Stochastic PDEs.

1st part: Week 1: White noise, Gaussian measures

2: Heat eq°, Additive stochastic PDEs.

3/4: Stochastic integrals, Multiplicative SPDEs

2nd part: Week 5 to 8

* prop. of solutions to multi. stoch. heat eq°
* CVce of discrete models
* Approximations of SPDEs.

3rd week, 12th May 2–3 pm only.

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Motivating example.

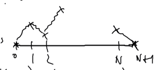
chain of N oscillators $X_1(t), \dots, X_N(t)$

with nearest neighbours interaction

random ext. forcing: Brown. motion

B_i, $i = 1 \dots N$

indpdt Brownian Motions



$$dX_i(t) = (X_{i+1}(t) - X_i(t))dt + (X_{i-1}(t) - X_i(t))dt + dB_i(t)$$
$$= \Delta X_i(t)dt + dB_i(t)$$

Q°: What happens when $N \to \infty$?

$\to$ Need a rescaling: $u(t,x) = \frac{1}{\sqrt{N}} X_{\lfloor xN \rfloor}(tN^2)$, $t \geq 0$, $x \in [0,1]$

$$d_t u(t,x) = \frac{N^2}{\sqrt{N}} d_t X_{\lfloor xN \rfloor}(tN^2)$$

$$\partial_x^2 u(t,x) \simeq \frac{N^2}{\sqrt{N}} \Delta X_{\lfloor xN \rfloor}(tN^2)$$

$$d_t u(t,x) = \partial_x^2 u(t,x) + \underbrace{N^{3/2} d_t B_{\lfloor xN \rfloor}(tN^2)}$$

Behaviour of $N^{3/2} d_t B_{\lfloor xN \rfloor}(tN^2)$

Scaling ppty of BM $\left(\frac{1}{N} B_i(tN^2), t \geq 0\right) \overset{(d)}{=} (W_i(t), t \geq 0)$ BM

$\to$ $\left(\frac{N^2}{N} d_t B_i(tN^2), t \geq 0\right) \overset{(d)}{=} (d_t W_i(t), t \geq 0)$

<u>So</u> $N^{3/2} d_t B_{\lfloor xN \rfloor}(tN^2) \overset{(d)}{=} \left(\sqrt{N} d_t W_{\lfloor xN \rfloor}(t), t \geq 0\right)$

Take $\varphi, \psi \in \mathcal{C}_c^\infty((0,\infty) \times (0,1))$

$$\mathbb{E}\left[\langle \varphi, \sqrt{N} d_t W_{\lfloor xN \rfloor} \rangle \langle \psi, \sqrt{N} d_t W_{\lfloor xN \rfloor}(t) \rangle\right]$$
$$= \mathbb{E} \int_{x=0}^1 \int_{y=0}^1 \int_{t=0}^\infty \int_{s=0}^\infty \varphi(x,t)\psi(y,s) \, N \cdot d_t W_{\lfloor xN \rfloor}(t) \, d_t W_{\lfloor yN \rfloor}(s)$$
$$\approx \mathbb{E} \sum_{i=1}^N \sum_{j=1}^N \int_{t,s} \frac{1}{N^2} \varphi\left(\frac{i}{N}, t\right) \psi\left(\frac{j}{N}, s\right) N \, d_t W_i(t) \, d_t W_j(s)$$
$$\simeq \sum_{i=1}^N \int_{t=0}^\infty \varphi\left(\frac{i}{N}, t\right) \psi\left(\frac{i}{N}, t\right) \frac{N}{N^2} \mathbb{E}\left[d_t W_i(t) \, d_t W_i(t)\right]$$
$$\simeq \frac{1}{N} \sum_{i=1}^N \int_{t=0}^\infty \varphi\left(\frac{i}{N}, t\right) \psi\left(\frac{i}{N}, t\right) dt \simeq \langle \varphi, \psi \rangle_{L^2}$$

<u>Guess</u> limiting eq°

$$\partial_t u = \partial_x^2 u + \xi + F(u)$$

ξ: Gaussian, covariance

$$\mathbb{E}\left[\langle \xi, \varphi\rangle \langle \xi, \psi\rangle\right] = \langle \varphi, \psi\rangle_{L^2(\mathbb{R}_+ \times (0,1))}$$

