$$\begin{cases} \partial_t u = \frac{1}{2}\partial_x^2 u + u \cdot \xi & x \in \mathbb{R} \\ & t > 0 \\ u(t=0,\cdot) = u_0(\cdot) \end{cases}$$

<u>Th</u>: Let $u_0 : \mathbb{R} \to \mathbb{R}$ be in $L^2(\mathbb{R}, dx)$.
Then there exists a unique solution which is in $\mathcal{P}_T$, for any $T > 0$.

$$\mathcal{P}_T = \left\{ f : \begin{array}{l} \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R} \\ (t, x, \omega) \mapsto f(t,x,\omega) \end{array} \quad \int_t \int_x \mathbb{E}\left[f(t,x)^2\right] dt\, dx < \infty \right\}.$$

<u>Proof</u>: * Existence / Uniqueness of Mild sol°

$$u(t,x) = P_t * u_0(x) + \int_0^t P_{t-s} * (u_s \cdot \xi(ds, dy))(x)$$

Picard <u>I</u>terations:

Let's define $v_0(t,x) = P_t * u_0(x)$.

Recursively, we set

$$v_{n+1}(t,x) := v_0(t,x) + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y) v_n(s,y) \xi(ds,dy)$$

We have the map:

$$\mathcal{M}_{T,u_0} : \mathcal{P}_T \longrightarrow \mathcal{P}_T$$
$$v \longmapsto \left(v_0 + \int_0^t \int_{\mathbb{R}} P_{t-s} * (v(s,\cdot)\xi(ds,\cdot)) \right)$$

Want a fixed point to this map. $\quad t \in [0,T]$

Let's show that we have a contraction:

$$\int_0^T \int_{x \in \mathbb{R}} \mathbb{E}\left[|v_{n+1}(t,x) - v_n(t,x)|^2 \right] dt\, dx$$
$$= \int_0^T \int_x \mathbb{E}\left[\left(\int_0^t \int_{y \in \mathbb{R}} P_{t-s}(x-y)(v_n(s,y) - v_{n-1}(s,y)) \xi(ds,dy) \right)^2 \right] dt\, dx$$
$$= \int_{t=0}^T \int_x \int_{s=0}^t \int_y P_{t-s}^2(x-y) \mathbb{E}\left[|v_n(s,y) - v_{n-1}(s,y)|^2 \right] ds\, dy\, dt\, dx$$
$$= \int_{s=0}^T \int_y \int_{t=s}^T \int_x P_{t-s}^2(x-y) \mathbb{E}\left[|v_n(s,y) - v_{n-1}(s,y)|^2 \right] dt\, dx\, ds\, dy$$

(we have $\int_x P_{t-s}^2(x-y)\, dx = \frac{1}{\sqrt{2\pi 2(t-s)}}$

$$= \int_{s=0}^T \int_y \mathbb{E}\left[|v_n(s,y) - v_{n-1}(s,y)|^2 \right] \int_{t=s}^T \frac{dt}{2\sqrt{\pi(t-s)}}\, dy\, ds$$
$$\lesssim \sqrt{T}\, \| v_n - v_{n-1} \|_{\mathcal{P}_T}^2$$

Choose T^* small enough, we have

$$\| \mathcal{M}_{T^*,u_0}(v_n) - \mathcal{M}_{T^*,u_0}(v_{n-1}) \|_{\mathcal{P}_{T^*}} < \frac{1}{2} \| v_n - v_{n-1} \|_{\mathcal{P}_{T^*}}$$

Strictly contractive map in the Banach space $\mathcal{P}_T \Rightarrow$ unique fixed point u^*

Consider $\mathcal{M}_{T^*, u^*_{T^*}}$ is contractive in $\mathcal{P}_{T^*}$. So it has a unique fixed point u^{**}.

Now we set: $u(t,x) = \begin{cases} u^*(t,x) & \text{if } t \leq T^* \\ u^{**}(t-T^*,x) & \text{if } t > T^* \end{cases}$

* Check that u is a fixed point for $\mathcal{M}_{2T^*,u_0}$

This yields the existence of a solution, on any arbitrary interval $[0,T]$.

<u>Uniqueness</u>: if u is a fixed point of $\mathcal{M}_{T,u_0}$, then necessarily, u is a fixed point of $\mathcal{M}_{T^*,u_0}$. So $u \equiv u^*$ on $[0,T^*]$ by the uniqueness of the point there. Then, if we set $v(t,x) = u(t+T^*,x)$ (check that) v is a fixed point of $\mathcal{M}_{T^*,u_{T^*}^*}$. (+Iterate).

$*$ Weak form / Mild form –

Weak form: $\varphi \in C_c^\infty(\mathbb{R})$

$$\langle u_t, \varphi\rangle = \langle u_0, \varphi\rangle + \frac{1}{2}\int_0^t \langle u_s, \varphi''\rangle ds$$
$$+ \int_0^t \int_X u_s(x)\varphi(x)\,\xi(ds,dx).$$

Suppose that u is a solution of the weak form. Let's show that u is the mild solution.

