}^d$. In (14), $\hat{\phantom{x}}$ denotes the Fourier transform on $\Lambda$, i.e.

$$\widehat{g}(\xi) = \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} f(x^1, x^2, \ldots, x^d) \cdot e^{-ix^1 \cdot \xi^1 - ix^2 \cdot \xi^2 - \cdots - ix^d \cdot \xi^d} \, dx^d \cdots dx^2 \, dx^1, \qquad (15)$$

whenever $g \in L^2(\Lambda)$. Let us note that the Fourier transform on $\Lambda_d$, given in (7), and the Fourier transform on $\Lambda$, given in (15), are denoted in the same way. From context, it will be possible to distinguish which Fourier transform we are considering.

Similarly, the analogues on $\Lambda$ of the operations given in (8), (9), (10), and (13) are defined by simply setting $\theta_1 = \theta_2 = \cdots = \theta_d = 1$ in the definitions given above. In what follows, we will denote the operations of Fourier transform, spacetime Fourier transform, collision, and fractional differentiation acting on density matrices on $\Lambda_d$ and on $\Lambda$ in the same way. It will typically be clear from context which operation we are using.

We associate to $\Delta_Q$ the operator $\mathcal{U}_Q^{(k)}(t)$, which acts on density matrices of order $k$ on $\Lambda$ by:

$$\mathcal{U}_Q^{(k)}(t) \gamma^{(k)} := e^{it \sum_{j=1}^k \Delta_{Q, x_j}} \gamma^{(k)} e^{-it \sum_{j=1}^k \Delta_{Q, x_j'}}. \qquad (16)$$

Here, $\Delta_{Q, x_j}$ denotes the operator $\Delta_Q$ acting in the $x_j$ variable and $\Delta_{Q, x_j'}$ denotes the operator $\Delta_Q$ acting in the $x_j'$ variable. Let us note that, unlike $\mathcal{U}^{(k)}(t)$, *the operator $\mathcal{U}_Q^{(k)}(t)$ is in general not periodic in time.*

We will now define a method of rescaling density matrices. Given $k \in \mathbb{N}$ and $\gamma^{(k)} : \Lambda^k \times \Lambda^k \to \mathbb{C}$, we define $\widetilde{\gamma}^{(k)} : \Lambda_d^k \times \Lambda_d^k \to \mathbb{C}$ as:

$$\widetilde{\gamma}^{(k)}(x_1, x_2, \ldots, x_k; x_1', x_2', \ldots, x_k') := \gamma^{(k)}(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_k; \widetilde{x}_1', \widetilde{x}_2', \ldots, \widetilde{x}_k') \qquad (17)$$

where, for $j \in \mathbb{N}$, given $x_j, x_j' \in \Lambda_d$, we define:

$$\widetilde{x}_j := \left( \theta_1 \cdot (x_j)^1, \theta_2 \cdot (x_j)^2, \cdots, \theta_d \cdot (x_j)^d \right) \in \Lambda$$

and
$$\widetilde{x}'_j := \big(\theta_1 \cdot (x'_j)^1, \theta_2 \cdot (x'_j)^2, \cdots, \theta_d \cdot (x'_j)^d\big) \in \Lambda.$$
The mapping defined in (17) gives a bijection between density matrices of order $k$ on $\Lambda$ and on $\Lambda_d$. By the Chain Rule, it follows that, with the above rescaling:

$$\Big(\big(\Delta_{\vec{x}_k} - \Delta_{\vec{x}'_k}\big)\widetilde{\gamma}^{(k)}\Big)(\vec{x}_k; \vec{x}'_k) = \Big((\Delta_{Q,\vec{x}_k} - \Delta_{Q,\vec{x}'_k})\gamma^{(k)}\Big)(\widetilde{x}_1, \ldots, \widetilde{x}_k; \widetilde{x}'_1, \ldots, \widetilde{x}'_k). \quad (18)$$

Here, $\Delta_{Q,\vec{x}_k} := \sum_{j=1}^{k} \Delta_{Q,x_j}$ and $\Delta_{Q,\vec{x}'_k} := \sum_{j=1}^{k} \Delta_{Q,x'_j}$.

We can take Fourier transforms of both sides of (17) and obtain that, for all $\vec{\xi}_k = (\xi_1, \ldots, \xi_k), \vec{\xi}'_k := (\xi'_1, \ldots, \xi'_k) \in (\theta_1 \cdot \mathbb{Z} \times \theta_2 \cdot \mathbb{Z} \times \cdots \times \theta_d \cdot \mathbb{Z})^k$:

$$\widehat{\widetilde{\gamma}^{(k)}}(\vec{x}_k; \vec{x}'_k) = \frac{1}{\theta_1^{2k} \cdot \theta_2^{2k} \cdots \theta_d^{2k}} \cdot \widehat{\gamma^{(k)}}(\widetilde{\xi}_1, \widetilde{\xi}_2, \ldots, \widetilde{\xi}_k; \widetilde{\xi}'_1, \widetilde{\xi}'_2, \ldots, \widetilde{\xi}'_k). \quad (19)$$

Here,
$$\widetilde{\xi}_j := \Big(\frac{1}{\theta_1} \cdot (\xi_j)^1, \frac{1}{\theta_2} \cdot (\xi_j)^2, \cdots, \frac{1}{\theta_d} \cdot (\xi_j)^d\Big) \in \mathbb{Z}^d \quad (20)$$

and
$$\widetilde{\xi}'_j := \Big(\frac{1}{\theta_1} \cdot (\xi'_j)^1, \frac{1}{\theta_2} \cdot (\xi'_j)^2, \cdots, \frac{1}{\theta_d} \cdot (\xi'_j)^d\Big) \in \mathbb{Z}^d. \quad (21)$$

Here, and throughout the paper, we use the notational convention
$$\xi_j = \big((\xi_j)^1, (\xi_j)^2, \ldots, (\xi_j)^d\big) \text{ and } \xi'_j = \big((\xi'_j)^1, (\xi'_j)^2, \ldots, (\xi'_j)^d\big).$$

In addition to (1), we will also study the *Gross-Pitaevskii hierarchy on the spatial domain* $\Lambda$ *with a modified Laplacian:*

$$\begin{cases} i\partial_t \gamma^{(k)} + (\Delta_{Q,\vec{x}_k} - \Delta_{Q,\vec{x}'_k})\gamma^{(k)} = b_0 \cdot \sum_{j=1}^{k} B_{j,k+1}(\gamma^{(k+1)}) \\ \gamma^{(k)}|_{t=0} = \gamma_0^{(k)}. \end{cases} \quad (22)$$

Here, $\gamma^{(k)} : \mathbb{R}_t \times \Lambda^k \times \Lambda^k \to \mathbb{C}$ and $\gamma_0^{(k)} : \Lambda^k \times \Lambda^k \to \mathbb{C}$. As above, $b_0 \in \mathbb{R}$ is a non-zero coupling constant. As in (1), we say that (22) is *defocusing* if $b_0 > 0$, and we say that it is *focusing* if $b_0 < 0$.

Suppose that $(\gamma^{(k)})_k = (\gamma^{(k)}(t))_k$ solves (22). For each $k \in \mathbb{N}$ and for each time $t$, we define $\widetilde{\gamma}^{(k)}(t)$ from $\gamma^{(k)}(t)$ according to (17), i.e.

$$\widetilde{\gamma}^{(k)}(t, \vec{x}_k; \vec{x}'_k) := \gamma^{(k)}(t, \widetilde{x}_1, \ldots, \widetilde{x}_k; \widetilde{x}'_1, \ldots, \widetilde{x}'_k).$$

A direct calculation then shows that, for all $k \in \mathbb{N}$, and for all $j \in \{1, 2, \ldots, k\}$:

$$\big[B_{j,k+1}(\gamma^{(k+1)})\big](t, \vec{x}_k; \vec{x}'_k) = \big[B_{j,k+1}(\widetilde{\gamma}^{(k+1)})\big](t, \widetilde{x}_1, \ldots, \widetilde{x}_k; \widetilde{x}'_1, \ldots, \widetilde{x}'_k). \quad (23)$$

From (18) and (23), we obtain:

**Lemma 2.1.** $(\widetilde{\gamma}^{(k)})_k$ *solves* (1) *if and only if* $(\gamma^{(k)})_k$ *solves* (22).

Hence, by using the rescaling (17), we obtain a *correspondence between solutions of* (1) *and* (22).

Given $\xi = (\xi^1, \xi^2, \ldots, \xi^d), \eta = (\eta^1, \eta^2, \ldots, \eta^d) \in \mathbb{Z}^d$, we define
$$Q(\xi, \eta) := \theta_1^2 \cdot \xi^1 \cdot \eta^1 + \theta_2^2 \cdot \xi^2 \cdot \eta^2 + \cdots + \theta_d^2 \cdot \xi^d \cdot \eta^d.$$

In the sequel, we will use the abbreviated notation:
$$Q(\xi) := Q(\xi, \xi).$$

In other words, we can write (14) as:

$$(\Delta_Q f)\widehat{\ }(\xi) = -Q(\xi) \cdot \widehat{f}(\xi)$$

for $\xi \in \mathbb{Z}^d$. Moreover, if $\vec{\xi}_k = (\xi_1, \ldots, \xi_k) \in (\mathbb{Z}^d)^k$, we define $Q(\vec{\xi}_k) := \sum_{j=1}^{k} Q(\xi_j)$.

In the paper, we will primarily focus on the case when $d = 2$ and $d = 3$. The two-dimensional problem will be considered in Section 3 and the three-dimensional problem will be considered in Section 4. A statement for general $d \geq 2$ is given in Remark 4.3, Remark 4.6, and in Section 5 below. Moreover, we will primarily be interested in the case when $\Lambda_d$ is a irrational torus in the sense that it is not possible to use a rescaling argument and apply the results from the setting of the classical torus $\mathbb{T}^d$ obtained in [45, 55].

## 2.2. Estimates for oscillatory sums.

Let us first recall a simple and well-known $L^4$-bound, cp. [14, Lemma 3.2], as well as [10, 48] for related results:

**Lemma 2.2.** *Let $I \subset \mathbb{R}$ be a bounded interval and let $\epsilon > 0$. There exists $c > 0$, depending on $\epsilon$, such that for all $b \in \mathbb{Z}$ and $N \in \mathbb{N}$,*

$$\int_I \Big| \sum_{m \in [b, b+N) \cap \mathbb{Z}} e^{itm^2} \Big|^4 dt \leq cN^{2+\epsilon}$$

*Proof.* Without loss it suffices to consider $I = [0, 2\pi]$. Then, by Plancherel,

$$L := \int_I \Big| \sum_{m \in [b, b+N) \cap \mathbb{Z}} e^{itm^2} \Big|^4 dt = \Big\| \sum_{m_1, m_2 \in [b, b+N) \cap \mathbb{Z}} e^{it(m_1^2 - m_2^2)} \Big\|_{L^2([0, 2\pi])}^2$$

$$= \sum_l \Big| \sum_{\substack{m_1, m_2 \in [b, b+N) \cap \mathbb{Z} \\ m_1^2 - m_2^2 = l}} 1 \Big|^2$$

Since there are at most $N^2$ different values for $l$ such that $m_1^2 - m_2^2 = l$ has a solution $(m_1, m_2)$ in the given interval, we obtain

$$L \leq N^2 \sup_l S_{l,b}(N)^2,$$

where

$$S_{l,b}(N) = \#\{(m_1, m_2) \in [b, b+N)^2 : (m_1 - m_2)(m_1 + m_2) = l\}.$$

Hence, it remains to give a uniform estimate for $S_{l,b}(N)$. Setting $k_1 := m_1 - m_2$, $k_2 := m_1 + m_2 - 2b$, we observe that

$$S_{l,b}(N) = \#\{(k_1, k_2) \in [-N, N) \times [0, 2N) : k_1(k_2 + 2b) = l\}.$$

We discuss two cases separately.

*Case 1:* $-10N^2 \leq b \leq 10N^2$. Then, if solutions exist, we must have $l \leq cN^3$, and the classical estimate on the number-of-divisors function $d$ [49, Thm. 315] yields

$$S_{l,b}(N) \leq d(l) \leq cl^\delta \leq cN^{3\delta},$$

for any $\delta > 0$.

*Case 2:* $|b| > 10N^2$. In this case, for any given $l$, there is at most one solution to $k_1(k_2 + 2b) = l$. Indeed, suppose that $(k_1, k_2)$ and $(k_1', k_2')$ are two distinct solutions, hence $k_1 \neq k_1'$. Then,

$$k_1 k_2 - k_1' k_2' = 2b(k_1' - k_1) \Rightarrow |k_1 k_2 - k_1' k_2'| > 20N^2,$$

a contradiction to $|k_1 k_2 - k_1' k_2'| \leq 4N^2$. $\qquad\square$

**Corollary 2.3.** *Let $p \geq 4$, $\epsilon > 0$ and $I \subset \mathbb{R}$ be a bounded interval. There exists $c > 0$, depending on $\epsilon$, such that for all $b \in \mathbb{Z}$ and $N \in \mathbb{N}$,*

$$\int_I \Big| \sum_{m \in [b, b+N) \cap \mathbb{Z}} e^{itm^2} \Big|^p dt \leq cN^{p-2+\epsilon}.$$

*Proof.* Let us note that, for all $b \in \mathbb{R}$ and $N \in \mathbb{N}$, it is obvious that

$$\Big\| \sum_{m \in [b, b+N) \cap \mathbb{Z}} e^{itm^2} \Big\|_{L_t^\infty(I)} \leq N.$$

The corollary now follows from Lemma 2.2 and interpolation. $\qquad\square$

We remark that in the case $p > 4$ the result of Corollary 2.3 holds with $\epsilon = 0$, see [10, 14, 48], but we do not need this here.