Today: def of ξ white noise

<u>def</u>: The white noise is a linear map ξ from $L^2(\mathbb{R}^d, dx)$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that: $\forall f \in L^2(\mathbb{R}^d, dx)$

$$\xi(f) = \langle \xi, f\rangle \sim \mathcal{N}(0, \|f\|_{L^2}^2)$$

<u>Prop</u>: (i) Isometry from $L^2(\mathbb{R}^d, dx)$ into $L^2(\Omega)$ preserves the inner product.

(ii) $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $\xi(A) = \xi(\mathbb{1}_A)$ if $\mathrm{Leb}(A) < \infty$

Then, if $(A_n)_{n\geq 1}$ seq. of disjoint Borel sets then

$$\xi\Big(\bigcup_{n\geq 1} A_n\Big) \overset{L^2(\Omega)}{=} \sum_n \xi(A_n)$$

<u>Proof</u> (i) Isometry ✓

Inner product: $f, g \in L^2(\mathbb{R}^d)$

$$\xi(f+g) = \xi(f) + \xi(g)$$

$$\mathbb{E}\left[(\xi(f+g))^2\right] = \|f+g\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + 2\langle f, g\rangle$$

$$\mathbb{E}\left[(\xi(f) + \xi(g))^2\right] = \mathbb{E}[\xi(f)^2] + \mathbb{E}[\xi(g)^2] + 2\mathbb{E}[\xi(f)\xi(g)]$$

$$\mathbb{E}[\xi(f)\xi(g)] = \langle f, g\rangle_{L^2} \quad ✓$$

(ii) $(A_n)_{n\geq 1}$ disjoint Borel sets of $\mathbb{R}^d$

$\mathbb{1}_{A_n} \perp \mathbb{1}_{A_m}$ $n \neq m$

$$\mathbb{E}\left[\Big(\sum_1^N \xi(A_n)\Big)^2\right] = \sum_1^N \mathbb{E}\left(\xi(A_n)^2\right) = \sum_1^N \mathrm{Leb}(A_n)$$

if $\mathrm{Leb}\Big(\bigcup_{n\geq 1}^{\infty} A_n\Big) < \infty$.

$$\mathbb{E}\left[\Big(\sum_N^{\infty} \xi(A_n)\Big)^2\right] < \infty$$

By linearity, I know $\xi\Big(\bigcup_1^N A_n\Big) = \sum_1^N \xi(A_n)$

Pass to the limit on $\mathbb{E}\left(\Big(\sum_N^{\infty} \xi(A_n)\Big)^2\right)$ as $N \to \infty$ $\square$.

<u>References</u>:

* Introduction to SPDEs, Martin Hairer
* St Flour lecture notes on SPDEs, Walsh 1984.
* Stochastic eq° in infinite dimension, Da Prato - Zabczyk.

There are 2 viewpoints for considering random processes:

1] A collection of random variables

$X_t, t \in \mathcal{T}$,

$\forall t \in \mathcal{T} \quad X_t : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Endowed with the Kolmogorov's σ-algebra, ie, smallest σ-algebra s.t. $\forall n \geq 1, \forall t_1, \dots, t_n \in \mathcal{T}$

$(X_{t_1}, \dots, X_{t_n}) : \Omega \longrightarrow \mathbb{R}^n$

is measurable.

$\rightarrow$ Finite-dimensional marginals are sufficient to characterize completely the law of a random process w.r.t. Kolmogorov's σ-alg

$\mathcal{C} = \left\{ (X_{t_1}, \dots, X_{t_n})^{-1}(A) : \begin{array}{l} n \geq 1, t_1, \dots, t_n \in \mathcal{T} \\ A \in \mathcal{B}(\mathbb{R}^n) \end{array} \right\}$

stable under intersections.

So, if μ and ν are 2 laws of random processes

then $\mathcal{M} = \{ A \in \Omega, \ \mu(A) = \nu(A) \}$

if $\mathcal{M} \supset \mathcal{C}$, then $\mathcal{M} \supset \sigma(\mathcal{C})$.

