MA4E0 Exercise Sheet 1

1. Show $\mathrm{SO}(n)$ is a Lie group of dimension $\frac{n(n-1)}{2}$.

2. Show $\mathrm{Sp}(2n,\mathbb{R})$ is a Lie group of dimension $2n^2+n$.

3. Show $\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n,\mathbb{R})$.

4. Any element in $\mathrm{SO}(m)$, $m = 2n$ or $2n+1$, is similar to an element in

$$\begin{pmatrix} \cos t_1 & -\sin t_1 & & & & \\ \sin t_1 & \cos t_1 & & & & \\ & & \ddots & & & \\ & & & & \cos t_n & -\sin t_n \\ & & & & \sin t_n & \cos t_n \end{pmatrix}, \begin{pmatrix} \cos t_1 & -\sin t_1 & & & & & \\ \sin t_1 & \cos t_1 & & & & & \\ & & \ddots & & & & \\ & & & & \cos t_n & -\sin t_n & \\ & & & & \sin t_n & \cos t_n & \\ & & & & & & 1 \end{pmatrix}.$$

   Show that these elements are path-connected to the identity matrix within $\mathrm{SO}(m)$. Use this to show that $\mathrm{SO}(m)$ is path connected.

5. Show both Lie groups $\mathrm{U}(2)$ and $\mathrm{SU}(2) \times S^1$ are diffeomorphic to $S^3 \times S^1$. Calculate the center of both groups, and show that they have different Lie group structures.

6. Show the commutator of the vector fields on a smooth manifold satisfies the conditions of a Lie algebra, *i.e.*

   (a) $[X,Y] = -[Y,X]$ (skew symmetry).

   (b) $[aX+bY,Z] = a[X,Z]+b[Y,Z]$ ($\mathbb{R}$-linearity).

   (c) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] = 0$ (Jacobi identity).

7. Show that both the spaces of symmetric matrices and Hermitian matrices are not Lie algebras with standard commutator.

8. When $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, show the Lie bracket on $\mathfrak{g}$ induces a Lie bracket on the quotient $\mathfrak{g}/\mathfrak{h}$.