## 3. The two-dimensional problem

In this section, we consider the two-dimensional problem. In other words, we fix $d = 2$ in the notation given above. Throughout the section, $\Lambda$ will denote the two-dimensional classical torus $\mathbb{T}^2$. In Subsection 3.1, we will prove a sharp spacetime estimate for the free evolution $\mathcal{U}_Q^{(k)}(t)$ associated to the classical torus $\Lambda = \mathbb{T}^2$. We will use this result and a scaling argument to prove a conditional uniqueness result on $\Lambda_2$ in Subsection 3.2. Finally, in Subsection 3.3, we will use the conditional uniqueness result in order to obtain a rigorous derivation of the defocusing cubic NLS on the two-dimensional general rectangular torus, as was done in the setting of the two-dimensional classical torus in [55].

### 3.1. The spacetime estimate in two dimensions. The following spacetime estimate, which extends a result of [55], will be key in our analysis:

**Proposition 3.1.** *Let $\alpha > \frac{1}{2}$ be given. There exists $C > 0$, which depends only on $\alpha, \theta_1, \theta_2$ such that, for all $k \in \mathbb{N}$, and for all $\gamma_0^{(k+1)} : \Lambda^{k+1} \times \Lambda^{k+1} \to \mathbb{C}$, the following estimate holds:*

$$\|S^{(k,\alpha)} B_{j,k+1} \mathcal{U}_Q^{(k+1)}(t) \gamma_0^{(k+1)}\|_{L^2([0,1] \times \Lambda^k \times \Lambda^k)} \leq C \|S^{(k+1,\alpha)} \gamma_0^{(k+1)}\|_{L^2(\Lambda^{k+1} \times \Lambda^{k+1})}.$$
(24)

The range of regularity exponents $\alpha > \frac{1}{2}$ in Proposition 3.1 is sharp due to the following:

**Proposition 3.2.** *For $\kappa \in \mathbb{N}$ sufficiently large, there exists $\gamma_0^{(2)} : \Lambda^2 \times \Lambda^2 \to \mathbb{C}$, such that for $\delta > 0$ sufficiently small:*

$$\|S^{(1,\frac{1}{2})} B_{1,2} \mathcal{U}_Q^{(2)}(t) \gamma_0^{(2)}\|_{L^2([0,\delta] \times \Lambda \times \Lambda)} \gtrsim_\delta \sqrt{\ln \kappa} \cdot \|S^{(2,\frac{1}{2})} \gamma_0^{(2)}\|_{L^2(\Lambda^2 \times \Lambda^2)}.$$

*Here, $\delta$ is independent of $\kappa$.*

Let us first prove Proposition 3.1.

*Proof of Proposition 3.1.* Let $\psi \in \mathcal{S}(\mathbb{R})$ be such that $\psi \geq 1$ on $[0,1]$. It suffices to show that:

$$\|\psi(t) S^{(k,\alpha)} B_{1,k+1}^+ \mathcal{U}_Q^{(k+1)}(t) \gamma_0^{(k+1)}\|_{L^2(\mathbb{R} \times \Lambda^k \times \Lambda^k)}$$

is bounded by the expression on the right-hand side in (24). Here, the operator $B_{j,k+1}^\pm$ is defined as in (11) and (12) when we set $\theta_1 = \theta_2 = 1$. The estimate when $B_{1,k+1}^+$ is replaced by general $B_{j,k+1}^\pm$ is proved in the same way.

We compute

$$
\begin{aligned}
&\big(\psi(t)S^{(k,\alpha)}B^{+}_{1,k+1}\mathcal{U}^{(k+1)}_Q(t)\,\gamma^{(k+1)}_0\big)^{\sim}(\tau,\vec{\xi}_k;\vec{\xi}'_k) \\
&= \sum_{\xi_{k+1},\xi'_{k+1}\in\mathbb{Z}^2} \widehat{\psi}\big(\tau + Q(\xi_1-\xi_{k+1}+\xi'_{k+1}) + Q(\vec{\xi}_{k+1}) - Q(\xi_1) - Q(\vec{\xi}'_{k+1})\big)\cdot \\
&\qquad\cdot \prod_{j=1}^{k+1}\langle\xi_j\rangle^\alpha \cdot \prod_{j=1}^{k+1}\langle\xi'_j\rangle^\alpha \cdot \widehat{\gamma}^{(k+1)}_0(\xi_1-\xi_{k+1}+\xi'_{k+1},\xi_2,\dots,\xi_{k+1};\xi'_1,\dots,\xi'_{k+1}).
\end{aligned}
$$

Hence, by the Cauchy-Schwarz inequality:

$$
\Big|\big(\psi(t)S^{(k,\alpha)}B^{+}_{1,k+1}\mathcal{U}^{(k+1)}_Q(t)\,\gamma^{(k+1)}_0\big)^{\sim}(\tau,\vec{\xi}_k;\vec{\xi}'_k)\Big| \le (\Sigma_1)^{\frac12}\cdot(\Sigma_2)^{\frac12}
$$

where

$$
\Sigma_1
:= \sum_{\xi_{k+1},\xi'_{k+1}\in\mathbb{Z}^2} \frac{\big|\widehat{\psi}\big(\tau + Q(\xi_1-\xi_{k+1}+\xi'_{k+1}) + Q(\vec{\xi}_{k+1}) - Q(\xi_1) - Q(\vec{\xi}'_{k+1})\big)\big|^2\cdot\langle\xi_1\rangle^{2\alpha}}{\langle\xi_1-\xi_{k+1}+\xi'_{k+1}\rangle^{2\alpha}\cdot\langle\xi_{k+1}\rangle^{2\alpha}\langle\xi'_{k+1}\rangle^{2\alpha}}
$$

and

$$
\begin{aligned}
\Sigma_2 := \sum_{\xi_{k+1},\xi'_{k+1}\in\mathbb{Z}^2} &\langle\xi_1-\xi_{k+1}+\xi'_{k+1}\rangle^{2\alpha}\cdot\prod_{j=2}^{k+1}\langle\xi_j\rangle^{2\alpha}\cdot\prod_{j=1}^{k+1}\langle\xi'_j\rangle^{2\alpha} \\
&\cdot|\widehat{\gamma}^{(k+1)}_0(\xi_1-\xi_{k+1}+\xi'_{k+1},\xi_2,\dots,\xi_{k+1};\xi'_1,\dots,\xi'_{k+1})|^2.
\end{aligned}
$$

In particular, the claim will follow if we show that $\Sigma_1$ is uniformly bounded in $(\tau,\vec{\xi}_k;\vec{\xi}'_k)$. Let us analyze $\Sigma_1$ more closely. We note that it can be decomposed as $\Sigma_1 = \sum_{\ell\in\mathbb{Z}}\Sigma_1(\ell)$ for

$$
\Sigma_1(\ell)
:= \sum_{\xi_{k+1},\xi'_{k+1}\in\mathbb{Z}^2:*} \frac{\big|\widehat{\psi}\big(\tau + Q(\xi_1-\xi_{k+1}+\xi'_{k+1}) + Q(\vec{\xi}_{k+1}) - Q(\xi_1) - Q(\vec{\xi}'_{k+1})\big)\big|^2\langle\xi_1\rangle^{2\alpha}}{\langle\xi_1-\xi_{k+1}+\xi'_{k+1}\rangle^{2\alpha}\langle\xi_{k+1}\rangle^{2\alpha}\langle\xi'_{k+1}\rangle^{2\alpha}},
$$

where we sum with respect to the constraint $*$ given by

$$
\tau + Q(\xi_1-\xi_{k+1}+\xi'_{k+1}) + Q(\vec{\xi}_{k+1}) - Q(\xi_1) - Q(\vec{\xi}'_{k+1}) \in [\ell,\ell+1].
$$

We can choose $\psi$ such that $\widehat{\psi}$ is increasing on $(-\infty,0)$ and decreasing on $(0,+\infty)$ (for example, we can take $\psi(t) = ce^{-t^2}$ for an appropriate choice of $c>0$). Then,

$$
\Sigma_1(\ell)
\lesssim \sum_{\ell\in\mathbb{Z}}|\widehat{\psi}(\ell)|^2\cdot\sup_{m\in\mathbb{Z}}\Big[\sum_{\xi_{k+1},\xi'_{k+1}\in\mathbb{Z}^2:**} \frac{\langle\xi_1\rangle^{2\alpha}}{\langle\xi_1-\xi_{k+1}+\xi'_{k+1}\rangle^{2\alpha}\cdot\langle\xi_{k+1}\rangle^{2\alpha}\cdot\langle\xi'_{k+1}\rangle^{2\alpha}}\Big]
$$

with respect to the constraint $**$ gived by

$$
\tau + Q(\xi_1-\xi_{k+1}+\xi'_{k+1}) + Q(\vec{\xi}_{k+1}) - Q(\xi_1) - Q(\vec{\xi}'_{k+1}) \in [m,m+1].
$$

Hence,

$\Sigma_1(\ell)$

$$\lesssim \sup_{m\in\mathbb{Z}} \sum_{\xi_{k+1},\xi'_{k+1}\in\mathbb{Z}^2} \frac{\widetilde{\delta}\big(\tau - m + Q(\xi_1 - \xi_{k+1} + \xi'_{k+1}) + Q(\vec{\xi}_{k+1}) - Q(\xi_1) - Q(\vec{\xi}'_{k+1})\big) \cdot \langle\xi_1\rangle^{2\alpha}}{\langle\xi_1 - \xi_{k+1} + \xi'_{k+1}\rangle^{2\alpha} \cdot \langle\xi_{k+1}\rangle^{2\alpha} \cdot \langle\xi'_{k+1}\rangle^{2\alpha}}.$$

Here, $\widetilde{\delta} := \chi_{[0,1]}$ is the characteristic function of the interval $[0,1]$. We rewrite

$$\tau + Q(\xi_1 - \xi_{k+1} + \xi'_{k+1}) + Q(\vec{\xi}_{k+1}) - Q(\xi_1) - Q(\vec{\xi}'_{k+1})$$

$$= \big(\tau + \sum_{\ell=2}^{k} Q(\xi_\ell) - \sum_{\ell=1}^{k} Q(\xi'_\ell)\big) + Q(\xi_1 - \xi_{k+1} + \xi'_{k+1}) + Q(\xi_{k+1}) - Q(\xi'_{k+1}).$$

After changing variables $\tau \mapsto \tau + \sum_{\ell=2}^{k} Q(\xi_\ell) - \sum_{\ell=1}^{k} Q(\xi'_\ell)$, and letting $p := \xi_1, n := \xi_{k+1}, m := -\xi_{k+1}$, it follows that we need to bound:

$$I(\tau,p) := \sum_{m,n\in\mathbb{Z}^2} \frac{\widetilde{\delta}\big(\tau + Q(p - n - m) + Q(n) - Q(m)\big) \cdot \langle p\rangle^{2\alpha}}{\langle p - n - m\rangle^{2\alpha} \cdot \langle n\rangle^{2\alpha} \cdot \langle m\rangle^{2\alpha}} \tag{25}$$

uniformly in $\tau \in \mathbb{R}$ and $p \in \mathbb{Z}^2$. By using the same calculations as in [45, equation (48)], we can rewrite $I(\tau,p)$ as

$$I(\tau,p) = \sum_{m,n\in\mathbb{Z}^2} \frac{\widetilde{\delta}\big(\tau + Q(p) - 2Q(n,m)\big) \cdot \langle p\rangle^{2\alpha}}{\langle m - p\rangle^{2\alpha} \cdot \langle n - p\rangle^{2\alpha} \cdot \langle p - n - m\rangle^{2\alpha}}. \tag{26}$$

Obviously, the contribution of $m = 0$ or $n = 0$ is uniformly bounded if $\alpha > \frac{1}{2}$, so we can restrict the sum to $\mathbb{Z}^2 \setminus \{0\} \times \mathbb{Z}^2 \setminus \{0\}$. The advantage of writing $I(\tau,p)$ as in (26) instead of as in (25) is that, in (26), for a fixed $m \neq 0$, one sums over the set of all $n$ which lie in a neighborhood of a fixed hyperplane.

