$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u + \xi \\ u(t,0) = u(t,1) = 0 \end{cases}$$

$$\xi(t,\cdot) = \sum_{k\geq 1} B_t^{(k)} \cdot e_k,$$

$$e_k(x) = \sqrt{2}\sin(\pi k x)$$

$$\dot{u}_t^{(k)} = -\underbrace{\frac{(\pi k)^2}{2}}_{\lambda_k} u_t^{(k)} + B_t^{(k)}$$

$$\bigotimes_k N\left(0, \frac{1}{2\lambda_k}\right)$$

$$\beta = \sum_k \beta^{(k)} \cdot e_k,$$

$$\beta^{(k)} \sim N\left(0, \frac{1}{2\lambda_k}\right)$$

$$E\langle \beta, f\rangle\langle \beta, g\rangle$$

$$f = \sum_k a_k e_k, \quad g = \sum_k b_k e_k$$

$$\Rightarrow \underbrace{\mathbb{E}\langle \beta, f\rangle\langle \beta, g\rangle}_{\langle Qf, g\rangle} = \sum \frac{1}{2\lambda_k} \cdot a_k b_k$$

$$\partial_t u = (\Delta) u + \xi$$

$$\partial_t u = -(-\Delta)^{\frac{\beta}{2}} u + \xi$$

$$\widehat{(-\Delta) f}(\xi) = 4\pi^2 |\xi|^2 \cdot \hat{f}(\xi)$$

$$\widehat{(-\Delta)^{\frac{\beta}{2}} f}(\xi) = |\xi|^\beta \cdot \hat{f}(\xi)$$

take Fourier inversion.

$$(-\Delta)^{\frac{\beta}{2}} u = f$$

$$\widehat{(-\Delta)^{\frac{\beta}{2}} u}(\xi) = |\xi|^\beta \hat{u}(\xi) = \hat{f}(\xi)$$

$$\hat{u}(\xi) = |\xi|^{-\beta} \cdot \hat{f}(\xi)$$

$$u(x) = (K * f)(x)$$

where $\hat{K}(\xi) = |\xi|^{-\beta}$

$$K(x) = \int |\xi|^{-\beta} \cdot e^{2\pi i x \cdot \xi} d\xi$$

$$K(\lambda x) = \lambda^{-d} \int \lambda^{\beta} |\lambda\xi|^{-\beta} \cdot e^{2\pi i x \cdot \lambda \xi} d\lambda \xi$$

$$= \lambda^{-d+\beta} \cdot \int |\xi|^{-\beta} e^{-2\pi i x \xi} d\xi$$

$$= \lambda^{-d+\beta} \cdot K(x) \Rightarrow K(x) \sim |x|^{-d+\beta}$$

$$\partial_t u = -(-\Delta)^{\frac{\beta}{2}} u + \xi$$

$\forall \eta \in \mathbb{R}^d,\ t \in \mathbb{R}^+$.

$$\hat{u}'(t,\eta) = -|\eta|^\beta \hat{u}(t,\eta) + \hat{\xi}(t,\eta)$$

$$\hat{u}(t,\eta) = \int_0^t e^{-(t-s)|\eta|^\beta} \hat{\xi}(s,\eta)\, ds$$

$$\Rightarrow u(t,x) = \int_0^t (K_{t-s} * \xi_s)(x)\, ds$$

$$= K * \xi$$

space-time convolution.

$$\hat{K_t}(\eta) = e^{-t|\eta|^\beta}$$

If f is s.t. $\hat{f}(\eta) = e^{-|\eta|^\beta}$.

then $K_t(x) = t^{-\frac{d}{\beta}} \cdot f(x/t^{1/\beta})$

$$K(t,x) = t^{-\frac{d}{\beta}} \cdot f(x/t^{1/\beta})$$

$$K(\lambda^\beta t, \lambda x) = (\lambda^\beta t)^{-\frac{d}{\beta}} \cdot f(x/t^{1/\beta})$$

$$= \lambda^{-d} \cdot K(t,x)$$

$$|(t,x)| \sim |t|^{1/\beta} + |x|$$

$$K(\lambda z) = \underbrace{\lambda^{-(d+\beta)+\beta}} \cdot K(z)$$

$$\partial_t u = -(-\Delta)^{\frac{\beta}{2}} u + \xi$$

$$\mathbb{E}\, \xi(s,x)\xi(t,y) = \delta(s-t)\cdot\delta(x-y)$$

under parabolic metric,

$$\xi \in C^{-\frac{d}{2}-1-}$$

now, $\xi \in C^{-\frac{d}{2}-\frac{\beta}{2}-}$

$$\Rightarrow u \in C^{\frac{\beta-d}{2}-}$$