2] Random process as a law on a space of functions of $t \in \mathcal{T}$.

Here ξ is a process indexed by L^2

Existence of the white noise:

- Kolmogorov's extension th ✓
- More explicit: take $(e_n)_{n \geq 1}$ orthonormal basis of $L^2(\mathbb{R}^d, dx)$, take $(Z_n)_{n \geq 1}$ iid $\mathcal{N}(0,1)$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

Set $\langle \xi, f \rangle = \sum_{n \geq 1} \langle e_n, f \rangle Z_n$

Easy to check that ξ is a white noise. Informally $\xi = \sum_{n \geq 1} Z_n \cdot e_n$

<u>Examples</u>:

$d = 1$, $\left(\xi(\mathbb{1}_{[0,t]}), t \geq 0 \right)$ is a Brownian Motion.

$d = 2$, $\left(\xi(\mathbb{1}_{[0,X] \times [0,Y]}), X \geq 0, Y \geq 0 \right)$ Brownian sheet.

<u>SPDEs</u> we will be interested in the case where $L^2(\underbrace{(0,\infty)}_{\text{time}} \times \underbrace{\mathbb{R}^d}_{\text{space}}, dt \times dx)$

ξ white noise on $L^2((0,\infty) \times \mathbb{R}^d)$: will be called space-time white noise.

Take $(e_n)_{n \geq 1}$ orthonormal basis $L^2(\mathbb{R}^d, dx)$

$W_t^n := \xi(\mathbb{1}_{[0,t]} \times e_n), \quad n \geq 1$

$(W^n, n \geq 1)$ is a seq of indepdt BM.

<u>Prop</u>: 1] $\forall f \in L^2(\mathbb{R}^d)$, $W_t(f) = \sum_{n \geq 1} W_t^n \langle e_n, f \rangle$ is a well-defined r.v.

2] $\mathbb{E}[W_t(f) W_s(g)] = t \wedge s \langle f, g \rangle_{L^2}$

<u>Proof</u> (1) $\sum_{n=1}^{N} W_t^n \langle f, e_n \rangle$

$$\mathbb{E}\left[\left(\sum_{N}^{N'} W_t^n \langle f, e_n\rangle\right)^2\right] = \sum_{n=N}^{N'} t \langle f, e_n\rangle^2$$

Since $f \in L^2$, this is a Cauchy Seq. in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

(2) $\mathbb{E}\left[W_t(f) \cdot W_s(g)\right] = \mathbb{E}\left(\sum_n \sum_m W_t^n W_s^m \langle f, e_n\rangle\langle g, e_m\rangle\right]$

$$= \sum_n t \wedge s \langle f, e_n\rangle \langle g, e_n\rangle$$

$$= t \wedge s \langle f, g\rangle_{L^2}$$

$$W_t(f) = \sum_{n \geq 1} W_t^n \langle f, e_n\rangle.$$

One would like to write $W_t = \sum_{n\geq 1} W_t^n e_n$ ....

$\rightarrow$ $(W_t, t \geq 0)$ cylindrical Wiener process.

Want to have more info about the regularity of ξ (or W_t).

White nois on $L^2(\mathbb{R}^d, dx)$, $(e_n)_{n\geq 1}$ basis

$$\mathcal{H}^\alpha = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \sum_{n\geq 1} |\langle f, e_n\rangle|^2 n^{2\alpha} < \infty \right)$$

$\alpha \in \mathbb{R}$ Sobolev space.

$\mathcal{H}^0 = L^2(\mathbb{R}^d)$; $\mathcal{H}^\alpha$ more regular $\alpha > 0$

less regular $\alpha < 0$.

<u>Prop</u> * ξ admits a modification which is almost surely in $\mathcal{H}^{-\alpha}$, $\alpha > 1/2$.

* $\mathbb{P}(\xi \in L^2) = 0$.

<u>Proof</u>:

$\lambda > 0$, $\mathbb{E}\left[e^{-\lambda \sum_1^N \xi(e_n)^2}\right] = e^{-\lambda^2 \frac{N}{2}}$

Then $N \to \infty$

$$\mathbb{P}\left(\sum_1^N \xi(e_n)^2 < \infty\right) = 0$$

L^2-norm of ξ is ∞. a.s.