Given $j = (j_1, j_2, j_3) \in \mathbb{N}_0^3$, $\tau \in \mathbb{R}$, and $p \in \mathbb{Z}^2$, we let $E_{\tau,p}(j) \subseteq \mathbb{Z}^2 \setminus \{0\} \times \mathbb{Z}^2 \setminus \{0\}$ be the set of all pairs $(m,n)$ such that

$$\begin{cases} \tau + Q(p) - 2Q(n,m) \in [0,1] \\ |m - p| \sim 2^{j_1}, |n - p| \sim 2^{j_2}, |p - n - m| \sim 2^{j_3}. \end{cases}$$

Here, by $|x| \sim 2^j$, we mean $2^{j-1} \le |x| < 2^j$ when $j \ge 1$ and $|x| < 1$ when $j = 0$. In what follows, we will order $j_1, j_2, j_3$ as $j_{\min} \le j_{\mathrm{med}} \le j_{\max}$. We would like to prove that

$$\#E_{\tau,p}(j) \lesssim_\epsilon 2^{(1+\epsilon)j_{\min}+(1+\epsilon)j_{\mathrm{med}}} \tag{27}$$

for all $\epsilon > 0$. The calculations in [45, equations (54) and (55)] then imply the claim.

Let us fix $\tau, p, j$ as above. We will now estimate $\#E_{\tau,p}(j)$. We first argue by using an idea from [10]. In particular, let us fix $\phi \in C_0^\infty(\mathbb{R})$ such that $\widehat{\phi} \ge 0$ on all of $\mathbb{R}$ and $\widehat{\phi} \ge 1$ on $[0,1]$, see Lemma A.1.

By using the same argument to deduce [10, formula (1.1.8')], it follows that:

$$\#E_{\tau,p}(j) \le \int \Big[ \sum_{\substack{n,m\in\mathbb{Z}^2 \\ |m-p|\sim 2^{j_1}, |n-p|\sim 2^{j_2}, |p-m-n|\sim 2^{j_3}}} e^{2iQ(n,m)t} \Big] \cdot e^{-i(\tau+Q(p))t} \cdot \phi(t)\, dt.$$

$$\tag{28}$$

This is the case since:

$$\int e^{2iQ(n,m)t} \cdot e^{-i(\tau+Q(p))t}\phi(t)\,dt = \widehat{\phi}\big(\tau + Q(p) - 2Q(m,n)\big).$$

**Remark 3.3.** *Let us note that the estimate in* (28) *holds if we replace the sum in $m$ and $n$ by the sum over a larger set in $m$ and $n$. This follows from the fact that $\widehat{\phi} \geq 0$ on all of $\mathbb{R}$. We will use this observation several times in the discussion that follows.*

We note that:

$$2Q(n,m) = \frac{Q(n+m) - Q(n-m)}{2}.$$

So, the right-hand side of (28) equals:

$$\int \Big[ \sum_{\substack{n,m\in\mathbb{Z}^2 \\ |m-p|\sim 2^{j_1}, |n-p|\sim 2^{j_2}, |p-m-n|\sim 2^{j_3}}} e^{\frac{1}{2}it\big(Q(n+m)-Q(n-m)\big)} \Big] \cdot e^{-i(\tau+Q(p))t} \cdot \phi(t)\,dt.$$

$$\tag{29}$$

Let $\eta := n - m$ and $\eta' := n + m$. This is a one-to-one change of variables. We would like to find the localization properties of the $\eta$ and $\eta'$ variables thus defined.

We first note that:

$$\eta = n - m =$$

$$= \underbrace{(n-p)}_{2^{j_2}} - \underbrace{(m-p)}_{2^{j_1}} = \underbrace{p-m-n}_{2^{j_3}} + \underbrace{2n-2p}_{2^{j_2+1}} + p = \underbrace{-p+n+m}_{2^{j_3}} - \underbrace{(2m-2p)}_{2^{j_1+1}} - p.$$

Here, by $\underbrace{x}_{2^j}$, we mean that $|x| \leq 2^j$. Let $a \vee b := \max\{a,b\}$. Since

$$2^{k_1} + 2^{k_2} \leq 2^{k_1\vee k_2 + 1} \text{ for all } k_1, k_2 \geq 0,$$

it follows that:

$$\eta \in B_{2^{j_1\vee j_2+1}}(0) \cap B_{2^{j_2\vee j_3+2}}(p) \cap B_{2^{j_1\vee j_3+2}}(-p). \tag{30}$$

Similarly,

$$\eta' = n + m = \underbrace{(-p+n+m)}_{2^{j_3}} + p = \underbrace{(n-p)}_{2^{j_2}} + \underbrace{(m-p)}_{2^{j_1}} + 2p.$$

Hence:

$$\eta' \in B_{2^{j_3}}(p) \cap B_{2^{j_1\vee j_2+1}}(2p). \tag{31}$$

We now apply this change of variables in (29) and deduce that:

$$\#E_{\tau,p}(j)$$

$$\leq \int \Big[ \sum_{\substack{\eta,\eta'\in\mathbb{Z}^2 \\ (30),(31)}} e^{\frac{1}{2}it\big(Q(\eta)-Q(\eta')\big)} \Big] \cdot e^{-i(\tau+Q(p))t} \cdot \phi(t)\,dt$$

$$\leq \int \Big[ \sum_{\substack{\eta,\eta'\in\mathbb{Z}^2 \\ (33),(34)}} e^{\frac{1}{2}it\big(Q(\eta)-Q(\eta')\big)} \Big] \cdot e^{-i(\tau+Q(p))t} \cdot \phi(t)\,dt \tag{32}$$

where

$$\eta \in C_{2^{j_1 \vee j_2 + 1}}(0) \cap C_{2^{j_2 \vee j_3 + 2}}(p) \cap C_{2^{j_1 \vee j_3 + 2}}(-p), \tag{33}$$

$$\eta' \in C_{2^{j_3}}(p) \cap C_{2^{j_1 \vee j_2 + 1}}(2p). \tag{34}$$

Here $C_{2^j}(q)$ denotes a two-dimensional square, centered at $q$, whose sides are parallel to the coordinate axes, and who have sidelength $2^{j+1}$. Let us note that, in the above calculation, we used Remark 3.3. By the triangle inequality and the fact that $\phi \in C_0^\infty(\mathbb{R})$, it follows that:

$$\#E_{\tau,p}(j) \leq c \cdot \Big\| \sum_{\substack{\eta, \eta' \in \mathbb{Z}^2 \\ (33),(34)}} e^{\frac{1}{2}it\left(Q(\eta) - Q(\eta')\right)} \Big\|_{L_t^1(I)} \tag{35}$$

for some $c > 0$ and for some finite interval $I \subseteq \mathbb{R}$.

Let us now use (35) to show (27). We will argue by considering all of the possible cases for the relative sizes of $j_1, j_2, j_3$. In particular, we consider:

**Case 1:** $j_3 = \min\{j_1, j_2, j_3\}$.

By (33), it follows that, in this case, $\eta$ is localized to a cube of sidelength $O(2^{j_{\mathrm{med}}})$. Moreover, by (34), it follows that $\eta'$ is localized to a cube of sidelength $O(2^{j_{\mathrm{min}}})$. Hence, in this case, we need to estimate[1]:

$$\int_I \Big| \sum_{\substack{\eta_1, \eta_2 \in \mathbb{Z} \\ \eta_j \in I_j^{\mathrm{med}}}} \sum_{\substack{\eta'_1, \eta'_2 \in \mathbb{Z} \\ \eta'_j \in I_j^{\mathrm{min}}}} e^{\frac{1}{2}it\left(\theta_1^2 \eta_1^2 + \theta_2^2 \eta_2^2 - \theta_1^2 (\eta'_1)^2 - \theta_2^2 (\eta'_2)^2\right)} \Big| dt. \tag{36}$$

Here $I_1^{\mathrm{med}}, I_2^{\mathrm{med}}$ are fixed intervals of size $\sim 2^{j_{\mathrm{med}}}$ and $I_1^{\mathrm{min}}, I_2^{\mathrm{min}}$ are fixed intervals of size $\sim 2^{j_{\mathrm{min}}}$. By Hölder's inequality, this expression is:

$$(36) \leq \Big( \int_I \Big| \sum_{\substack{\eta_1 \in \mathbb{Z} \\ \eta_1 \in I_1^{\mathrm{med}}}} e^{\frac{1}{2}it\theta_1^2 \eta_1^2} \Big|^4 dt \Big)^{\frac{1}{4}} \cdot \Big( \int_I \Big| \sum_{\substack{\eta_2 \in \mathbb{Z} \\ \eta_2 \in I_2^{\mathrm{med}}}} e^{\frac{1}{2}it\theta_2^2 \eta_2^2} \Big|^4 dt \Big)^{\frac{1}{4}} \cdot$$

$$\cdot \Big( \int_I \Big| \sum_{\substack{\eta'_1 \in \mathbb{Z} \\ \eta'_1 \in I_1^{\mathrm{min}}}} e^{\frac{1}{2}it\theta_1^2 (\eta'_1)^2} \Big|^4 dt \Big)^{\frac{1}{4}} \cdot \Big( \int_I \Big| \sum_{\substack{\eta'_2 \in \mathbb{Z} \\ \eta'_2 \in I_2^{\mathrm{min}}}} e^{\frac{1}{2}it\theta_2^2 (\eta'_2)^2} \Big|^4 dt \Big)^{\frac{1}{4}}.$$

For each of the four factors, we rescale in time and apply Lemma 2.2 to deduce that this product is:

$$\lesssim_\epsilon \left(2^{(2+\epsilon)j_{\mathrm{med}}}\right)^{\frac{1}{4}} \cdot \left(2^{(2+\epsilon)j_{\mathrm{med}}}\right)^{\frac{1}{4}} \cdot \left(2^{(2+\epsilon)j_{\mathrm{min}}}\right)^{\frac{1}{4}} \cdot \left(2^{(2+\epsilon)j_{\mathrm{min}}}\right)^{\frac{1}{4}}$$

$$\lesssim 2^{(1+\epsilon)j_{\mathrm{min}} + (1+\epsilon)j_{\mathrm{med}}}$$

for all $\epsilon > 0$. This is a good bound. We note that the implied constants depend on $\theta_1$ and $\theta_2$.

**Case 2:** $j_1 = \min\{j_1, j_2, j_3\}$.

---

[1] Similarly as in Remark 3.3, it is possible to replace the sum in $\eta, \eta'$ in (35) by the sum over a larger set. Moreover, for simplicity of notation, we write $\eta = (\eta_1, \eta_2), \eta' = (\eta'_1, \eta'_2) \in \mathbb{Z}^2$.

Let us note that, in this case, we can no longer deduce that $\eta'$ is localized to a ball of radius $O(2^{j_{\min}})$. In order to reduce to the case where one of the variables has a localization of the order of the smallest frequency, we will apply a covering argument similar to that which was used in [45, Proof of Proposition 3.1, Case 3]. In particular, given $k = (k_1, k_2) \in \mathbb{Z}^2$, we consider the rectangle:

$$B_k := [2^{j_1} k_1, 2^{j_1} k_1 + 2^{j_1} - 1] \times [2^{j_1} k_2, 2^{j_1} k_2 + 2^{j_1} - 1].$$

By using the arguments from Case 1, it follows that for all $(k, k') \in \mathbb{Z}^2 \times \mathbb{Z}^2$, it is the case that:

$$\#\big(E_{\tau,p}(j) \cap (B_k \times B_{k'})\big) \lesssim_\epsilon 2^{(2+\epsilon)j_1}$$

for all $\epsilon > 0$. Namely, in $E_{\tau,p}(j) \cap (B_k \times B'_k)$, the variables $m$ and $n$, and hence $\eta$ and $\eta'$ are localized to sets of diameter $O(2^{j_1})$.

Hence, in order to prove (27) in this case, it suffices to show that:

$$\#\{(k, k') \in \mathbb{Z}^2 \times \mathbb{Z}^2, \, E_{\tau,p}(j) \cap (B_k \times B'_k) \neq \emptyset\} \lesssim 2^{\min\{j_2, j_3\} - j_1}.$$

Let us recall

**Lemma 3.4** (Lemma 3.6 from [45]). *Let $Y \subset \mathbb{R}^D$ be any set, and let $Y_s$ be the set of points in $\mathbb{R}^D$ which are of distance at most $s$ to the set $Y$. Let $Z \subset \mathbb{R}^D$ be any $r$-separated set (meaning that $|x - x'| \geq r$ whenever $x, x'$ are distinct points in $Z$). Then for any $r' > 0$:*

$$\#(Y_{r'} \cap Z) \lesssim (\min\{r, r'\})^{-D} |Y_{r'}|.$$

*Here, $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^D$. The implied constant depends only on the dimension.*

We use Lemma 3.4 to deduce that:

$$\#\{(k, k') \in \mathbb{Z}^2 \times \mathbb{Z}^2, \, E_{\tau,p}(j) \cap (B_k \times B_{k'}) \neq \emptyset\} \lesssim 2^{-4j_1} |X_{2^{j_1}}|.$$

Here, $X_{2^{j_1}}$ denotes the set of all points in $\mathbb{R}^2 \times \mathbb{R}^2$ which are of distance at most $2^{j_1}$ from the set $E_{\tau,p}(j)$. More precisely, $E_{\tau,p}(j) \cap (B_k \times B_{k'}) \neq \emptyset$ implies that $(k, k')$ belongs to a $2^{j_1}$ thickening of $E_{\tau,p}(j)$. We then apply Lemma 3.4 with $Z$ being the set of all the centers of $B_k \times B_{k'}$ (which is $2^{j_1}$-separated), with $Y$ being $E_{\tau,p}(j)$, and with $r = 2^{j_1}, r' = 2^{j_1}$. We recall that the dimension $D$ is equal to 4.

