MA4E0 Exercise Sheet 4

1. Let T be a maximal torus of the compact connected Lie group G. A Lie algebra is called abelian if $[X, Y] = 0$ for all X, Y in the Lie algebra. Show that $\mathfrak{t}$ is a maximal abelian Lie subalgebra of $\mathfrak{g}$.

2. Show that each homomorphism from $T^n = \mathbb{R}^n / \mathbb{Z}^n$ to S^1 has the form
$$f(v_1, \cdots, v_n) = e^{2\pi i(\alpha_1 v_1 + \cdots + \alpha_r v_r)}, \quad \alpha_j \in \mathbb{Z}.$$

3. Let $S \subset G$ be a closed subgroup such that $G = \cup_{g \in G} gSg^{-1}$. Show that S contains a maximal torus.

4. Let G be a (not necessarily compact) Lie group and $H \subset G$ a 1-PSG which is not closed. Show that $\bar{H}$ is a torus.

5. Show that the exponential map on $\mathrm{GL}(n, \mathbb{C})$ is surjective.

6. (a) Show that the elements of symplectic group $\mathrm{Sp}(n) \subset \mathrm{U}(2n)$ have the form $\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$.

   (b) We have canonical inclusion $\mathrm{U}(n) \to \mathrm{Sp}(n), A \mapsto \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}$. Show that the image of a maximal torus in $\mathrm{U}(n)$ is a maximal torus in $\mathrm{Sp}(n)$.

   (c) Show that the Weyl group of $\mathrm{Sp}(n)$ is $G(n)$, the group of permutations ϕ of the set $\{-n, \cdots, -1, 1, \cdots, n\}$ with $\phi(-k) = -\phi(k)$ for all $1 \leq k \leq n$.