1st lecture:

$\lambda>0$, $\mathbb{E}\left[e^{-\lambda\sum_1^N \xi(e_n)^2}\right] = \left(\frac{1}{2\lambda+1}\right)^N$

rest of the argument works the same

Linear SPDE

$$\partial_t u = \tfrac{1}{2}\partial_x^2 u + \xi \qquad x\in(0,1),\ t>0$$

ξ space-time W.N.

Weak form

$W_t(\varphi) = \langle \xi, \mathbb{1}_{[0,t]}\varphi\rangle$

$\varphi \in C_c^\infty(0,1)$

$$\langle u_t,\varphi\rangle = \langle u_0,\varphi\rangle + \tfrac{1}{2}\int_0^t \langle u_s,\varphi''\rangle ds + W_t(\varphi)$$

Orthonormal basis of $L^2(0,1)$

$e_n(x) = \sqrt{2}\sin(n\pi x) \quad x\in(0,1),\ n\geq 1$

Taking $\varphi = e_n$ $\quad \hat{u}_t(n) = \langle u_t, e_n\rangle$

$$\hat{u}_t(n) = \hat{u}_0(n) - \frac{(n\pi)^2}{2}\int_0^t \hat{u}_s(n)ds + W_t(e_n)$$

Since $(W_t(e_n), t\geq 0)_{n\geq 1}$ iid sequence of B.M.

This way, we get a collection of SDE's driven by indep't BMs.

$\forall n\geq 1$, $(\hat{u}_t(n), t\geq 0)$ is an Ornstein-Uhlenbeck process.

$$dX_t = -\lambda X_t dt + dB_t, \quad X_0 = x\in\mathbb{R}$$

Solution to such an SDE:

$$X_t = e^{-\lambda t}x + \int_0^t e^{-\lambda(t-s)} dB_s$$

(Apply Itô to $e^{\lambda t}\cdot X_t$)

Fact: $X_t \sim \mathcal{N}\left(e^{-\lambda t}x, \frac{1-e^{-2\lambda t}}{2\lambda}\right)$

$\forall x\in\mathbb{R}$, $X_t \xRightarrow[t\to\infty]{(d)} \mathcal{N}\left(0,\frac{1}{2\lambda}\right)$

Inv. measure for $(X_t, t\geq 0)$:

μ on $\mathbb{R}$ $\quad \int P_t f(x)\,\mu(dx) = \int f(x)\,\mu(dx)$

where P_t semigroup, $f\in C_b(\mathbb{R})$.

Prop: $\mu_\lambda = \mathcal{N}\left(0,\frac{1}{2\lambda}\right)$ is the unique inv. measure for X.

Proof $\quad \forall x\in\mathbb{R}$, $\int f\mu_\lambda = \lim_{s\to\infty} P_{t+s}f(x)$

$= \lim_{s\to\infty} P_s\circ P_t f(x) = \int P_t f(x)\mu_\lambda(dx)$

So μ_λ invariant.

Uniq: If ν is inv,

$$\int f\,d\nu = \int P_t f(x)\,\nu(dx) \xrightarrow[t\to\infty]{} \int_x\int_y f(y)\,d\mu_\lambda(y)\,\nu(dx) = \int f\,d\mu_\lambda \quad \square$$

$\forall n\geq 1$, $(\hat{u}_t(n), t\geq 0)$ admits $\mu_{\frac{(n\pi)^2}{2}}$ its inv. meas.

$\mu = \bigotimes_{n\geq 1}\mu_{\frac{(n\pi)^2}{2}}$ meas. on $L^2(0,1)$.

inv for SPDE $\partial_t u = \frac{1}{2}\partial_x^2 u + \xi$

Exercise: Show that $\mu \sim$ law of the Brownian bridge on $(0,1)$

<u>Goal</u> $\partial_t u = \frac{1}{2}\partial_x^2 u + u \cdot \xi$

→ Stochastic integrals.

Even for SDE's, $\partial_t X = X_t \cdot dB_t$

Issue with $\int B_s \, dB_s$ $B \sim \mathcal{C}^{\frac{1}{2}-}$

<u>Th (Young)</u> $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \longrightarrow \mathcal{C}^{\alpha \wedge \beta}$

$\mathcal{C}^{\alpha}$: Hölder space reg α $(dB_s, B_s) \longmapsto B_s \, dB_s$

this map is continuous iff $\boxed{\alpha + \beta > 0}$.