Thus, we would like to show that:

$$\big|X_{2^{j_1}}\big| \lesssim 2^{3j_1 + \min\{j_2, j_3\}}.$$

Let $(m, n) \in E_{\tau,p}(j)$.

Then, we know that $m, n \neq 0$ and:

$$\begin{cases} |m - p| \sim 2^{j_1}, \, |n - p| \sim 2^{j_2}, \, |p - n - m| \sim 2^{j_3} \\ \tau + Q(p) - 2Q(n, m) \in [0, 1] \end{cases}$$

We note that $m$ is allowed to vary over a ball of radius $O(2^{j_1})$. For fixed $m$, the $n$ coordinate is allowed to vary over a ball of radius $O(2^{\min\{j_2, j_3\}})$. Furthermore, since $\tau + Q(p) - 2Q(n, m) \in [0, 1]$, it follows that $n$ lies within an $O(1)$ distance of a fixed line in $\mathbb{R}^2$. Namely, we know that:

$$\frac{\tau + Q(p)}{2|m|} - Q\Big(n, \frac{m}{|m|}\Big) = \frac{\tau + Q(p)}{2|m|} - n_1 \cdot \frac{\theta_1^2 \, m_1}{|m|} - n_2 \cdot \frac{\theta_2^2 \, m_2}{|m|} \in \Big[0, \frac{1}{2|m|}\Big]$$

and $|m| \geq 1$. Hence, *n lies in the intersection of a ball of radius $O(2^{\min\{j_2, j_3\}})$ and an $O(1)$ neighborhood of a fixed line in $\mathbb{R}^2$.*

Let us now consider the thickening $X_{2^{j_1}}$. If $(x, y) \in X_{2^{j_1}}$, it follows from the previous arguments that $x$ lies in a ball of radius $O(2^{j_1})$, and for a fixed $x$, the $y$ coordinate lies in the intersection of a ball of radius $O(2^{\min\{j_2, j_3\}})$ and an $O(2^{j_1})$ neighborhood of a fixed line. In deducing the localization for $(x, y)$ from the localization of $(m, n)$ we used the fact that $j_1 = \min\{j_1, j_2, j_3\}$. Let us note that, in order to deduce the localization properties of the $y$ coordinate, we thicken by an amount of $\sim 2^{j_1}$ in the direction perpendicular to the line and parallel to the line separately.

Consequently, the Lebesgue measure of the set to which $x$ is localized is $\lesssim 2^{2j_1}$. For a fixed $x$, the Lebesgue measure of the set to which $y$ is localized is:

$$\lesssim 2^{j_1} \cdot 2^{\min\{j_2, j_3\}} = 2^{j_1 + \min\{j_2, j_3\}}.$$

We may hence conclude that:

$$\left| X_{2^{j_1}} \right| \lesssim 2^{2j_1} \cdot 2^{j_1 + \min\{j_2, j_3\}} = 2^{3j_1 + \min\{j_2, j_3\}}.$$

This is the bound that we wanted to show.

**Case 3:** $j_2 = \min\{j_1, j_2, j_3\}$.

By symmetry in $m$ and $n$ in the sum defining $I(\tau, p)$, this case is analogous to Case 2. Proposition 3.1 now follows. □

**Remark 3.5.** *If we apply the rescaling (17) in inequality (24), we can deduce a spacetime estimate on $\Lambda_2$. More precisely, we can deduce that, given $\alpha > \frac{1}{2}$, there exists $C_1 > 0$ depending on $\alpha, \theta_1, \theta_2$ such that for all $k \in \mathbb{N}$ and for all density matrices $\gamma_0^{(k)}$ of order $k$ on $\Lambda_d$, it is the case that:*

$$\|S^{(k,\alpha)} B_{j,k+1} \mathcal{U}^{(k+1)}(t) \gamma_0^{(k+1)}\|_{L^2([0,1] \times \Lambda_2^k \times \Lambda_2^k)} \leq C_1^k \|S^{(k+1,\alpha)} \gamma_0^{(k+1)}\|_{L^2(\Lambda_2^{k+1} \times \Lambda_2^{k+1})}.$$
$$(37)$$

*The reason why we obtain a k-th power of $C_1$ is that the Sobolev norms which we are using are inhomogeneous. Due to the additional $k$ dependence in the constant, it is not possible to directly use (37) and prove a conditional uniqueness result for (1). We will circumvent this difficulty by first applying (24) in order to prove a conditional uniqueness result for (22). Then, we will use the rescaling (17) and the correspondence of solutions to (1) and (22) given by Lemma 2.1 in order to deduce a conditional uniqueness result for (1). The details of this approach will be given in Subsection 3.2 below.*

We will now consider the endpoint case in $2D$. In other words, we set $\alpha = \frac{1}{2}$. Before we prove Proposition 3.2, let us first note some preliminaries. If $\alpha = \frac{1}{2}$, then it is the case that:

$$I(\tau, p) = \sum_{m, n \in \mathbb{Z}^2} \frac{\widetilde{\delta}(\tau + Q(p) - 2Q(n, m)) \cdot \langle p \rangle}{\langle m - p \rangle \cdot \langle n - p \rangle \cdot \langle p - n - m \rangle}$$

where we recall that $\widetilde{\delta} = \chi_{[0,1]}$ is the characteristic function of the interval $[0, 1]$.

Let us now show that $I(\tau, p)$ *is not uniformly bounded in $\tau$ and $p$ in the endpoint case.* In order to do this, let $\kappa \gg 1$ be an integer and let $p := (\kappa, 0)$. We choose

$\tau \in \mathbb{R}$ such that $|\tau + Q(p)| \le 2$. Finally, we consider only the part of the sum $I(\tau, p)$ in which $n = p$. We note that we are then summing over all $m = (m_1, m_2) \in \mathbb{Z}^2$ such that:

$$\tau + Q(p) - 2Q(p, m) \in [0, 1]$$
$$\Rightarrow -2Q(p, m) \in [-2, 3]$$
$$\Rightarrow \theta_1^2 \kappa \cdot m_1 \in [-2, 1].$$

If $\kappa$ is chosen to be sufficiently large, it follows that $m_1 = 0$. In particular, for $p = (\kappa, 0)$ and $|\tau + Q(p)| \le 1$, it follows that:

$$I(\tau, p) \gtrsim \sum_{m_2 \in \mathbb{Z}} \frac{\kappa}{\sqrt{1 + \kappa^2 + m_2^2} \cdot \sqrt{1 + m_2^2}} \gtrsim \int_{-\infty}^{+\infty} \frac{\kappa}{\sqrt{1 + \kappa^2 + x^2} \cdot \sqrt{1 + x^2}} \, dx$$

$$\gtrsim \int_{1 \le |x| \le \kappa} \frac{\kappa}{\sqrt{1 + \kappa^2 + x^2} \cdot \sqrt{1 + x^2}} \, dx \gtrsim \int_{1 \le |x| \le \kappa} \frac{dx}{|x|} \sim \ln \kappa.$$

(38)

**Remark 3.6.** *We know from the calculation in* (38) *and duality that there exists a sequence* $(c_{m_2}) \in \ell^2(\mathbb{Z})$ *such that* $c_{m_2} \ge 0$ *for all* $m_2 \in \mathbb{Z}$ *and:*

$$\sum_{m_2 \in \mathbb{Z}} \frac{\kappa^{\frac{1}{2}}}{(1 + \kappa^2 + m_2^2)^{\frac{1}{4}} \cdot (1 + m_2^2)^{\frac{1}{4}}} \cdot c_{m_2} \gtrsim \sqrt{\ln \kappa} \cdot \Big( \sum_{m_2 \in \mathbb{Z}} c_{m_2}^2 \Big)^{\frac{1}{2}}$$

(39)

In what follows, we will directly use the lower bound obtained in (39) to show that the spacetime estimate from Proposition 3.1 does not hold in the endpoint case $\alpha = \frac{1}{2}$. In other words, we will not directly refer to the fact that $I(\tau, p)$ is not uniformly bounded in $\tau$ and $p$. Unlike in the arguments on the classical torus (c.f. [45, Proposition 3.12]), where it was sufficient to get a pointwise lower bound in the $\tau$ variable (and hence it was possible to directly apply the unboundedness property of $I$), in the setting of the general torus, we need to integrate in the $\tau$ variable over a finite interval and then estimate the obtained integral from below. The reason for this change is the fact that $\mathcal{U}_Q^{(k)}(t)$ *is in general no longer periodic in time.* However, in the analysis, we can reduce to the case where the integrand in the $\tau$ variable is bounded from below uniformly on a finite interval. This is done in (41) below. We can then bound the obtained integral by using (39). This is done in (42) below.

Let $\zeta \in L^1(\mathbb{R})$ be a function such that $\widehat{\zeta} \ge 0$ on $\mathbb{R}$ and $\widehat{\zeta} \ge 1$ on $[-1, 1]$. Such a function can be shown to exist (for example by applying Lemma A.1 below). For $\gamma_0^{(2)} : \Lambda^2 \times \Lambda^2 \to \mathbb{C}$, we compute:

$$\big( \zeta(t) S^{(1, \frac{1}{2})} B_{1,2} \mathcal{U}_Q^{(2)}(t) \, \gamma_0^{(2)} \big)^{\sim}(\tau, p; q)$$
$$= \big( \zeta(t) S^{(1, \frac{1}{2})} B_{1,2}^+ \mathcal{U}_Q^{(2)}(t) \, \gamma_0^{(2)} \big)^{\sim}(\tau, p; q) - \big( \zeta(t) S^{(1, \frac{1}{2})} B_{1,2}^- \mathcal{U}_Q^{(2)}(t) \, \gamma_0^{(2)} \big)^{\sim}(\tau, p; q)$$
$$= \langle p \rangle^{\frac{1}{2}} \cdot \langle q \rangle^{\frac{1}{2}} \cdot \sum_{m, n \in \mathbb{Z}^2} \int [d_{n,m}^+(\tau_1, p; q) - d_{n,m}^-(\tau_1, p; q)] \widehat{\zeta}(\tau - \tau_1) d\tau_1$$

where

$$d_{n,m}^+(\tau_1, p; q) := \widetilde{\delta}\big( \tau_1 - Q(q) + Q(p) - 2Q(n, m) \big) \cdot (\gamma_0^{(2)})^{\frown}(p - m, p - n; q, p - n - m),$$
$$d_{n,m}^-(\tau_1, p; q) := \widetilde{\delta}\big( \tau_1 + Q(p) - Q(q) - 2Q(n, m) \big) \cdot (\gamma_0^{(2)})^{\frown}(p, q - n - m; q - n, q - m).$$

Here, and in the discussion that follows, $\widetilde{\cdot}$ applied to density matrices denotes the spacetime Fourier transform. Let us now choose a specific $\gamma_0^{(2)}$. With the sequence $(c_{m_2})$ as in Remark 3.6, we choose $\gamma_0^{(2)}$ such that:

$$\langle (\kappa, -m_2) \rangle^{\frac{1}{2}} \cdot \langle (0, -m_2) \rangle^{\frac{1}{2}} \cdot (\gamma_0^{(2)})^\frown \Big( (\kappa, -m_2), (0,0); (0,0), (0,-m_2) \Big) = c_{m_2}, \quad (40)$$

for all $m_2 \in \mathbb{Z}$, and such that $(\gamma_0^{(2)})^\frown = 0$ at all frequencies which are not of the form $\Big( (\kappa, -m_2), (0,0); (0,0), (0,-m_2) \Big)$ for some $m_2 \in \mathbb{Z}$.