By Itô formula, and a density argument, we can show that $\forall\, \psi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$, we have:

$$\langle u_t, \psi(t,\cdot)\rangle = \langle u_0, \psi(0,\cdot)\rangle + \int_0^t \langle u_s, \partial_t \psi(s,\cdot)\rangle ds$$
$$+ \frac{1}{2}\int_0^t \langle u_s, \partial_x^2 \psi(s,\cdot)\rangle ds + \int_0^t \int_X u_s(x)\psi_s(x)\,\xi(ds,dx).$$

Take $\varphi \in C_c^\infty(\mathbb{R})$, $t > 0$, and set

$$\psi(s,x) = \begin{cases} P_{t-s} * \varphi(x) & \text{if } s \in [0,t] \\ \varphi(x) & \text{if } s > t. \end{cases}$$

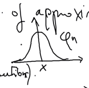
Apply the formula above with this test function:

$$\langle u_t, \varphi\rangle = \langle u_0, P_t * \varphi\rangle + \int_0^t \langle u_s, \partial_s P_{t-s} * \varphi\rangle ds$$
$$+ \frac{1}{2}\int_0^t \langle u_s, \partial_x^2 P_{t-s} * \varphi\rangle ds + \int_0^t \int_X u_s(x) P_{t-s} * \varphi\,\xi(ds,dx)$$

$$\partial_s P_{t-s} + \frac{1}{2}\partial_x^2 P_{t-s} \equiv 0 \quad \forall\, t-s > 0.$$

<u>So</u>: $\langle u_t, \varphi\rangle = \langle u_0, P_t * \varphi\rangle + \int_0^t \int_X u_s(x) P_{t-s} * \varphi(x)\,\xi()$

Take φ_n to be a seq. of approximations to the Dirac mass δ_x.



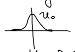
$\varphi_n \xrightarrow[n\to\infty]{} \delta_x$ (distribution).

Lebesgue differentiation th: $(n \to \infty)$

$$u_t(x) = \langle u_0, P_t(x-\cdot)\rangle + \int_0^t \int_y u_s(y) P_{t-s}(x-y)\,\xi()$$

$$\frac{1}{\lambda(B(\frac{1}{n},x))}\int_{B(\frac{1}{n},x)} |u_t(y) - u_t(x)|\,dy \to 0.$$

$$\begin{cases} \partial_t u = \partial_x^2 u + u \cdot \xi & x \in \mathbb{R},\ t>0 \\ u(t=0,\cdot) = u_0(\cdot) \end{cases}$$

Q^0: If u_0 is non-negative, non-zero, compactly supported , then what can we say on the support of the solution at $t>0$?

Is it cpctly supported? supp. unbounded? is $\operatorname{supp} u(t,\cdot) = \mathbb{R}$?

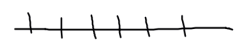
Intuition: 2 effects $\begin{cases} \text{smoothing } \partial_x^2 \\ \text{random fluct. of } u\cdot\xi \end{cases}$

① Assume $\xi \equiv 0$. $\partial_t u = \partial_x^2 u$

Then $u(t,x) = P_t * u_0(x)$.

so $\operatorname{supp}(u(t,\cdot)) = \mathbb{R}$.

② Assume that $\cancel{\partial_x^2 u}$: "$\partial_t u = u \cdot \xi$"

Consider $[n, n+1)$ 

$u(t,x) \equiv u^{(n)}(t)\quad \forall x \in [n,n+1)$

and consider:

$$\partial_t u^{(n)} = u^{(n)}\, dB_t^{(n)}$$

$$B_t^{(n)} = \langle \xi, \mathbb{1}_{[0,t]} \times \mathbb{1}_{[n,n+1)} \rangle$$

Boils down to the SDE: $\begin{cases} dX_t = X_t \cdot dB_t \\ X_0 \text{ given} \end{cases}$

A unique solution:

$$X_t = \begin{cases} X_0 e^{B_t - \frac{t}{2}} & (\text{Itô}) \\ X_0 e^{B_t} & (\text{Stratonovich}) \end{cases}$$

$\{X_t > 0\ \forall t>0\}$ iff $X_0 > 0$.

Th (Mueller 1991) If u_0 is non-neg. non-zero, compactly supported then, $\forall t>0$, a.s. $u(t,x)>0\ \forall x \in \mathbb{R}$.

Hopf-Cole: $-\log u(t,x)$ solves KPZ (formally).

Proof Let's show that $\forall t>0, \forall M>0$ $\forall \delta \in (0,1)$ $\mathbb{P}\big(u_t(x)>0 \text{ on } (-M,M)\big) > 1-\delta$

Assume now that $u_0(x) \geq \beta \cdot \mathbb{1}_{(a,b)}(x)\quad x \in \mathbb{R}$.