In the case of BM, $\alpha = -\frac{1}{2}-$ $\beta = \frac{1}{2}-$

$\alpha + \beta < 0$.

No canonical way of defining $B_s \, dB_s$

→ Need another way to define $\int B_s \, dB_s$

→ Different ways of doing that: Itô, Stratonovich

ξ white noise on $\mathbb{R}_+ \times \mathbb{R}^d$

$W_t = \sum_{n \geq 1} W_t^n e_n$ $(e_n)_{n \geq 1}$ basis $L^2(\mathbb{R}^d)$

$W_t(\cdot) = \xi(\mathbb{1}_{[0,t]} \cdot)$

$(\mathcal{F}_t, t \geq 0)$ natural filtration associated to $(W_t, t \geq 0)$.

<u>Elementary process</u> $f : (t, x, \omega) \longmapsto Z(\omega) \mathbb{1}_{(a,b]}(t) \mathbb{1}_A(x)$

for $0 \leq a < b$, Z bounded, $\mathcal{F}_a$-meas.

$A \in \mathcal{B}(\mathbb{R}^d)$, $\mathrm{Leb}(A) < \infty$

Linear combinations of elemt. process: simple processes.

<u>def</u> Let $\mathcal{P}_T$ be the Banach space of $f : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ s.t.

(1) f meas. in σ-field generated by all simple processes.

(2) $\exists (f_n)_{n \geq 1}$ simple process s.t.:

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} (f - f_n)^2(t, x, \omega) \, dt \, dx\right] \xrightarrow[n \to \infty]{} 0$$

For elementary process $f = \mathbb{1}_{(a,b]} \mathbb{1}_A Z$

$$\int_0^T \int_{\mathbb{R}^d} f(t,x) \, \xi(dt, dx) = Z \cdot \left(W_{T \wedge b}(A) - W_{T \wedge a}(A)\right)$$

$$\int_0^T \int_{\mathbb{R}^d} f(t,x) \, dW_t(dx)$$

<u>Lemma</u> $\left(\int_0^T \int_{\mathbb{R}^d} f(t,x) \, dW_t(dx), \, T > 0\right)$ is an $\mathcal{F}_T$-mgale, in $L^2(\Omega)$.

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} f(t,x) \, dW_t(dx)\right)^2\right] = \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left[f(t,x)^2\right] dt \, dx$$

Proof: exercise.

def/Prop Let $f \in \mathcal{P}_T$.

We define $\int_0^T \int_{\mathbb{R}^d} f(t,x)\, dW_t(dx)$

$$= \lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} f_n(t,x)\, dW_t(dx)$$

where $(f_n)_{n\geq 1}$ seq. of simple processes $\xrightarrow{\mathcal{P}_T} f$.

This is an $\mathcal{F}_T$-mgale, and

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} f(t,x)\, dW_t(dx)\right)^2\right] = \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left[f(t,x)^2\right] dt\, dx$$

Proof: Limit L^2 ✓

Mgale: limit of mgales.

$B \in \mathcal{F}_t$ $\mathbb{E}\left[M^n_{t+s} \mathbb{1}_B\right] = \mathbb{E}\left[M^n_t \mathbb{1}_B\right]$

$$\mathbb{E}\left[M_{t+s} \cdot \mathbb{1}_B\right] = \mathbb{E}\left[M_t \cdot \mathbb{1}_B\right] \checkmark$$

Define

$$(*) \begin{cases} \partial_t u = \frac{1}{2} \partial_x^2 u + u \cdot \xi \\ u(0,\cdot) = u_0(\cdot) \\ x \in \mathbb{R} \end{cases}$$

Th: Let $u_0 : \mathbb{R} \to \mathbb{R}$ in $L^2(\mathbb{R}, dx)$.
There exists a unique solution to $(*)$ which lives in $\mathcal{P}_T$, for any $T > 0$.

Proof * Existence (mild form)

Look for a fixed point of the map

$$\mathcal{M}_{T,u_0} : \mathcal{P}_T \longrightarrow \mathcal{P}_T$$

$$v \longmapsto \left\{ (t,x) \longmapsto \int_{\mathbb{R}} P_t(x-y) u_0(y)\, dy + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y) v(s,y) \xi(ds\, dy) \right.$$