Same calculations

$$\mathbb{E}\left[e^{-\lambda \sum_1^N \xi(e_n)^2 n^{-2\alpha}}\right] = e^{-\lambda \sum_1^N n^{-2\alpha}}$$

$\alpha > 1/2$

Pass to the limit $N \to \infty$,

<u>So</u>: Pass to the limit $\lambda \to 0$.

$$\mathbb{P}\left(\sum_1^\infty \xi(e_n)^2 n^{-2\alpha} < \infty\right) = 1.$$

On an event: $\tilde{\Omega} = \{\omega : \sum_1^\infty \xi(e_n)^2 n^{-2\alpha} < \infty\}$ of proba 1, I can define

$$\tilde{\xi} := \begin{cases} \sum_1^\infty \xi(e_n) \cdot e_n & \text{on } \tilde{\Omega} \\ 0 & \text{on } \tilde{\Omega}^c \end{cases}$$

Almost surely, $\|\xi\|_{\mathcal{H}^{\alpha}} < \infty$.

$\sum_1^\infty n^{-2\alpha} < \infty$. The structure behind is that the embedding of $L^2 \hookrightarrow \mathcal{H}^{-\alpha}$ is Hilbert-Schmidt.

Ccl: ξ can be viewed as a random element of a space of distributions.
We get the 2^{nd} viewpoint,
We get into the topic of Gaussian measures.

Def: (Gaussian Measures). Let $\mathcal{B}$ be a Banach space, separable. Then μ is a Gauss. meas. on $\mathcal{B}$ if ⊛ $\forall \ell \in \mathcal{B}^*$ (space of continuous linear maps on $\mathcal{B}$) the pushforward of μ through ℓ is Gaussian. ⊛ if μ is a proba measure on $\mathcal{B}$ and

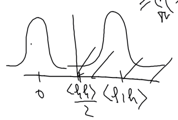
In our setting, the white noise can be seen as a Gaussian measure on $\mathcal{H}^{-\alpha}$.
Then $\mathcal{B}^* = \mathcal{H}^{+\alpha}$.
$\forall \ell \in \mathcal{H}^{+\alpha}$, $\ell^* \mu$ is a Gaussian measure on $\mathbb{R}$, variance given by $\|\ell\|_{L^2}^2$

Th (Cameron-Martin Theorem).
Let $h \in \mathcal{H}^{-\alpha}$. $T_h : \mathcal{H}^{-\alpha} \to \mathcal{H}^{-\alpha}$, $f \mapsto f+h$

$$\mu \sim T_h^* \mu \quad \text{iff} \quad h \in L^2(\mathbb{R}^d).$$

Proof:
* Let $h \notin L^2$. Then $\ell = \sum_1^{n_0} \ell(n) \cdot e_n$
where $\ell(n) := \dfrac{\langle h, e_n \rangle}{\left(\sum_1^{n_0} h(n)^2\right)^{1/2}}$

Then, $\ell^* \mu \sim \mathcal{N}(0, \|\ell\|_{L^2}^2 = 1)$
want to show $\ell^* T_h^* \mu \sim \mathcal{N}(\langle \ell, h \rangle, \|\ell\|_{L^2}^2 = 1)$
$d_{TV}(\mu, T_h^* \mu) = 1$
$\geq d_{TV}(\ell^* \mu, \ell^* T_h^* \mu)$



$\xrightarrow[n_0 \to \infty]{} 1$

* if $h \in L^2$, $h^* \mu$ law of $\langle \xi, h \rangle \sim \mathcal{N}(0, \|h\|_{L^2}^2)$

$e^{i\langle h, \cdot \rangle}$ is integrable against μ.

$$\underbrace{e^{i\langle h, \cdot \rangle - \frac{1}{2}\|h\|_{L^2}^2}}_{\text{Radon-Nikodym derivative}} d\mu = d\mu_h$$

Compute $\hat{\mu}_h$ and $\widehat{T_h^* \mu}$.