Let us now fix $\kappa \gg 1$ sufficiently large as above and let us take $\bar{p} := (\kappa, 0), \bar{q} := (0,0)$. Furthermore, we choose $\bar{\tau} \in \mathbb{R}$ such that $|\bar{\tau} - Q(\bar{q}) + Q(\bar{p})| = |\bar{\tau} + Q(\bar{p})| \leq 1$. Since $(\gamma_0^{(2)})^\frown \geq 0$, $\widehat{\zeta} \geq 0$ on $\mathbb{R}$, and $\widehat{\zeta} \geq 1$ on $[-1,1]$, it follows that:

$$\big( \zeta(t) S^{(1,\frac{1}{2})} B_{1,2}^+ \mathcal{U}_Q^{(2)}(t)\, \gamma_0^{(2)} \big)^\thicksim (\bar{\tau}, \bar{p}; \bar{q})$$

$$\gtrsim \int_{|\tau_1 - \bar{\tau}| \leq 1} \sum_{m,n \in \mathbb{Z}^2} d_{n,m}^+(\tau_1, \bar{p}, \bar{q}) \cdot \langle \bar{p} \rangle^{\frac{1}{2}} \cdot \langle \bar{q} \rangle^{\frac{1}{2}} d\tau_1$$

In the above expression, we are integrating over the set of $\tau_1$ for which $|\tau_1 + Q(\bar{p})| \leq 2$. By our choice of $\kappa \gg 1$ as before, it follows that the above integrand is:

$$\gtrsim \sum_{m_2 \in \mathbb{Z}} \int_{|\tau_1 - \bar{\tau}| \leq 1} \kappa^{\frac{1}{2}} \cdot (\gamma_0^{(2)})^\frown \big( (\kappa, -m_2), (0,0); (0,0), (0,-m_2) \big)\, d\tau_1 \qquad (41)$$

$$\gtrsim \sum_{m_2 \in \mathbb{Z}} \frac{\kappa^{\frac{1}{2}} \cdot \langle (\kappa, -m_2) \rangle^{\frac{1}{2}} \cdot \langle (0, -m_2) \rangle^{\frac{1}{2}}}{(1 + \kappa^2 + m_2^2)^{\frac{1}{4}} \cdot (1 + m_2^2)^{\frac{1}{4}}} \cdot (\gamma_0^{(2)})^\frown \big( (\kappa, -m_2), (0,0); (0,0), (0,-m_2) \big)$$

$$= \sum_{m_2 \in \mathbb{Z}} \frac{\kappa^{\frac{1}{2}}}{(1 + \kappa^2 + m_2^2)^{\frac{1}{4}} \cdot (1 + m_2^2)^{\frac{1}{4}}} \cdot c_{m_2}$$

$$\gtrsim \sqrt{\ln \kappa} \cdot \big( \sum_{m_2 \in \mathbb{Z}} c_{m_2}^2 \big)^{\frac{1}{2}} \qquad (42)$$

by using (39) from Remark 3.6. By construction of $\gamma_0^{(2)}$, it follows that the above expression equals $\sqrt{\ln \kappa} \cdot \| S^{(2,\frac{1}{2})} \gamma_0^{(2)} \|_{L^2(\Lambda^2 \times \Lambda^2)}$. Consequently, for all $\bar{\tau}$ with $|\bar{\tau} + Q(\bar{p})| \leq 1$, it is the case that:

$$\big( \zeta(t) S^{(1,\frac{1}{2})} B_{1,2}^+ \mathcal{U}_Q^{(2)}(t)\, \gamma_0^{(2)} \big)^\thicksim (\bar{\tau}, \bar{p}; \bar{q}) \gtrsim \sqrt{\ln \kappa} \cdot \| S^{(2,\frac{1}{2})} \gamma_0^{(2)} \|_{L^2(\Lambda^2 \times \Lambda^2)}. \qquad (43)$$

Let us now look at the contribution from $B_{1,2}^-$.

$$\big( \zeta(t) S^{(1,\frac{1}{2})} B_{1,2}^- \mathcal{U}_Q^{(2)}(t)\, \gamma_0^{(2)} \big)^\thicksim (\bar{\tau}, \bar{p}; \bar{q})$$

$$= \langle \bar{p} \rangle^{\frac{1}{2}} \cdot \langle \bar{q} \rangle^{\frac{1}{2}} \cdot \sum_{m,n \in \mathbb{Z}^2} \int d_{n,m}^-(\tau_1, \bar{p}; \bar{q}) \cdot \widehat{\zeta}(\bar{\tau} - \tau_1)\, d\tau_1$$

By construction, the summand corresponding to $(m,n) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ equals zero unless:

$$\begin{cases} \bar{p} = (\kappa, -\widetilde{m}_2) \\ \bar{q} - m - n = (0,0) \\ \bar{q} - n = (0,0) \\ \bar{q} - m = (0, -\widetilde{m}_2) \end{cases}$$

for some $\widetilde{m}_2 \in \mathbb{Z}$. In particular, it must be the case that $m = n = \widetilde{m}_2 = (0,0)$. Consequently:

$$\big(\zeta(t) S^{(1,\frac{1}{2})} B_{1,2}^- \mathcal{U}_Q^{(2)}(t)\,\gamma_0^{(2)}\big)^{\sim}(\bar{\tau}, \bar{p}; \bar{q})$$
$$= \langle \bar{p} \rangle^{\frac{1}{2}} \cdot \langle \bar{q} \rangle^{\frac{1}{2}} \cdot \int \widetilde{\delta}\big(\tau_1 + Q(\bar{p})\big) \cdot (\gamma_0^{(2)})^{\widehat{}}\big(\bar{p}, (0,0); (0,0), (0,0)\big) \cdot \widehat{\zeta}(\bar{\tau} - \tau_1)\, d\tau_1.$$

Hence:

$$\big|\big(\zeta(t) S^{(1,\frac{1}{2})} B_{1,2}^- \mathcal{U}_Q^{(2)}(t)\,\gamma_0^{(2)}\big)^{\sim}(\bar{\tau}, \bar{p}; \bar{q})\big| \lesssim \|S^{(2,\frac{1}{2})}\gamma_0^{(2)}\|_{L^2(\Lambda^2 \times \Lambda^2)}. \tag{44}$$

From (43) and (44), it follows that:

$$\big|\big(\zeta(t) S^{(1,\frac{1}{2})} B_{1,2} \mathcal{U}_Q^{(2)}(t)\,\gamma_0^{(2)}\big)^{\sim}(\bar{\tau}, \bar{p}; \bar{q})\big| \gtrsim \sqrt{\ln \kappa} \cdot \|S^{(2,\frac{1}{2})}\gamma_0^{(2)}\|_{L^2(\Lambda^2 \times \Lambda^2)}. \tag{45}$$

We recall that $\kappa \gg 1$ is chosen to be sufficiently large $\bar{p} := (\kappa, 0), \bar{q} := (0,0)$ and $|\bar{\tau} + Q(\bar{p})| \le 1$. By Plancherel's Theorem and (45), it follows that:

$$\|\zeta(t) S^{(1,\frac{1}{2})} B_{1,2} \mathcal{U}_Q^{(2)}(t)\,\gamma_0^{(2)}\|_{L^2(\mathbb{R} \times \Lambda \times \Lambda)}$$
$$\sim \|(\zeta(t) S^{(1,\frac{1}{2})} B_{1,2} \mathcal{U}_Q^{(2)}(t)\,\gamma_0^{(2)})^{\sim}\|_{L^2(\mathbb{R})\ell^2(\mathbb{Z}^2 \times \mathbb{Z}^2)}$$
$$\gtrsim \sqrt{\ln \kappa} \cdot \|S^{(2,\frac{1}{2})}\gamma_0^{(2)}\|_{L^2(\Lambda^2 \times \Lambda^2)}. \tag{46}$$

We can now prove Proposition 3.2.

*Proof of Proposition 3.2.* Let $\gamma_0^{(2)}$ be as in (40). We take $\kappa \gg 1$ as above and $\delta$ and $\zeta$ as in Lemma A.1. The lemma follows from (46) as well as from the fact that $\operatorname{supp} \zeta \subseteq [-\delta, \delta]$ and $|\zeta| \lesssim_\delta 1$. We note that $\delta$ is indeed independent of $\kappa$.  □

3.2. **A conditional uniqueness result.** Let us fix $\alpha > \frac{1}{2}$. Let us first consider the following class of density matrices on the torus $\Lambda = \mathbb{T}^2$:

**Definition 3.7.** *Let $\mathcal{A}$ denote the class of all time-dependent sequences $\Gamma(t) = (\gamma^{(k)}(t))$, where each $\gamma^{(k)} : \mathbb{R}_t \times \Lambda^k \times \Lambda^k \to \mathbb{C}$ satisfies:*

    i) $\gamma^{(k)}(t, x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x'_{\sigma(1)}, \ldots, x'_{\sigma(k)}) = \gamma^{(k)}(t, x_1, \ldots, x_k; x'_1, \ldots, x'_k)$,
       *for all $t \in \mathbb{R}, x_1, \ldots, x_k, x'_1, \ldots, x'_k \in \Lambda$, and for all $\sigma \in S^k$.*

    ii) *There exist positive and continuous functions $f, g : \mathbb{R} \to \mathbb{R}$, which are independent of $k$, such that for all $t \in \mathbb{R}$ and for all $j \in \{1, 2, \ldots, k\}$:*

$$\int_{t-g(t)}^{t+g(t)} \|S^{(k,\alpha)} B_{j,k+1}(\gamma^{(k+1)})(s)\|_{L^2(\Lambda^k \times \Lambda^k)} ds \le f^{k+1}(t).$$

For future reference, we will define the class $\mathcal{A}$ analogously on $\mathbb{T}^d$ for $d \ge 2$.

It is possible to argue as in [45, 55, 56] to deduce that:

**Proposition 3.8.** *Solutions to the Gross-Pitaevskii hierarchy on $\Lambda$ with a modified Laplacian (22) are unique in the class $\mathcal{A}$. More precisely, given two solutions in $\mathcal{A}$ with the same initial data, these two solutions have to be equal.*

*Proof.* Let us observe that the boardgame argument from [56] still applies on $\Lambda = \mathbb{T}^2$ with the Laplacian $\Delta_Q$. Namely, in the boardgame argument, one interchanges the different $\Lambda$ variables without interchanging their components. Hence, the fact that the operator $\Delta_Q$ acts differently in each component does not affect the argument. The rest of the proof then follows by using the spacetime bound from Proposition 3.1 analogously as in [45, 55, 56]. We will omit the details.  □

We are more interested in obtaining uniqueness for the Gross-Pitaevskii hierarchy (1) on $\Lambda_2$. Let us recall the scaling transformation given by (17). In this way, we obtain two sequences $\Gamma(t) = (\gamma^{(k)}(t))$, a sequence of density matrices on $\Lambda$ and $\widetilde{\Gamma}(t) = (\widetilde{\gamma}^{(k)}(t))$, a sequence of density matrices on $\Lambda_2$. From Lemma 2.1, we know that $\Gamma(t)$ solves (22) if and only if $\widetilde{\Gamma}(t)$ solves (1). Let us now note another correspondence result between $\Gamma$ and $\widetilde{\Gamma}$:

**Lemma 3.9.** $\Gamma$ *belongs to the class* $\mathcal{A}$ *if and only if* $\widetilde{\Gamma}$ *belongs to the class* $\widetilde{\mathcal{A}}$.

*Proof.* We will show that $\Gamma \in \mathcal{A}$ implies that $\widetilde{\Gamma} \in \widetilde{\mathcal{A}}$. The reverse implication is proved in an analogous way. Suppose that $\Gamma \in \mathcal{A}$. We will show that $\widetilde{\Gamma} \in \widetilde{\mathcal{A}}$. The fact that condition $i$) is satisfied is immediate. We need to check the a priori bound given by $ii$). In order to do this, we compute, for all $t$:

$$
\begin{aligned}
&\big(B_{1,k+1}^{+}\widetilde{\gamma}^{(k+1)}\big)^{\widehat{}}(t,\xi_1,\ldots,\xi_k;\xi_1',\ldots,\xi_k') \\
&= \sum_{\eta,\eta'\in\theta_1\mathbb{Z}\times\theta_2\mathbb{Z}} (\widetilde{\gamma}^{(k+1)})^{\widehat{}}(t,\xi_1-\eta+\eta',\xi_2,\ldots,\xi_k,\eta;\xi_1',\xi_2',\ldots,\xi_k',\eta') \\
&= \sum_{\widetilde{\eta},\widetilde{\eta}'\in\mathbb{Z}^2} \frac{1}{(\theta_1\theta_2)^{2k+2}} \cdot (\gamma^{(k+1)})^{\widehat{}}\Big(t,\widetilde{\xi}-\widetilde{\eta}+\widetilde{\eta}',\widetilde{\xi}_2,\ldots,\widetilde{\xi}_k,\widetilde{\eta};\widetilde{\xi}_1',\widetilde{\xi}_2',\ldots,\widetilde{\xi}_k',\widetilde{\eta}'\Big) \\
&= \frac{1}{(\theta_1\theta_2)^{2k+2}} \big(B_{1,k+1}^{+}\gamma^{(k+1)}\big)^{\widehat{}}(t,\widetilde{\xi}_1,\ldots,\widetilde{\xi}_k;\widetilde{\xi}_1',\ldots,\widetilde{\xi}_k').
\end{aligned}
$$

Here, we have used (19). By an analogous calculation, it follows that:

$$
\begin{aligned}
&\big(B_{j,k+1}\widetilde{\gamma}^{(k+1)}\big)^{\widehat{}}(t,\xi_1,\ldots,\xi_k;\xi_1',\ldots,\xi_k') \\
&= \frac{1}{(\theta_1\theta_2)^{2k+2}} \big(B_{j,k+1}\gamma^{(k+1)}\big)^{\widehat{}}(t,\widetilde{\xi}_1,\ldots,\widetilde{\xi}_k;\widetilde{\xi}_1',\ldots,\widetilde{\xi}_k'),
\end{aligned}
$$

for all $j \in \{1,2,\ldots,k\}$. Consequently,

$$
\|S^{(k,\alpha)}B_{j,k+1}\widetilde{\gamma}^{(k+1)}(t)\|_{L^2(\Lambda_2^k\times\Lambda_2^k)} \leq C_1^k \cdot \|S^{(k,\alpha)}B_{j,k+1}\gamma^{(k+1)}(t)\|_{L^2(\Lambda^k\times\Lambda^k)}
$$

for some constant $C_1$ which depends only on $\alpha, \theta_1, \theta_2$. The claim now follows. $\square$

**Remark 3.10.** *The result of Lemma 3.9 extends in general to $d$ dimensions.*

We can now deduce the main conditional uniqueness result of this section:

**Theorem 3.11.** *Solutions to the Gross-Pitaevskii hierarchy* (1) *on $\Lambda_2$ are unique in the class* $\widetilde{\mathcal{A}}$.