$m \in \mathbb{N}$, $k \in \{1, \dots, m\}$, $\beta > 0$

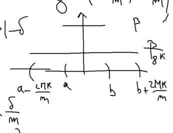
$$E_k := \left\{\omega \in \Omega : u\left(\frac{kt}{m}, x\right) \geq \frac{\beta}{8^k} \mathbb{1}_{\left(a - \frac{2Mk}{m},\, b + \frac{2Mk}{m}\right)}\right\}$$

Objective: $\mathbb{P}(E_m) > 1-\delta$

Suffices to get:

$$\mathbb{P}\left(E_{k+1}^c \mid E_1 \cap \dots \cap E_k\right) < \frac{\delta}{m}$$

$\forall k \in \{0, 1, \dots, m-1\}$.



$$u\left(\frac{t(k+1)}{m}, x\right) = \int_Y P_{\frac{t}{m}}(x-y)\, u\left(\frac{kt}{m}, y\right) dy + \int_0^{\frac{t}{m}} \int_Y P_{\frac{t}{m}-s}(x-y)\, u_s(y)\, \xi(ds, dy).$$

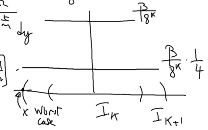
① Lower bound for the spread of the mass

On $E_1 \cap \dots \cap E_{k}$,

$$\int P_{\frac{t}{m}}(x-y)\, u\left(\frac{kt}{m}, x-y\right) dy \underset{\text{Comparison pcple}}{\geq} \int P_{\frac{t}{m}}(x-y) \frac{\beta}{8^k} \mathbb{1}_{I_k}(y)\, dy$$

$$\int_{y \in \mathbb{R}} \frac{1}{\sqrt{4\pi \frac{t}{m}}} e^{-\frac{(y-x)^2}{4\frac{t}{m}}} \frac{\beta}{8^k} \mathbb{1}_{I_k}(y)\, dy$$

$$\geq \int_{\left[\frac{2M}{m},\, b-a+\frac{4Mk}{m}+\frac{2M}{m}\right]} \frac{1}{\sqrt{4\pi\frac{t}{m}}} \frac{\beta}{8^k} e^{-\frac{y^2}{4\frac{t}{m}}} dy$$



$$\geq \frac{\beta}{8^k} \mathbb{P}\left(\mathcal{N}(0,1) \in \left(M \cdot \sqrt{\frac{2}{tm}},\ \frac{b-a}{\sqrt{\frac{2t}{m}}}\right)\right)$$

$$\geq \frac{\beta}{8^k} \cdot \frac{1}{4} \quad \text{for } m \text{ large}$$

---

② Control $\left| \int_0^{\frac{t}{m}} P_{\frac{t}{n}-s}(x-y)\, u_s(y)\, \xi(ds, dy) \right| \leq \frac{\beta}{8^k} \cdot \frac{1}{8}$ with high probab.

<u>Comparison pcple</u>: if $u_0^{(1)}(\cdot) \geq u_0^{(2)}(\cdot)$

Then $u_t^{(1)}(\cdot) \geq u_t^{(2)}(\cdot)\ \forall t > 0$.

<u>Proof</u>:
* Discretize eq° $u^{(i,n)}$
* Prove the ordering ppty $u^{(i,n)}$
* Prove uniform estimates $|u^{(i,n)} - u^{(i)}|$

<u>Discretisation</u>

$$\begin{cases} \partial_t \overline{P}^{(n)} = \Delta^{(n)} \overline{P}^{(n)} \\ \overline{P}^{(n)}\left(0, \frac{k}{n}\right) = \begin{cases} n & \text{if } k = 0. \\ 0 & \text{otherwise} \end{cases} \end{cases} \qquad \Delta^{(n)} f(x) = n^2\left(f\left(x+\tfrac{1}{n}\right) - 2f(x) + f\left(x-\tfrac{1}{n}\right)\right)$$

Yields $\overline{P}^{(n)}\left(t, \frac{k}{n}\right)$ $k \in \mathbb{Z}$

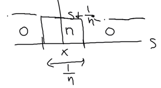
$$\overline{P}^{(n)}(t, x, y) = \overline{P}^{(n)}\left(t, \frac{k-l}{n}\right)$$

when $\frac{2k-1}{n} \leq x < \frac{2k+1}{n}$; $\frac{2l-1}{n} \leq y < \frac{2l+1}{n}$

$$P^{(n)}(s, t, x, y) = \overline{P}^{(n)}\left(\frac{[n^2 t] - [n^2 s]}{n^2}, x, y\right)$$

$P^{(n)}$ is a step fct° in each of its variables.

$$u^{(n)}(t, x) = \int_Y P^{(n)}(0, t, x, y)\, u^{(n)}(0, y)\, dy + \int_0^t \int_Y P^{(n)}(s, t, x, y)\, u^{(n)}(s, y)\, \xi(ds, dy)$$



$$d_t u^{(n)}(t, x) = \frac{1}{\sqrt{n}} u^{(n)}(t, x)\, dB_t^{(n)}(x)$$

Whenever $t \in \left[\frac{k}{n^2}, \frac{k+1}{n^2}\right)$