$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + F(u_\varepsilon) + \sqrt{2}\,\xi$$
$$- \varepsilon^2 \cdot \partial_x^4 u_\varepsilon + \varepsilon \cdot g(u_\varepsilon) \cdot \partial_x^2 u_\varepsilon + \varepsilon \cdot h(u_\varepsilon) \cdot (\partial_x u_\varepsilon)^2$$

$u: \mathbb{R}^+ \times S^1 \longrightarrow \mathbb{R}$

$F: \mathbb{R} \longrightarrow \mathbb{R}, \ \in C^1$

Thm: As $\varepsilon \to 0$, $u_\varepsilon \to u$, which solves

$$\partial_t u = \partial_x^2 u + \bar{F}(u) + \sqrt{2}\,\xi,$$

$$\bar{F}(u) = F(u) + \frac{1}{2} \cdot (h(u) - g'(u))$$

$$(*)\ \partial_t u = \partial_x^2 u + F(u) + \xi.$$
$$\partial_t \Psi = \partial_x^2 \Psi + \xi.$$

Claim: $u - \Psi$ is smoother than Ψ.

1st Claim: $(*)$ has a unique solution in $C([0,T], H^\alpha(S^1))$ for any $\alpha < \frac{1}{2}$.

$$u(t,x) = \int_0^t (P_{t-s} * F(u_s))(x)\, ds + \Psi(t,x)$$

$$(MU)(t) = \int_0^t P_{t-s} * F(U_s)\, ds + \Psi_t$$

- If $f \in H^\alpha$, then $\|P_t * f\|_\alpha < C \cdot \|f\|_\alpha$ where C is independent of t.

$$\|MU - MV\| \underset{\substack{\uparrow \\ \text{want}}}{\leq} \rho \cdot \|U - V\|, \quad \rho < 1$$

$$\|U - V\| = \sup_{t \leq T} \|U_t - V_t\|_\alpha$$

$$\|(MU - MV)(t)\|_\alpha$$
$$= \left\| \int_0^t (P_{t-s} * (F(U_s) - F(V_s)))\, ds \right\|_\alpha$$
$$\leq \int_0^t \|P_{t-s} * (F(U_s) - F(V_s))\|_\alpha\, ds$$
$$< C \cdot \int_0^t \|F(U_s) - F(V_s)\|_\alpha\, ds$$
$$< C \int_0^t \|U_s - V_s\|_\alpha\, ds$$
$$< C \cdot t \cdot \|U - V\|$$
$$\Rightarrow \|MU - MV\| < C \cdot t \cdot \|U - V\|$$

$\Rightarrow$ If T is small enough, then M is a contraction, and $\exists!$ in $C([0,T], H^\alpha(S^1))$, $\alpha < \frac{1}{2}$

$$\partial_t u = \partial_x^2 u + F(u) + \xi.$$
$$\partial_t \Psi = \partial_x^2 \Psi + \xi.$$

$$v = u - \Psi$$
$$\Rightarrow \partial_t v = \partial_x^2 v + F(v + \Psi)$$
$$(MV)(t) = \int_0^t P_{t-s} * F(V_s + \Psi_s)\, ds$$

Claim: If $f \in C^\alpha$, and $\beta > \alpha$,

then $\|P_t * f\|_\beta \lesssim t^{-\frac{\beta-\alpha}{2}} \cdot \|f\|_\alpha$

$$f = \sum_k \hat{f}(k) \cdot e_k, \quad e_k = \frac{e^{ikx}}{\sqrt{2\pi}}$$
$$f' = \sum_k ik \cdot \hat{f}(k) \cdot e_k$$

$$\|f\|_2^2 = \sum_k |\hat{f}(k)|^2$$
$$\|f'\|_2^2 = \sum_k k^2 \cdot |\hat{f}(k)|^2$$
$$\|f\|_{H^\alpha}^2 = \sum_k k^{2\alpha} \cdot |\hat{f}(k)|^2$$
$$g = P_t * f, \quad \Rightarrow \hat{g}(k) = e^{-tk^2} \cdot \hat{f}(k)$$
$$\|g\|_\beta^2 = \sum_k e^{-tk^2} \cdot |\hat{f}(k)|^2 \cdot |k|^{2\beta}$$
$$= \sum_k e^{-tk^2} \cdot |k|^{2(\beta-\alpha)} \cdot |\hat{f}(k)|^2 \cdot |k|^{2\alpha}$$
$$\leq \left(\sup_k e^{-tk^2} |k|^{2(\beta-\alpha)}\right) \cdot \|f\|_\alpha^2$$

$$h(x) = e^{-tx} \cdot x^{\beta-\alpha}$$
$$h'(x) = -t \cdot \cancel{e^{-tx}} \cdot x^{\cancel{\beta-\alpha}} + (\beta-\alpha) \cdot \cancel{e^{-tx}} \cdot \cancel{x}^{\beta-\alpha-1} = 0$$
$$x^* \Rightarrow \frac{\beta-\alpha}{t}$$
$$h(x^*) = C \cdot \left(\frac{\beta-\alpha}{t}\right)^{\beta-\alpha} = C \cdot t^{-(\beta-\alpha)}$$

$$\Rightarrow \|P_t * f\|_\beta^2 \lesssim t^{-(\beta-\alpha)} \cdot \|f\|_\alpha^2$$
$$\Rightarrow \|P_t * f\|_\beta \lesssim t^{-\frac{\beta-\alpha}{2}} \cdot \|f\|_\alpha$$

$$(MV)(t) = \int_0^t P_{t-s} * F(V_s + \Psi_s)\, ds$$
$$\|MV(t)\|_\beta \lesssim \int_0^t (t-s)^{-\frac{\beta-\alpha}{2}} \cdot \underline{\|F(V_s + \Psi_s)\|_\alpha}\, ds$$