*Proof.* In order to prove this fact, we first recall that $\Gamma(t)$ and $\widetilde{\Gamma}(t)$ are related by the scaling transformation (17). We then apply Lemma 2.1, Lemma 3.9, and Proposition 3.8. The claim follows. $\square$

3.3. **A rigorous derivation of the defocusing cubic nonlinear Schrödinger equation on a general two-dimensional torus.** In this subsection, we will obtain a rigorous derivation of the defocusing cubic nonlinear Schrödinger equation from many-body quantum systems on general two-dimensional tori. Let us recall that, in [55], this result was obtained on the classical torus $\Lambda = \mathbb{T}^2$. We will now extend it to the case of the spatial domain $\Lambda_2$, which can, in particular, be an *irrational torus*. We will prove the following result:

**Theorem 3.12.** *Let $V \in W^{2,\infty}(\Lambda_2)$ be such that $V \geq 0$, $\int_{\Lambda_2} V(x)\,dx = b_0 > 0$, and let $\beta \in (0, \frac{3}{4})$. Suppose that $(\psi_N)_N \in \bigoplus_N L^2(\Lambda_2^N)$ satisfies the properties of bounded energy per particle* (5) *and asymptotic factorization* (6)*. Then, there exists a sequence $N_j \to \infty$ such that for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$:*

$$Tr\big|\gamma_{N_j,t}^{(k)} - |\phi_t\rangle\langle\phi_t|^{\otimes k}\big| \to 0 \ \text{ as } j \to \infty,$$

*where $\phi_t$ solves the defocusing cubic nonlinear Schrödinger equation on $\Lambda_2$ with initial data $\phi$:*

$$\begin{cases} i\partial_t\phi_t + \Delta\phi_t = b_0|\phi_t|^2\phi_t \\ \phi_t\big|_{t=0} = \phi. \end{cases}$$

*Proof.* The proof of Theorem 3.12 follows from the arguments given in [55] combined with the uniqueness result of Theorem 3.11. Namely, we recall that the limiting arguments presented in [55, Sections 3-6] are originally given in the setting of the classical torus. Nevertheless, since these arguments do not depend on any Diophantine properties of the frequencies, but just on Sobolev embedding type results, they directly carry over to the setting of a general torus. More precisely, the only place in the analysis of [55] where the authors use the rationality of the torus is in the proof of the conditional uniqueness result [55, Theorem 7.4], whose analogue on a general torus we have proven in Theorem 3.11 above.

In particular, the analogue of [55, Theorem 5.2] on $\Lambda_2$ holds. This result implies that the density matrices obtained according to the limiting procedure belong to the class $\widetilde{\mathcal{A}}$ for $\alpha < 1$ *with $\widetilde{f}$ and $\widetilde{g}$ being positive constant functions.* Consequently, we obtain that the limit has to be the factorized solution $(|\phi_t\rangle\langle\phi_t|^{\otimes k})_k$. We refer the reader to [55, Section 2] for a more precise outline of this procedure.     $\square$

## 4. The three-dimensional problem

In this section, we will consider the three-dimensional problem. Throughout the section, $\Lambda$ will denote the three-dimensional classical torus $\mathbb{T}^3$. In Subsection 4.1, we will prove a three-dimensional analogue of the sharp spacetime estimate proved in Subsection 3.1 above. In Subsection 4.2, we will prove the corresponding conditional uniqueness result. In particular, this extends the uniqueness result in [45] to general tori. Finally, in Subsection 4.3, we will prove an unconditional uniqueness result, which will allow us to obtain a rigorous derivation of the defocusing cubic NLS on the irrational torus as was done in the setting of the classical torus in [72].

4.1. **The spacetime estimate in three dimensions.** We will now prove a conditional uniqueness result for the three-dimensional problem. In particular, we will extend the uniqueness result of [45] to general three-dimensional tori. We will start by proving a spacetime estimate, which is the three-dimensional analogue of Proposition 3.1.

**Proposition 4.1.** *Let $\alpha > 1$ be given. There exists $C > 0$, which depends only on $\alpha, \theta_1, \theta_2$ such that, for all $k \in \mathbb{N}$, and for all $\gamma_0^{(k+1)} : \Lambda^{k+1} \times \Lambda^{k+1} \to \mathbb{C}$, the following estimate holds:*

$$\|S^{(k,\alpha)}B_{j,k+1}\,\mathcal{U}_Q^{(k+1)}(t)\,\gamma_0^{(k+1)}\|_{L^2([0,1]\times\Lambda^k\times\Lambda^k)} \leq C\|S^{(k+1,\alpha)}\gamma_0^{(k+1)}\|_{L^2(\Lambda^{k+1}\times\Lambda^{k+1})}.$$

As was the case in Proposition 3.1, the range of regularity exponents $\alpha > 1$ in Proposition 4.1 is sharp due to the following result:

**Proposition 4.2.** *For $\kappa \in \mathbb{N}$ sufficiently large, there exists $\gamma_0^{(2)} : \Lambda^2 \times \Lambda^2 \to \mathbb{C}$, such that for $\delta > 0$ sufficiently small:*

$$\|S^{(1,1)} B_{1,2} \mathcal{U}_Q^{(2)}(t)\, \gamma_0^{(2)}\|_{L^2([0,\delta] \times \Lambda \times \Lambda)} \gtrsim_\delta \sqrt{\ln \kappa} \cdot \|S^{(2,1)} \gamma_0^{(2)}\|_{L^2(\Lambda^2 \times \Lambda^2)}.$$

We will first prove Proposition 4.1.

*Proof of Proposition 4.1.* The proof will be similar to that of Proposition 3.1. We will just outline the key differences. As in (26), it suffices to obtain a uniform bound on:

$$I(\tau, p) = \sum_{m,n \in \mathbb{Z}^3} \frac{\widetilde{\delta}\big(\tau + Q(p) - 2Q(n,m)\big) \cdot \langle p \rangle^{2\alpha}}{\langle m-p \rangle^{2\alpha} \cdot \langle n-p \rangle^{2\alpha} \cdot \langle p-n-m \rangle^{2\alpha}}, \tag{47}$$

whenever $\alpha > 1$. Here, the sum is over elements of $\mathbb{Z}^3$ and the notation has been adapted to the three-dimensional setting. Again, since $\alpha > 1 > \frac{3}{4}$, the contributions of $m = 0$ or $n = 0$ to the sum are uniformly bounded.

As before, given $j = (j_1, j_2, j_3) \in \mathbb{N}_0^3$, $\tau \in \mathbb{R}$, and $p \in \mathbb{Z}^2$, we let $E_{\tau,p}(j) \subseteq \mathbb{Z}^3 \setminus \{0\} \times \mathbb{Z}^3 \setminus \{0\}$ be the set of all pairs $(m, n)$ such that:

$$\begin{cases} \tau + Q(p) - 2Q(n,m) \in [0,1] \\ |m-p| \sim 2^{j_1}, |n-p| \sim 2^{j_2}, |p-n-m| \sim 2^{j_3}. \end{cases}$$

The bound that we would like to prove in three dimensions is:

$$\#E_{\tau,p}(j) \lesssim_\epsilon 2^{(2+\epsilon)j_{\min} + (2+\epsilon)j_{\mathrm{med}}} \tag{48}$$

for all $\epsilon > 0$. As in the proof of Proposition 3.1, the claim follows.

Let us now prove (48). We note that the three-dimensional analogue of (35) holds:

$$\#E_{\tau,p}(j) \le c \cdot \Big\| \sum_{\substack{\eta, \eta' \in \mathbb{Z}^3 \\ \eta \in C_{2^{j_1 \vee j_2 + 1}}(0) \cap C_{2^{j_2 \vee j_3 + 2}}(p) \cap C_{2^{j_1 \vee j_3 + 2}}(-p) \\ \eta' \in C_{2^{j_3}}(p) \cap C_{2^{j_1 \vee j_2 + 1}}(2p)}} e^{\frac{1}{2}it\left(Q(\eta) - Q(\eta')\right)} \Big\|_{L_t^1(I)} \tag{49}$$

for some $c > 0$ and for some finite interval $I \subseteq \mathbb{R}$. Similarly as in (35), in (49) $C_{2^j}(q)$ denotes a three-dimensional cube, centered at $q$, whose sides are parallel to the coordinate axes, and who have sidelength $2^{j+1}$. As before, we need to consider several cases, depending on the relative sizes of $j_1, j_2, j_3$. We recall that $j_1, j_2, j_3$ are ordered as $j_{\max} \ge j_{\mathrm{med}} \ge j_{\min}$.

**Case 1:** $j_3 = \min\{j_1, j_2, j_3\}$.

This case follows once we estimate the following analogue of the expression in (36):

$$\int_I \Big| \sum_{\substack{\eta_1, \eta_2, \eta_3 \in \mathbb{Z} \\ \eta_j \in I_j^{\mathrm{med}}}} \sum_{\substack{\eta_1', \eta_2', \eta_3' \in \mathbb{Z} \\ \eta_j' \in I_j^{\min}}} e^{\frac{1}{2}it\left(\theta_1^2 \eta_1^2 + \theta_2^2 \eta_2^2 + \theta_3^2 \eta_3^2 - \theta_1^2 (\eta_1')^2 - \theta_2^2 (\eta_2')^2 - \theta_3^2 (\eta_3')^2\right)} \Big| dt.$$

Here, $I_1^{\mathrm{med}}, I_2^{\mathrm{med}}, I_3^{\mathrm{med}}$ are fixed intervals of size $\sim 2^{j_{\mathrm{med}}}$, and $I_1^{\min}, I_2^{\min}, I_3^{\min}$ are fixed intervals of size $\sim 2^{j_{\min}}$. By Hölder's inequality, this expression is:

$$\leq \Big(\int_I \Big| \sum_{\substack{\eta_1 \in \mathbb{Z} \\ \eta_1 \in I_1^{\mathrm{med}}}} e^{\frac{1}{2} it\theta_1^2 \eta_1^2} \Big|^6 dt\Big)^{\frac{1}{6}} \cdot \Big(\int_I \Big| \sum_{\substack{\eta_2 \in \mathbb{Z} \\ \eta_2 \in I_2^{\mathrm{med}}}} e^{\frac{1}{2} it\theta_2^2 \eta_2^2} \Big|^6 dt\Big)^{\frac{1}{6}} \cdot \Big(\int_I \Big| \sum_{\substack{\eta_3 \in \mathbb{Z} \\ \eta_3 \in I_3^{\mathrm{med}}}} e^{\frac{1}{2} it\theta_3^2 \eta_3^2} \Big|^6 dt\Big)^{\frac{1}{6}} \cdot$$

$$\cdot \Big(\int_I \Big| \sum_{\substack{\eta_1' \in \mathbb{Z} \\ \eta_1' \in I_1^{\min}}} e^{\frac{1}{2} it\theta_1^2 (\eta_1')^2} \Big|^6 dt\Big)^{\frac{1}{6}} \cdot \Big(\int_I \Big| \sum_{\substack{\eta_2' \in \mathbb{Z} \\ \eta_2' \in I_2^{\min}}} e^{\frac{1}{2} it\theta_2^2 (\eta_2')^2} \Big|^6 dt\Big)^{\frac{1}{6}} \cdot \Big(\int_I \Big| \sum_{\substack{\eta_3' \in \mathbb{Z} \\ \eta_3' \in I_3^{\min}}} e^{\frac{1}{2} it\theta_3^2 (\eta_3')^2} \Big|^6 dt\Big)^{\frac{1}{6}}.$$

For each of the six factors, we rescale in time and apply Corollary 2.3 with $p = 6$ to deduce that this product is:

$$\lesssim_\epsilon \big(2^{(4+\epsilon)j_{\mathrm{med}}}\big)^{\frac{1}{6}} \cdot \big(2^{(4+\epsilon)j_{\mathrm{med}}}\big)^{\frac{1}{6}} \cdot \big(2^{(4+\epsilon)j_{\mathrm{med}}}\big)^{\frac{1}{6}} \cdot \big(2^{(4+\epsilon)j_{\min}}\big)^{\frac{1}{6}} \cdot \big(2^{(4+\epsilon)j_{\min}}\big)^{\frac{1}{6}} \cdot \big(2^{(4+\epsilon)j_{\min}}\big)^{\frac{1}{6}}.$$

$$\lesssim 2^{(2+\epsilon)j_{\min} + (2+\epsilon)j_{\mathrm{med}}}$$

for all $\epsilon > 0$. This is a good bound.

