MA3D9 Example Sheet 5

1. Let v and w be tangent vector fields along a curve $\gamma : I \to S$. Prove that

$$\frac{d}{dt} < v(t), w(t) > = < \nabla_\gamma v, w > + < v, \nabla_\gamma w > .$$

Solution:

$$\begin{aligned} \tfrac{d}{dt} < v(t), w(t) > &= < \tfrac{dv(t)}{dt}, w(t) > + < v(t), \tfrac{dw(t)}{dt} > \\ &= < \nabla_\gamma v + (\tfrac{dv}{dt} \cdot \mathbf{N})\mathbf{N}, w > + < v, \nabla_\gamma w + (\tfrac{dw}{dt} \cdot \mathbf{N})\mathbf{N} > \\ &= < \nabla_\gamma v, w > + < v, \nabla_\gamma w > \end{aligned}$$

since $\mathbf{N}$ is orthogonal to the tangent plane $T_\gamma S$.

2. Let $\gamma(s)$ be a curve parametrized by arclength s, with nonzero curvature. Consider the parametrized surface

$$\sigma(s, v) = \gamma(s) + v\mathbf{b}(s), s \in I, -\epsilon < v < \epsilon, \epsilon > 0,$$

where $\mathbf{b}$ is the binormal vector of γ. Prove that if ϵ is small, $\sigma(I \times (-\epsilon, \epsilon)) = S$ is a regular surface over which $\gamma(I) = \sigma(I \times 0)$ is a geodesic.

Solution:

At $\sigma(I \times 0)$, $\sigma_s = \mathbf{t}(s) - v\tau\mathbf{n}(s)$, $\sigma_v = \mathbf{b}(s)$. So $\sigma_v \times \sigma_s = \mathbf{n}(s) + v\tau\mathbf{t}(s) \neq 0$. Since it is continuous, when ϵ is small, $\sigma_v \times \sigma_s$ is still nonzero at $\sigma(I \times (-\epsilon, \epsilon)) = S$. So S is a regular surface.

When $v = 0$, the normal vector of surface $\mathbf{N} = \mathbf{n}$. So $\gamma'' = \kappa\mathbf{n}$, which is perpendicular to the tangent plane. So $\gamma(I) = \sigma(I \times 0)$ is a geodesic.

3. Let T be a torus of revolution which we shall assume to be parametrized by

$$\sigma(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u).$$

Prove that

a. If a geodesic is tangent to the parallel $u = \frac{\pi}{2}$, then it is entirely contained in the region of T given by $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$.

b. A geodesic that intersects the parallel $u = 0$ under an angle θ $(0 < \theta < \frac{\pi}{2})$ also intersects the parallel $u = \pi$ if $\cos\theta < \frac{a-r}{a+r}$.

Hint: use Clairaut's relation.

Solution:

a.) Clairaut's relation says that for a geodesic, let R be the radius of the parallel of intersections and θ be the angle with the parallel, $R\cos\theta$ is fixed, which is a in this case (notice $\theta = 0$). So for $R\cos\theta = a$ to hold in general, $R \geq a$. So $\cos u \geq 0$, which is $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$.

b.) In this case $R\cos\theta = (a+r)\cos\theta$. To intersect $u = \pi$, $a - r \geq (a-r)\cos\theta_\pi = (a+r)\cos\theta$. But if the equality holds, the geodesic must tangent to $u = \frac{\pi}{2}$. Notice $u = \frac{\pi}{2}$ is also a geodesic since it is a critical point of $f(u) = r\cos u + a$. It contradicts to the fact that there is unique geodesic passing through a given point and tangent to a given direction. Hence $\cos\theta < \frac{a-r}{a+r}$.

4. *Surface of Liouville* are those surfaces for which it is possible to obtain a system of local coordinates $\sigma(u, v)$ such that the coefficients of the first fundamental form are given as

$$E = G = U(u) + V(v), F = 0.$$

Prove that

a. The geodesic of a surface of Liouville may be obtained by

$$\int \frac{du}{\sqrt{U-c}} = \pm \int \frac{dv}{\sqrt{V+c}} + c_1,$$

where c and c_1 are constants that depend on the initial conditions.

b. If θ, $0 \leq \theta \leq \frac{\pi}{2}$, is the angle which a geodesic makes with the curve $v = const$, then

$$U\sin^2\theta - V\cos^2\theta = const.$$

(this is the analogue of Clairaut's relation.)

Solution:

a.) We only need to check that the curves given are geodesics since c_1 and c gives the initial conditions for point and tangent direction respectively. To check they are geodesics, we play the following trick: introduce new coordinates

$$du' = \sqrt{U-c}\,du \pm \sqrt{V+c}\,dv$$
$$dv' = \frac{du}{\sqrt{U-c}} \mp \frac{dv}{\sqrt{V+c}}$$

The first fundamental form becomes $du'^2 + (U-c)(V+c)dv'^2$. In this fundamental form, $v' = const$ (the so called u' curves) are geodesics by geodesic equations. These are exactly the curves in our statement.

b.) Suppose our geodesic is unit-speed, so $(U+V)(u'^2 + v'^2) = 1$. So

$$v'^2 = \frac{1}{(U+V)(1+(\frac{du}{dv})^2)} = \frac{V+c}{(U+V)^2}$$

$$u'^2 = \frac{1}{(U+V)(1+(\frac{dv}{du})^2)} = \frac{U-c}{(U+V)^2}$$

We calculate that $\cos^2\theta = (U+V)v'^2$ and $\sin^2\theta = (U+V)u'^2$. So

$$U\sin^2\theta - V\cos^2\theta = \frac{U(V+c)}{U+V} - \frac{V(U-c)}{U+V} = c.$$