$$\partial_t u = \partial_x^2 u + f(u) + \sqrt{2}\cdot \xi(t,x)$$
$$-\varepsilon^2 \partial_x^4 u + \varepsilon\cdot g(u)\cdot \partial_x^2 u + \varepsilon\cdot h(u)\cdot(\partial_x u)^2$$

$$\partial_t \phi = \partial_x^2 \phi + \xi.$$

$$\left.\begin{aligned} \xi &= \sum_k W_k(t)\cdot e_k \\ \phi &= \sum_k \hat{\phi}(k)\cdot e_k \end{aligned}\right\} \Rightarrow \hat{\phi}_t'(k) = -k^2 \hat{\phi}_t(k) + \dot{W}_k(t)$$

$$\Rightarrow \hat{\phi}(k) = \int_0^t e^{-(t-s)\cdot k^2}\cdot dW_k(t)$$

$$\partial_t \Psi = (-\varepsilon^2\partial_x^4 + \partial_x^2)\Psi + \sqrt{2}\cdot\xi$$

$$\xi = \sum_k W_k(t)\cdot e_k, \quad e_k = \frac{e^{ikx}}{\sqrt{2\pi}}$$

$$W_k = \overline{W_{-k}}$$

$$d\hat{\Psi}(k) = -(\varepsilon^2 k^4 + k^2)\cdot\hat{\Psi}(k)\,dt + dW_k(t)$$

$$\hat{\Psi}(k) \sim N\left(0, \frac{1}{k^2+\varepsilon^2 k^4}\right)$$

$$u = \sum_k \hat{u}(k)\cdot e_k$$

$$\Rightarrow \hat{u}(k) \simeq N\left(0, \frac{1}{k^2+\varepsilon^2 k^4}\right)$$

want to understand: $\varepsilon\cdot h(u)\cdot(\partial_x u)^2$

$$F(u) = \underbrace{h(u)}_{v}\cdot\underbrace{(\partial_x u)}_{w}{}^2$$

$$\sum_n \hat{F}(n)\cdot e_n = \left(\sum_k \hat{v}_k e_k\right)\left(\sum_l \hat{w}(l)\cdot e_l\right)\cdot\left(\sum_m \hat{w}(m)\, e_m\right)$$

- $e_k\cdot e_l\cdot e_m = \frac{1}{2\pi}\cdot e_{k+l+m}$

$$\Rightarrow \hat{F}(n) = \frac{1}{2\pi}\cdot\sum_{k+l+m=n} \hat{v}(k)\cdot\hat{w}(l)\cdot\hat{w}(m)$$

$$\underset{\substack{\uparrow \\ \text{heuristic}}}{\Longrightarrow} \hat{F}(n) \simeq \frac{\hat{v}(n)}{2\pi}\cdot\sum_k |\hat{w}(k)|^2$$

$$w = \partial_x u \simeq \partial_x \Psi$$

$$\hat{w}(k) = ik\cdot\hat{u}(k) \simeq ik\cdot\hat{\Psi}(k)$$

$$|\hat{w}(k)|^2 \sim \frac{k^2}{k^2+\varepsilon^2 k^4} = \frac{1}{1+(\varepsilon k)^2}$$

- $\varepsilon\cdot\hat{F}(n)$

$$= \varepsilon\cdot\frac{\hat{v}_n}{2\pi}\cdot\sum_k \frac{1}{1+(\varepsilon k)^2}$$

$$= \frac{\hat{v}_n}{2\pi}\cdot\sum_k \frac{\varepsilon}{1+(\varepsilon k)^2}$$

$$\simeq \frac{\hat{v}(n)}{2\pi}\cdot\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}\cdot\hat{v}(n)$$

$$\varepsilon F(u) = \varepsilon h(u) \cdot (\partial_x u)^2$$

$$\varepsilon \cdot \hat{F}(n) \approx \frac{1}{2} \hat{u}(n)$$

$$\varepsilon \cdot F(u) = \varepsilon \cdot \sum_n \hat{F}(n) \cdot e_n$$

$$\approx \frac{1}{2} \cdot \sum_n \hat{u}(n) \cdot e_n$$

$$= \frac{1}{2} \cdot h(u)$$

$$\varepsilon \cdot g(u) \cdot \partial_x^2 u$$

$$\langle g(u) \partial_x^2 u, \varphi \rangle$$

$$= \int g(u) \cdot u'' \varphi \, dx$$

$$= \int g(u) \cdot \varphi \cdot du'$$

$$= - \int u' \cdot d(g(u) \cdot \varphi)$$

$$= - \int g'(u) (\partial_x u)^2 \cdot \varphi \, dx$$

$$- \int g(u) \cdot \partial_x u \cdot \varphi' \, dx$$

$$\varepsilon \langle g(u) \cdot \partial_x^2 u, \varphi \rangle$$

$$= - \langle g'(u) \cdot (\partial_x u)^2, \varphi \rangle \cdot \varepsilon$$

$$- \langle g(u) \cdot \partial_x u, \varphi' \rangle \cdot \varepsilon$$

$$\varepsilon \cdot g(u) \cdot \partial_x^2 u = \varepsilon \cdot \partial_x (g(u) \cdot \partial_x u) \quad ee$$

$$- \varepsilon \cdot g'(u) \cdot (\partial_x u)^2$$

$$\downarrow$$

$$- \frac{1}{2} g'(u)$$

$$\Longrightarrow \bar{f}(u) = f(u) + \frac{1}{2}(h(u) - g'(u))$$

ξ T . u^2 $2u$

$$\partial_t u = \partial_x^2 u - u^3 + \xi$$