**Case 2:** $j_1 = \min\{j_1, j_2, j_3\}$.

Given $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, we let:

$$B_k := [2^{j_1} k_1, 2^{j_1} k_1 + 2^{j_1} - 1] \times [2^{j_1} k_2, 2^{j_1} k_2 + 2^{j_1} - 1] \times [2^{j_1} k_3, 2^{j_1} k_3 + 2^{j_1} - 1].$$

By using the arguments from Case 1, it follows that for all $(k, k') \in \mathbb{Z}^3 \times \mathbb{Z}^3$:

$$\#\big(E_{\tau,p}(j) \cap (B_k \times B_{k'})\big) \lesssim_\epsilon 2^{(4+\epsilon)j_1}$$

for all $\epsilon > 0$. Hence, in order to prove (48) in this case, it suffices to show that:

$$\{(k, k') \in \mathbb{Z}^3 \times \mathbb{Z}^3,\, E_{\tau,p}(j) \cap (B_k \times B_k') \neq \emptyset\} \lesssim 2^{2\min\{j_2,j_3\} - 2j_1}.$$

By applying [45, Lemma 3.6] when $D = 6$, we observe that:

$$\#\{(k, k') \in \mathbb{Z}^2 \times \mathbb{Z}^2,\, E_{\tau,p}(j) \cap (B_k \times B_{k'}) \neq \emptyset\} \lesssim 2^{-6j_1} \big|X_{2^{j_1}}\big|.$$

Here, $X_{2^{j_1}}$ denotes the set of all points in $\mathbb{R}^3 \times \mathbb{R}^3$ which are of distance at most $2^{j_1 + \frac{1}{2}}$ from the set $E_{\tau,p}(j)$. Namely, when $D = 6$, the fact that $E_{\tau,p}(j) \cap (B_k \times B_{k'}) \neq \emptyset$ implies that $(k, k')$ belongs to a $2^{j_1 + \frac{1}{2}}$ thickening of $E_{\tau,p}(j)$.

Thus, we would like to show that:

$$\big|X_{2^{j_1}}\big| \lesssim 2^{4j_1 + 2\min\{j_2,j_3\}}. \tag{50}$$

Let $(m, n) \in E_{\tau,p}(j)$. As in the two-dimensional problem, it is the case that $m$ is allowed to vary over a ball of radius $O(2^{j_1})$, and that for fixed $m$, $n$ varies over a ball of radius $O(2^{\min\{j_2,j_3\}})$. The fact that $\tau + Q(p) - 2Q(n, m) \in [0, 1]$ can be rewritten as:

$$\frac{\tau + Q(p)}{2|m|} - n_1 \cdot \frac{\theta_1^2 m_1}{|m|} - n_2 \cdot \frac{\theta_2^2 m_2}{|m|} - n_3 \cdot \frac{\theta_3^2 m_3}{|m|} \in \Big[0, \frac{1}{2|m|}\Big]$$

Hence, $n$ lies within an $O(1)$ distance of a fixed plane in $\mathbb{R}^3$.

Consequently, given $(x, y)$, which belongs to the thickening $X_{2^{j_1}}$, it follows that $x$ lies in a ball of radius $O(2^{j_1})$, and for a fixed $x$, the $y$ coordinate lies in the intersection of a ball of radius $O(2^{\min\{j_2,j_3\}})$ and an $O(2^{j_1})$ neighborhood of a fixed plane. It follows that the Lebesgue measure of the set to which $x$ is localized is

$\lesssim 2^{3j_1}$ and that, for a fixed $x$, the Lebesgue measure of the set to which $y$ is localized is $\lesssim 2^{j_1 + 2\min\{j_2, j_3\}}$. The bound (50) now follows.

**Case 3:** $j_2 = \min\{j_1, j_2, j_3\}$.

Case 3 is analogous to Case 2 due to the symmetry in $m$ and $n$ in the definition of $I(\tau, p)$ given in (47). □

Let us now prove Proposition 4.2. The proof will be very similar to the proof of Proposition 3.2, so we will just outline the main differences.

*Proof of Proposition 4.2.* If $\alpha = 1$, then:

$$I(\tau, p) = \sum_{m,n \in \mathbb{Z}^3} \frac{\widetilde{\delta}\big(\tau + Q(p) - 2Q(n, m)\big) \cdot \langle p \rangle^2}{\langle m - p \rangle^2 \cdot \langle n - p \rangle^2 \cdot \langle p - n - m \rangle^2}.$$

We will show that $I(\tau, p)$ is not uniformly bounded in $\tau, p$. Similarly as in the $2D$ setting, we let $\kappa \gg 1$ be an integer, we let $p := (\kappa, 0, 0)$, and we consider only the part of the sum $I(\tau, p)$ in which $n = p$. Hence, we sum over all $m = (m_1, m_2, m_3) \in \mathbb{Z}^3$ such that $\tau + Q(p) - 2Q(n, m) \in [0, 1]$. As in the $2D$ setting, it follows that for $\kappa$ sufficiently large, in this sum, $m_1 = 0$. Hence:

$$I(\tau, p) \gtrsim \sum_{m_2, m_3 \in \mathbb{Z}} \frac{\kappa^2}{(1 + \kappa^2 + m_2^2 + m_3^2) \cdot (1 + m_2^2 + m_3^2)}$$

$$\sim \int_{\mathbb{R}^2} \frac{\kappa^2}{(1 + \kappa^2 + |x|^2) \cdot (1 + |x|^2)} \, dx \gtrsim \ln \kappa.$$

The details of the above calculation can be found in the proof of [45, Lemma 3.11]. In particular, by duality, there exists a sequence $(d_{m_2, m_3}) \in \ell^2(\mathbb{Z}^2)$ such that $d_{m_2, m_3} \geq 0$ for all $(m_2, m_3) \in \mathbb{Z}^2$ and:

$$\sum_{m_2, m_3 \in \mathbb{Z}} \frac{\kappa}{\sqrt{1 + \kappa^2 + m_2^2 + m_3^2} \cdot \sqrt{1 + m_2^2 + m_3^2}} \cdot d_{m_2, m_3}$$

$$\gtrsim \sum_{m_2, m_3 \in \mathbb{Z}} \sqrt{\ln \kappa} \cdot \big( \sum_{m_2, m_3} d_{m_2, m_3}^2 \big)^{\frac{1}{2}}.$$

We now choose a specific $\gamma_0^{(2)} : \Lambda^2 \times \Lambda^2 \to \mathbb{C}$, similarly as in (40). In particular, with $(d_{m_2, m_3})$ as above, we choose $\gamma_0^{(2)}$ such that:

$$\langle (\kappa, -m_2, -m_3) \rangle \cdot \langle (0, -m_2, -m_3) \rangle \cdot (\gamma_0^{(2)})\widehat{\,}\Big( (\kappa, -m_2, -m_3), (0, 0); (0, 0), (0, -m_2, -m_3) \Big)$$

$$= d_{m_2, m_3},$$

for all $m_2, m_3 \in \mathbb{Z}$, and such that $(\gamma_0^{(2)})\widehat{\,} = 0$ at all frequencies which are not of the form $\Big( (\kappa, -m_2, -m_3), (0, 0); (0, 0), (0, -m_2, -m_3) \Big)$ for some $m_3 \in \mathbb{Z}$. Let $\zeta \in L^1(\mathbb{R})$ be such that $\widehat{\zeta} \geq 0$ on $\mathbb{R}$ and $\widehat{\zeta} \geq 1$ on $[-1, 1]$, cp. Lemma A.1. Then, by arguing as in the proof of (46)

$$\|\zeta(t) S^{(1,1)} B_{1,2} \mathcal{U}_Q^{(2)}(t) \gamma_0^{(2)}\|_{L^2(\mathbb{R} \times \Lambda \times \Lambda)} \gtrsim \sqrt{\ln \kappa} \cdot \|S^{(2,1)} \gamma_0^{(2)}\|_{L^2(\Lambda^2 \times \Lambda^2)}.$$

The proposition now follows. □

**Remark 4.3.** *More generally, in $d \geq 2$ dimensions, it is the case that for all $k \in \mathbb{N}$ and $\gamma_0^{(k+1)} : \Lambda^{k+1} \times \Lambda^{k+1} \to \mathbb{C}$:*

$$\|S^{(k,\alpha)} B_{j,k+1} \mathcal{U}_Q^{(k+1)}(t) \gamma_0^{(k+1)}\|_{L^2([0,1] \times \Lambda^k \times \Lambda^k)} \leq C \|S^{(k+1,\alpha)} \gamma_0^{(k+1)}\|_{L^2(\Lambda^{k+1} \times \Lambda^{k+1})}. \quad (51)$$

*whenever $\alpha > \frac{d-1}{2}$. Here, $\Lambda = \mathbb{T}^d$ is the classical d-dimensional torus and $\mathcal{U}_Q^{(k)}(t)$ is the free evolution operator obtained from the d-dimensional version of $\Delta_Q$. The constant $C > 0$ depends on d and $\alpha$.*

*Moreover, for $\kappa \in \mathbb{N}$ sufficiently large, there exists $\gamma_0^{(2)} : \Lambda^2 \times \Lambda^2 \to \mathbb{C}$, such that for $\delta > 0$ sufficiently small:*

$$\|S^{(1,\frac{d-1}{2})} B_{1,2} \mathcal{U}_Q^{(2)}(t) \gamma_0^{(2)}\|_{L^2([0,\delta] \times \Lambda \times \Lambda)} \gtrsim_\delta \sqrt{\ln \kappa} \cdot \|S^{(2,\frac{d-1}{2})} \gamma_0^{(2)}\|_{L^2(\Lambda^2 \times \Lambda^2)}. \quad (52)$$

*The estimate (51) is proved by using the geometric arguments given in Proposition 3.1 and Proposition 4.1, and applying Corollary 2.3. The estimate (52) is proved by using the same methods as in Proposition 3.2 and Proposition 4.2, and the fact that:*

$$\sum_{m_2,\ldots,m_d \in \mathbb{Z}} \frac{\kappa^{d-1}}{(1 + \kappa^2 + m_2^2 + \cdots + m_d^2)^{\frac{d-1}{2}} \cdot (1 + m_2^2 + \cdots + m_d^2)^{\frac{d-1}{2}}} \gtrsim \ln \kappa. \quad (53)$$

*The estimate (53) follows by using polar coordinates to see that the left-hand side is:*

$$\sim \int_\mathbb{R} \frac{\kappa^{d-1} \cdot r^{d-2}}{(1 + \kappa^2 + r^2)^{\frac{d-1}{2}} \cdot (1 + r^2)^{\frac{d-1}{2}}} \, dr \gtrsim \int_{1 + r^2 \leq \kappa^2} \frac{\kappa^{d-1} \cdot r^{d-2}}{\kappa^{d-1} \cdot r^{d-1}} \, dr \gtrsim \ln \kappa.$$

*We will omit the details of the proofs of (51) and (52).*

4.2. **A conditional uniqueness result.** We recall the classes $\widetilde{\mathcal{A}}$ and $\mathcal{A}$ given in Definitions 1.1 and 3.7 respectively. Here, we are considering $\widetilde{\mathcal{A}}$ and $\mathcal{A}$ in three dimensions. By arguing as in the proof of Theorem 3.11, we can deduce the following three-dimensional result:

**Theorem 4.4.** *Solutions to the Gross-Pitaevskii hierarchy on $\Lambda_3$ are unique in the class $\widetilde{\mathcal{A}}$ whenever $\alpha > 1$. Moreover, whenever $\alpha \geq 1$, the class $\widetilde{\mathcal{A}}$ is non-empty and it contains the factorized solutions $(|\phi_t\rangle\langle\phi_t|^{\otimes k})_k$.*

*Proof.* The first part of the theorem is proved analogously as in the two-dimensional setting in Theorem 3.11. Namely, we use the rescaling (17) in three dimensions. We then finish the argument as before by using the spacetime estimate given in Proposition 4.1.

For the second part of the theorem, we argue analogously as in the proof of [45, Theorem 1.3]. The analysis carries over to general tori once we recall the trilinear estimate given in [75, Proposition 4.1]. More precisely, it is possible to set $\epsilon = 1$ in [75, Proposition 4.1] and deduce that there exists a universal constant $\delta_0 > 0$ such that for all dyadic integers $N_1, N_2, N_3$ with $N_1 \geq N_2 \geq N_3 \geq 1$, and for any finite interval $I$, it is the case that:

$$\|P_{N_1} e^{it\Delta_Q} f_1 \cdot P_{N_2} e^{it\Delta_Q} f_2 \cdot P_{N_3} e^{it\Delta_Q} f_3\|_{L^2(I \times \Lambda)}$$
$$\lesssim N_2 N_3 \max\{\frac{N_3}{N_1}, \frac{1}{N_2}\}^{\delta_0} \cdot \|P_{N_1} f_1\|_{L^2(\Lambda)} \cdot \|P_{N_2} f_2\|_{L^2(\Lambda)} \cdot \|P_{N_3} f_3\|_{L^2(\Lambda)}. \quad (54)$$

Here, $P_N$ denotes the projection to frequencies $|\xi| \sim N$. The implied constant depends only on the length of the interval $I$. We remark that (54) can also be obtained from the more recent results in [11, 54]. In particular, from (54), it is possible to deduce the analogue of the trilinear estimate given in [51, Proposition 3.5], and following the arguments from [45, Section 5], it follows that the factorized solution $(|\phi_{Q,t}\rangle\langle\phi_{Q,t}|^{\otimes k})_k$ to the Gross-Pitaevskii hierarchy with modified Laplacian $\Delta_Q$ on $\Lambda$ in regularity $\alpha \geq 1$ belongs to the class $\mathcal{A}$. We then apply the scaling transformation (17) and Lemma 2.1 in order to deduce that the factorized solution $(|\phi_t\rangle\langle\phi_t|^{\otimes k})_k$ to the Gross-Pitaevskii hierarchy on $\Lambda_2$ in regularity $\alpha \geq 1$ belongs to the class $\widetilde{\mathcal{A}}$. Here, we note that $\phi_{Q,t}$ solves $i\partial_t\phi_{Q,t} + \Delta_Q\phi_{Q,t} = b_0|\phi_{Q,t}|^2\phi_{Q,t}$ on $\Lambda$ and $\phi_t$ solves $i\partial_t\phi_t + \Delta\phi_t = b_0|\phi_t|^2\phi_t$ on $\Lambda_2$, and the corresponding initial data are related by the scaling transformation (17). $\qquad\square$

**Remark 4.5.** *The result of Theorem 4.4 also holds in the focusing setting.*

**Remark 4.6.** *The conditional uniqueness results in Theorem 3.11 and Theorem 4.4 hold in general on $\Lambda_d$ for $d \geq 2$. More precisely, by using (51) from Remark 4.3, it follows that solutions to (1) are unique in the class $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(\alpha)$, defined on $\Lambda_d$ for $\alpha > \frac{d-1}{2}$.*

4.3. **An unconditional uniqueness result and a rigorous derivation of the defocusing cubic NLS on $\Lambda_3$.** It is also possible to prove an unconditional uniqueness result for the Gross-Pitaevskii hierarchy on $\Lambda_3$ in a class of density matrices in which the allowed range of regularity exponents is $\alpha \geq 1$. As was noted in [72], this type of result does not extend the conditional uniqueness result given in Theorem 4.4. However, as we will see below, it will allow us to obtain a rigorous derivation of the defocusing cubic nonlinear Schrödinger equation on $\Lambda_3$.

Given $\alpha \in \mathbb{R}$, we denote by $\mathfrak{H}^\alpha$ the set of all $(\gamma^{(k)})_k \in \bigoplus_k L^2(\Lambda_3^k \times \Lambda_3^k)$ such that:

   i) $\gamma^{(k)} \in L^2_{sym}(\Lambda_3^k \times \Lambda_3^k)$ and $\gamma(\vec{x}_k, \vec{x}'_k) = \overline{\gamma^{(k)}(\vec{x}'_k; \vec{x}_k)}$ for all $(\vec{x}_k, \vec{x}'_k)$ in $\Lambda_3^k \times \Lambda_3^k$.
   ii) $S^{(k,\alpha)}\gamma^{(k)}$ belongs to the trace class on $L^2(\Lambda_3^k \times \Lambda_3^k)$.
   iii) There exists $M > 0$, which is independent of $k$, such that $Tr\big(|S^{(k,\alpha)}\gamma^{(k)}|\big) \leq M^{2k}$.

The unconditional uniqueness result that we prove is the following:

**Theorem 4.7.** *Let $T > 0$ be fixed. If $(\gamma^{(k)}(t))_k \in L^\infty_{t\in[0,T]}\mathfrak{H}^1$ is a mild solution to the Gross-Pitaevskii hierarchy on $\Lambda_3$, for which there exist $\Gamma_{N,t} \in L^2_{sym}(\Lambda_3^N \times \Lambda_3^N)$, which are non-negative as operators and whose trace is equal to 1 such that:*

$$Tr_{k+1,\ldots,N}\Gamma_{N,t} \rightharpoonup^* \gamma^{(k)}(t)$$

*as $N$ tends to infinity in the weak-$*$ topology of the trace class on $L^2_{sym}(\Lambda_3^k)$. Then, the solution $(\gamma^{(k)}(t))_k$ is uniquely determined by the initial data $(\gamma_0^{(k)})_k$.*

For the precise terminology and notation, we refer the reader to [16] and [72].

In order to prove Theorem 4.7, one argues analogously as in the proof of [72, Theorem 4.6]. In other words, one applies the Weak Quantum de Finetti Theorem as in [16]. The only difference from [72] in the case of the irrational torus $\Lambda_3$ is the fact that one has to use a rescaled version of (54) on $\Lambda_3$. Such a trilinear estimate allows us to prove the analogue on $\Lambda_3$ of [72, Proposition 3.1] and hence the analogues of [72, inequalities (41) and (42)], which were crucial in the derivation

analysis on $\mathbb{T}^3$. The further details of the proof of Theorem 4.7 are then the same as in the setting of the classical torus [72, Section 4]. For a more detailed discussion of this approach in the context of $\mathbb{R}^3$, we refer the reader to [16, Sections 4-8]. By arguing as in [72], we can deduce the following derivation result:

**Theorem 4.8.** *Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a non-negative, smooth, compactly supported function with $\int_{\mathbb{R}^3} V(x)\, dx = b_0 > 0$, and let $\beta \in (0, \frac{3}{5})$ be given. Suppose that $(\psi_N)_N \in \bigoplus_N L^2(\Lambda_3^N)$ satisfies the assumption of bounded energy per particle (5) and that of asymptotic factorization (6). Then, there exists a sequence $N_j \to \infty$ such that for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$:*

$$Tr\big|\gamma_{N_j,t}^{(k)} - |\phi_t\rangle\langle\phi_t|^{\otimes k}\big| \to 0 \ \ as \ j \to \infty,$$

*where $\phi_t$ solves the defocusing cubic nonlinear Schrödinger equation on $\Lambda_3$ with initial data $\phi$:*

$$\begin{cases} i\partial_t \phi_t + \Delta\phi_t = b_0|\phi_t|^2\phi_t \\ \phi_t\big|_{t=0} = \phi. \end{cases}$$

Let us note that the limiting arguments in [34] carry over to the setting of $\Lambda_3$. Namely, these arguments rely on Sobolev embedding results and do not use any Diophantine properties. We can then combine the analogue on $\Lambda_3$ of the result of [34] with the unconditional uniqueness result from Theorem 4.7, and argue as in [72, Section 5] in order to deduce Theorem 4.8. We will omit the details of the proof, and we will refer the interested reader to [72] for a more detailed discussion in the case of the classical torus.

**Remark 4.9.** *It is possible to redo the analysis of this subsection in the two-dimensional setting if, instead of [75, Proposition 4.1], we use [75, Proposition 3.3]. In this way, we can also obtain a derivation of the defocusing cubic NLS on $\Lambda_3$, but by using an unconditional uniqueness result. We will not pursue this approach here. We recall that we have already obtained a derivation of the defocusing cubic NLS on $\Lambda_2$ in Theorem 3.12 by using the conditional uniqueness result given in Theorem 3.11.*

## 5. A consequence of the spacetime estimates; local-in-time solutions to the Gross-Pitaevskii hierarchy on general tori

In this section, let us recall that the spacetime estimates given in Theorems 3.1 and 4.1, as well as in (51) of Remark 4.3 allow us to construct local-in-time solutions to the Gross-Pitaevskii hierarchy with modified Laplacian $\Delta_Q$ on $\Lambda = \mathbb{T}^d$ for general $d \geq 2$. This is a direct consequence of the truncation method from the work of T. Chen and Pavlović [21], which relies on the combinatorial boardgame argument and on the spacetime estimate. Given $\alpha, \xi > 0$, we recall the definition of the space $\mathcal{H}_\xi^\alpha(\mathbb{T}^d) \subseteq \bigoplus_k L^2(\mathbb{T}^d \times \mathbb{T}^d)$, first introduced in [18]:

$$\big\|(\gamma_0^{(k)})_k\big\|_{\mathcal{H}_\xi^\alpha(\mathbb{T}^d)} := \sum_{k=1}^\infty \xi^k \cdot \|\gamma_0^{(k)}\|_{H^\alpha\left((\mathbb{T}^d)^k \times (\mathbb{T}^d)^k\right)}. \tag{55}$$

In particular, we can deduce that the following result holds:

**Proposition 5.1.** *Let $d \geq 2$ be given and we consider $\Lambda = \mathbb{T}^d$. Let us fix $\alpha > \frac{d-1}{2}$, and let $\alpha_0 > \alpha$. Furthermore, let $\xi, \xi' > 0$. Suppose that $(\gamma_0^{(k)})_k \in \mathcal{H}_{\xi'}^{\alpha_0}(\Lambda)$. Then, if*

$\frac{\xi}{\xi'}$ is sufficiently small depending on $d, \alpha, \alpha_0, \theta_1, \ldots, \theta_d$, there exists $T > 0$ depending on $d, \alpha, \alpha_0, \xi, \xi', \theta_1, \ldots, \theta_d$ and $(\gamma^{(k)})_k = (\gamma^{(k)}(t))_k \in L^\infty_{[0,T]} \mathcal{H}^\alpha_\xi(\Lambda)$, such that for all $k \in \mathbb{N}$:

$$\left\| \gamma^{(k)}(t) - \mathcal{U}_Q^{(k)}(t)\gamma_0^{(k)} + ib_0 \int_0^t \mathcal{U}_Q^{(k)}(t) B^{(k+1)} \gamma^{(k+1)}(s)\, ds \right\|_{L^\infty_{[0,T]} \mathcal{H}^\alpha_\xi(\Lambda)} = 0. \quad (56)$$

We interpret (56) as $(\gamma^{(k)})_k$ being a local-in-time solution of (22).

Furthermore, we can modify the definition given in (55) and define $\mathcal{H}^\alpha_\xi(\Lambda_d)$ on the general $d$-dimensional torus $\Lambda_d$. By using the scaling (17) and Lemma 2.1, we can deduce from Proposition 5.1 the following:

**Corollary 5.2.** *Let* $\alpha, \alpha_0, \xi, \xi'$ *be as in Proposition 5.1 and let* $(\widetilde{\gamma}_0^{(k)})_k \in \mathcal{H}^{\alpha_0}_\xi(\Lambda_d)$ *be given. Then, for the* $T$ *as in Proposition 5.1, there exists* $(\widetilde{\gamma}^{(k)})_k = (\widetilde{\gamma}^{(k)}(t))_k \in L^\infty_{[0,T]} \mathcal{H}^\alpha_\xi(\Lambda_d)$, *such that for all* $k \in \mathbb{N}$:

$$\left\| \widetilde{\gamma}^{(k)}(t) - \mathcal{U}^{(k)}(t)\widetilde{\gamma}_0^{(k)} + ib_0 \int_0^t \mathcal{U}^{(k)}(t) B^{(k+1)} \widetilde{\gamma}^{(k+1)}(s)\, ds \right\|_{L^\infty_{[0,T]} \mathcal{H}^\alpha_\xi(\Lambda_d)} = 0. \quad (57)$$

We interpret (57) as $(\widetilde{\gamma}^{(k)})_k$ being a local-in-time solution of (1).

## Appendix A. Auxiliary results

Let us recall the following result from Fourier analysis:

**Lemma A.1.** *For* $\delta > 0$ *sufficiently small, there exists* $\zeta \in C_0^\infty(\mathbb{R})$, *such that*

    i) $\operatorname{supp} \zeta \subseteq [-\delta, \delta]$
    ii) $\widehat{\zeta} \geq 0$ *on* $\mathbb{R}$
    iii) $\widehat{\zeta} \geq 1$ *on* $[-1, 1]$.

*Proof.* Let $\phi_1 := \frac{1}{2}\chi_{[-2\pi,2\pi]}$, where $\chi_{[-2\pi,2\pi]}$ denotes the characteristic function of the interval $[-2\pi, 2\pi]$. We compute:

$$\widehat{\phi_1}(\xi) = \frac{\sin(2\pi\xi)}{\xi}.$$

In particular, $\widehat{\phi_1} \geq 2$ on $[-C, C]$, for some fixed $C > 0$. By density, we can find $\phi_2 \in C_0^\infty(\mathbb{R})$ such that $\|\phi_2 - \phi_1\|_{L^1} \leq 1$. Then $\|\widehat{\phi_2} - \widehat{\phi_1}\|_{L^\infty} \leq \|\phi_2 - \phi_1\|_{L^1} \leq 1$, and hence $\widehat{\phi_2} \geq 1$ on $[-C, C]$. Let:

$$\phi_3 := \phi_2 * \mathcal{F}^{-1}(\overline{\widehat{\phi_2}}).$$

Here, $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then $\phi_3 \in C_0^\infty(\mathbb{R})$ and $\widehat{\phi_3} = |\widehat{\phi_2}|^2$ is non-negative on $\mathbb{R}$ and greater than or equal to 1 on $[-C, C]$. We now choose $m > 0$ sufficiently small such that:

$$\operatorname{supp} \phi_3\left(\frac{\cdot}{m\delta}\right) \subseteq [-\delta, \delta].$$

Let $\zeta := \frac{1}{m\delta}\phi_3(\frac{\cdot}{m\delta})$. Then $\zeta \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} \zeta \subseteq [-\delta, \delta]$ and:

$$\widehat{\zeta}(\xi) = \widehat{\phi_3}(m\delta\xi).$$

Hence, $\zeta \geq 0$ on $\mathbb{R}$ and $\widehat{\zeta} \geq 1$ whenever $|\xi| \leq \frac{C}{m\delta}$. We then choose $\delta > 0$ sufficiently small so that $\frac{C}{m\delta} \geq 1$. $\qquad\square$

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